



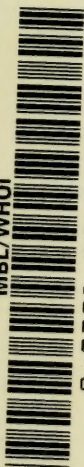




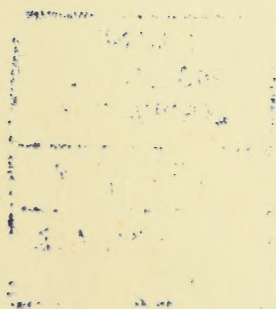
# NAVAL HYDRODYNAMICS

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Sixth Symposium

# NAVAL HYDRODYNAMICS

PHYSICS OF FLUIDS, MANEUVERABILITY AND OCEAN  
PLATFORMS, OCEAN WAVES, AND SHIP-GENERATED  
WAVES AND WAVE RESISTANCE

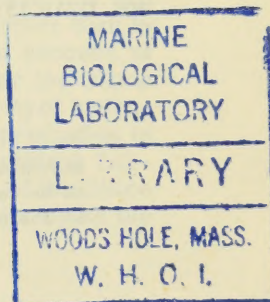
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**RALPH D. COOPER**  
**STANLEY W. DOROFF**  
*Editors*



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"Third Symposium on Naval Hydrodynamics: High-Performance Ships," Office of Naval Research, Department of the Navy, ACR-65, 1960; Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., Catalog No. D210.15:ACR-65; \$3.50

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"The Collected Papers of Sir Thomas Havelock on Hydrodynamics," Office of Naval Research, Department of the Navy, ACR-103, 1963; Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., Catalog No. D210.15:ACR-103; \$4.50

"Fifth Symposium on Naval Hydrodynamics: Ship Motions and Drag Reduction," Office of Naval Research, Department of the Navy, ACR-112, 1964; Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., Catalog No. D210.15:ACR-112; \$7.25

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## PREFACE

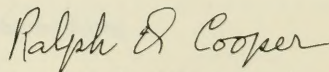
This Symposium is the sixth in a series each of which has been concerned with various aspects of Naval Hydrodynamics. The first (held in September 1956) presented critical surveys of Hydrodynamics that are of significance in naval science. Subsequent meetings were to be devoted to one or more topics selected on the basis of importance and need for research stimulation, or of particular current interest.

In keeping with this objective, the second symposium (August 1958) had for its subject the areas of hydrodynamic noise and cavity flow, the third (September 1960) was concerned with the area of high performance ships, the fourth (August 1962) emphasized the topics of propulsion and hydroelasticity, and the fifth (September 1964) addressed itself to coverage of the areas of ship motions and drag reduction.

Still continuing with the original plan, a threefold theme was selected for the present symposium, thus emphasizing, among other things, the interest in the current problems and latest accomplishments associated with physics of fluids, ocean waves, and maneuverability and ocean platforms. Important progress has been made in various aspects of ship maneuverability, and this meeting served to stimulate further effort and to emphasize a relatively little documented field of endeavor. The broad field of waves covers many aspects of ship waves and wave resistance and ocean or wind-generated waves. Physics of fluids covers many fundamental aspects of ship problems: cavitation, viscoelastic flows, turbulence, and turbulent boundary layers.

The international flavor of these meetings continues to be an outstanding feature and, in this case, has been enhanced by the large percentage of authors and chairmen from foreign countries.

The address of welcome by Dr. Davis and the speech opening the symposium by Captain Melson more than adequately describe the background and objectives of this meeting, thus leaving little more to be said other than to express our gratitude to all those who contributed so much to the success of this symposium. However, taking the liberty of speaking both for the Office of Naval Research, as well as the international scientific community of hydrodynamicists, I should like once again to express our deepest appreciation to Dr. J. P. Breslin, Mr. D. Savitsky, and Mr. J. S. Welker of the Davidson Laboratory, Stevens Institute of Technology, and to Mr. P. Granville of the Office of Naval Research for their outstanding efficiency and care in managing the many varied aspects of this symposium.



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Head, Fluid Dynamics Branch  
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# WELCOME

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*Stevens Institute of Technology*  
*Hoboken, New Jersey*

In behalf of the Stevens Institute of Technology it is a pleasure for me to be here and to welcome you to the Sixth Symposium on Naval Hydrodynamics. We at Stevens are happy to be the co-sponsor of these meetings, which have attracted participants from throughout the world.

In looking over your program I noticed that many of the papers are concerned with rather fundamental exploratory research. This was particularly interesting to me because I am associated with an educational institution, and colleges and universities have traditionally been the centers of what might be termed basic research.

Naturally, your program will also emphasize applications or activities directed toward the improvement and refinement of existing methods and techniques; however, the attention which is given to basic research seems to me to illustrate the broad range of opportunities for new developments and new applications which lie ahead in hydrodynamics.

During its 20-year existence, ONR has continually stressed the importance of exploratory studies in the spirit of free investigation. It has been most gratifying for Stevens to have had the understanding support of such an organization in its own studies in hydrodynamics. Such support to Stevens, as well as other institutions and organizations, has resulted in many advances in naval hydrodynamics including increases in the effectiveness of both surface and subsurface vessels.

Man is reaching out into space in his search for new knowledge. He is also exploring the oceans to uncover secrets which may someday become vital to his continued existence. The late President Kennedy remarked that "knowledge of the oceans is more than a matter of curiosity; our very survival may hinge upon it." For centuries man has depended upon the oceans for travel and commerce. Now he looks to the sea as a source of fresh water, as a source of increased food supplies, and as a seeding area for the control of turbulent weather.

Inevitably man's new relationship with the oceans must depend on his knowledge of hydrodynamics and engineering skills. Meetings, such as this, which foster an exchange of information on fundamental research and its applications provide a major step in amassing the knowledge man must gain to meet future needs.

Stevens has a long history of contributions to maritime research dating back to the last century when one of the members of the Stevens family perfected an early steamboat. Our Davidson Laboratory has been engaged in hydrodynamics research since 1935, and we take pride in the fact that many results of the work of the Laboratory have been reported in meetings of this kind in previous years. Today it is with special pride that we share the sponsorship of this symposium during the ONR vicentennial year. Again, it is a pleasure to be with you and to welcome you to this forum.

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## OPENING ADDRESS

Captain L. B. Melson  
*Assistant Chief for Research*  
*Office of Naval Research*  
*Washington, D.C.*

Brought together here are scientists and engineers from the far corners of the earth to review the progress that has been made in the demanding and exciting field of hydrodynamics. This progress will provide clues as to the concepts and technologies of the future. In the late 1870's the Superintendent of the U.S. Patent Office resigned his position, explaining he had seen all the ideas patented that could ever possibly be submitted and had no reason to continue his work. Fortunately, his philosophy was not correct and the future at any point in time will always be bright and challenging.

Rear Admiral Leydon has mentioned a few of the new areas that have opened within the hydrodynamic field. These intriguing avenues, through collective thought and research, will lead the way to newer and wider applications. The United States of America, recognizing the importance of the oceans, has recently created by law a Committee chaired by the Vice President of our country and charged with the responsibility of insuring that our national efforts are channeled into the support of oceanographic development. The United States Navy, in meeting the requirements of the present, has reorganized its oceanographic efforts to insure that its support maintains a proper balance with the advancing technology. The Oceanographer of the Navy is now responsible for all research and development in oceanography, assisted by the Chief of Naval Research and the Chief of Naval Development as his principal deputies. Through this organization, we know we will encourage the collection of knowledge and information needed to emphasize the sciences concerned with the oceans and hydrodynamics.

Progress in translating the truths of sciences into the practical applications of technology always seems slow. All of us are impatient in seeing new concepts tested, their feasibility ascertained, and improvements becoming the order of the day. Yet, when we look back only a few years, we realize that many concepts have proved feasible and have been placed into practice. The integration of the knowledge of any given scientific discipline, coupled with the advances in others, have permitted the development of techniques and vehicles in a remarkably short time. In my youthful days as a budding naval architect, I was told any design that might be conceived had to meet three limitations. Any ship to be built had to be low enough to pass under the Brooklyn Bridge, be narrow enough for the Panama Canal, and be shallow enough to transit the Suez Canal. These restrictions stagnated marine construction for more than forty years.



Since the removal of these restrictions, through the combined efforts of all persons interested in advancing hydrodynamics and ship construction, remarkable advances have been achieved. Twenty years ago any proposal for the construction of a 100,000-ton hull would have been greeted with skepticism. Today magazines routinely report the plans for 300,000-ton ships on the back pages. Submarine designs for years called for construction of boats that would barely submerge below the lower limits of the continental shelves. Today the deep submersible is capable of extended operations at 6000 feet or more.

The impetus of the petroleum industry's search for offshore oil fields has brought home a realization that dynamics of the ocean must be understood before safe and suitable platforms and drill rigs may be constructed. When we view the future, it is obvious hydrodynamics is a vital and rapidly moving science and the years to come will open new vistas for the benefits of humanity through the efforts of those assembled here today.

In preparing for this symposium, we wish to express our thanks to the efforts of the Davidson Laboratory of Stevens Institute in planning and managing the many and varied aspects of the symposium. We also wish to acknowledge the outstanding support given us by the Hydromechanics Laboratory of the David Taylor Model Basin\* for their participation in the initial planning of the technical program.

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\*The Naval Ship Research and Development Center as of March 1967.

## DINNER ADDRESS

Vice Admiral R. Brard  
*Director*  
*Bassin d'Essais des Carènes de la Marine*  
*Paris, France*

Several participants in this Sixth Symposium on Naval Hydrodynamics have requested me to deliver an address to our hosts. That is a great honor and privilege. Thus, I am speaking this evening on behalf of the foreign guests of the Office of Naval Research and of the Davidson Laboratory. I speak also on behalf of the ladies who have the good fortune to attend this symposium — as is the case of my wife.

Indeed, it is a pleasant duty to express to Admiral Leydon, President Davis, Mr. Ralph Cooper, Dr. John Breslin, Mr. Daniel Savitsky, and their staffs how grateful we are to them for their hospitality.

We very much appreciate all that they are doing for us during our stay in Washington. I should like also to congratulate and thank all those who were in charge of printing the papers before the start of the Symposium. We realize that it was a feat of strength.

In 1960, Dr. Weyl, speaking at the Third Symposium banquet, said that naval hydrodynamics knew, at the time, a new spring. That was — and is still today — my own feeling. In initiating these symposia, ONR is, for an impressive part, at the origin of the improvements made in naval hydrodynamics. The Office of Naval Research has the merit to have understood, ten years ago, that new requirements would appear very soon and that it was important to encourage and promote the sciences and technologies on which the necessary progress would depend.

The evolution of things has been so quick that it can be considered a revolution. The so-called conventional ships, whose designs are presently in hand, deeply differ from those which have been conceived less than ten years ago. New types of ships, of propellers, of steering devices also appear. Simultaneously, new vehicles and machines are conceived, particularly for exploring the oceans and the bottom of the seas.

In opening this symposium, Capt. Melson reminded us of the words of the late President Kennedy concerning this last purpose. In fact, many of the participants of this symposium have read the memorable book entitled "Legacy of a President." Many of us here had these words in remembrance. All of us, certainly, are proud to contribute more or less directly to the goals of this noble book, which will lead to an improvement of the standards of life of humanity.

So is it possible to say that oceans, which separate the continents from one another, may be a matter of concord between the nations? They are still today the most practical way by which men can exchange goods and products. They are also as a common weal of all the peoples because of the great treasures lying hidden in the seas. Moreover, they are, in themselves, a thrilling subject of scientific research.

Science has no mother country. Its mother country, if any, is the world as a whole. Because of that, ONR's intention was — and is always — that its symposia be of an international nature. The Office of Naval Research has reached its goal, as is proved by the great number of nations represented here. But this goal implies a generous and continuous support without which it would be unattainable. All of the guests of ONR are perfectly conscious of this fact.

Therefore, we warmly congratulate and thank again the Office of Naval Research for its action, and, on this day, which may be considered as the twentieth anniversary of its birthday, we cordially formulate our best wishes for its prosperity. Identical wishes are also formulated for the Stevens Institute of Technology and the Davidson Laboratory.

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Thursday, September 29, 1966

Morning Session

## PHYSICS OF FLUIDS

Chairman: G. B. Schubauer

National Bureau of Standards  
Washington, D.C.

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# HYDRODYNAMIC ASPECTS OF MACROMOLECULAR SOLUTIONS

Marshall P. Tulin  
*Hydronautics, Incorporated*  
*Laurel, Md.*

## INTRODUCTION

It is now common knowledge among us that tens or even several parts per million of innocuous large molecules in liquid solution *drastically* alter some turbulent flows (Refs. 1, 2, and 3, for example). This discovery was not only of very great potential practical importance, but it also meant that hydrodynamics and all of us with it have been brought to the end of a very long age of innocence. We have now discovered for ourselves what those physical chemists, chemical engineers, and mathematicians who call themselves rheologists, have been trying to tell us all along: that fluids *can* store strain energy; that, as a result, the normal pressure in even the simplest shear flow is *not* an isotropic quantity; that accompanying both shear *and* extension, fluids in general possess both viscosity *and* stiffness; and that these latter quantities as we would define them, are *not* physical properties of the fluid, but depend in a rather complicated way upon the flow.

Perhaps we should not be blamed too much for holding on so long to the gilt-edge security of an old blue-chip like Navier-Stokes, in preference to the frankly rather speculative and volatile issues that have been offered up to us. Now, however, we clearly face the necessity to finance our adventures in polymerland by selling out Navier-Stokes.

The present work attempts in very brief form to: review some important concepts involving disperse systems (such as thermal forces, entropy, and strain energy); discuss the dynamics of molecular response to the motion of the solvent; introduce and discuss the idea of critical strain rates, which arises naturally in the theory of molecular response; relate molecular strain to fluid stresses through the use of Mohr's circle; and discuss in a speculative way the role of fluid stiffness in turbulent flows, as for example in providing a mode for the extraction of turbulent energy from an eddy through the generation of elastic shear waves. Finally, it is speculated how radiation damping could result in a twofold or threefold thickening of the viscous sublayer.

## DISPERSE SYSTEMS

Rheological effects occur in all solutions containing a disperse phase — such as colloidal suspensions and polymer solutions. These include most fluids in

industry and biology (blood, for example). The role of the dispersed phase depends upon the particulate size  $\lambda$  and upon the time scales  $t$  within the flow. Thermal, viscous, and inertial forces act upon microscopic particles and depend linearly on particle displacement and its time derivatives:

| <u>thermal forces</u>                    | <u>viscous forces</u>    | <u>inertial forces</u>        |
|------------------------------------------|--------------------------|-------------------------------|
| $O\left(\frac{kT}{\lambda^2} x_i\right)$ | $O(\lambda\mu\dot{x}_i)$ | $O(\rho'\lambda^3\ddot{x}_i)$ |

where  $x_i$  is the displacement in the ensemble in reference to the state of maximum entropy ( $\dot{x}_i = dx_i/dt$ , etc.),  $\mu$  is the liquid viscosity,  $\rho'$  is the mass density of a particle relative to the volume  $\lambda^3$ ,  $k$  is Boltzmann's constant, and  $T$  is the absolute temperature.

In accordance, the relative magnitudes of these forces depend upon  $\lambda$  and  $t$  in the following way:

$$\frac{\text{viscous}}{\text{thermal}} \propto \frac{\lambda^3\mu}{kT t}$$

$$\frac{\text{inertial}}{\text{thermal}} \propto \frac{\rho'\lambda^5}{kT t^2}$$

$$\frac{\text{inertial}}{\text{viscous}} \propto \frac{\rho'\lambda^2}{\mu t}$$

Upon estimating the constants of proportionality implicit in relations such as those above, the regimes of importance for the three forces acting on a dispersed phase may be depicted in a  $\lambda - t$  diagram as illustrated in Fig. 1.

The inertia of sufficiently small particles (somewhat below the line  $\lambda \sim t^{2/5}$ ) may be neglected in comparison to viscous and thermal forces; all large molecules fall in this category. As the figure attempts to show, the ability of these inertialess particles to store thermal energy in strain increases as the time scale decreases, i.e., as the frequency of impressed motion increases; for sufficiently long time scales the molecular motions result mainly in viscous dissipation. In this connection, time scales must be compared with the so-called first relaxation time of the molecule (which is  $10^{-3}$  sec for the large molecule indicated by the circle on the line  $\lambda \sim t^{1/3}$ ).

## THERMAL FORCES, ENTROPY AND STRAIN ENERGY

What are these thermal forces? They arise through the bombardment of the dispersed particles by the agitated molecules *within the solvent*, i.e., through Brownian motion. In the case of very large molecules or macromolecules, the forces acting on them can be related to the entropy loss or to the ordering of the molecular configuration in relation to its most disordered state, which is ideally that of a spherical cloud (see Ref. 4, pages 399-494, for

# Hydrodynamic Aspects of Macromolecular Solutions

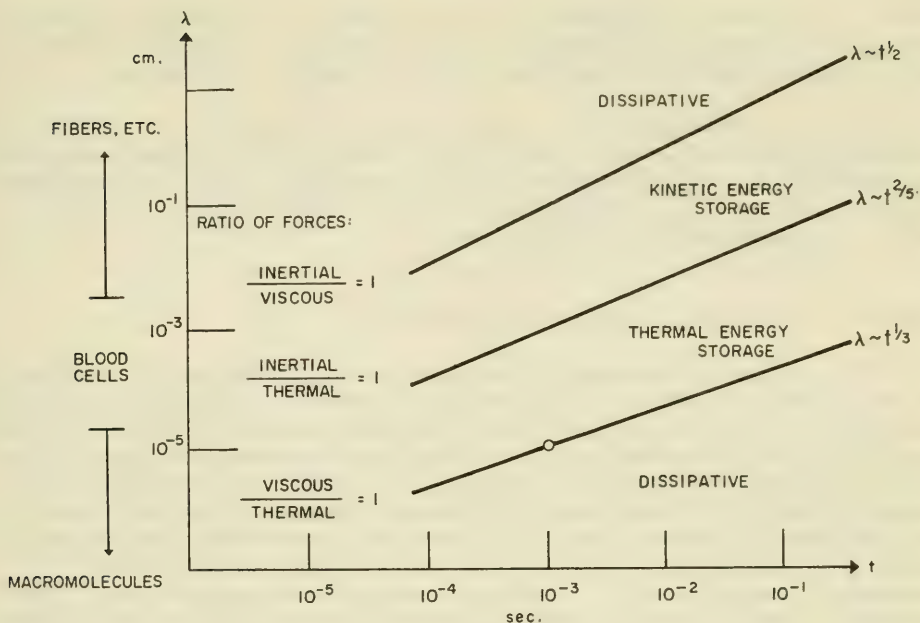


Fig. 1 - Effect of particle size ( $\lambda$ ) and straining time ( $t$ ) on flow processes

example). When strained relative to this state, free energy in an amount proportional to entropy change is stored within the molecule in vibrations of its links; we may properly call this the strain energy. It consists of a springlike portion due to the coherence of the chain, which resists its expansion, and another gaslike part which even a collection of separated particles possesses — and which resists compression. These ideas can be given a formal representation:

$$X_i = \frac{\partial \mathcal{L}}{\partial x_i},$$

where

$$\mathcal{L} = T \cdot S$$

and

$$\begin{aligned} S &= -k \ln [P(a_1, a_2, a_3)] \\ &= -\frac{k}{2} \underbrace{(a_1^2 + a_2^2 + a_3^2 - 3)}_{\text{springlike}} + k \underbrace{\ln (a_1 a_2 a_3)}_{\text{gaslike}}, \end{aligned}$$



where  $X_i$  are the thermal forces,  $\mathcal{L}$  is the Lagrangian or, in this case, the negative strain energy,  $S$  is the entropy,  $\alpha_i$  is the principal strain ( $x_i$ /relaxed molecular radius), and  $P$  is the principal strain probability distribution.

Almost all rheological effects exhibited by dilute polymer solutions are due to the existence of this strain energy. So is the elasticity of rubber, which is of precisely the same fundamental nature as the elasticity of dilute polymer solutions (5). Incidentally, the strain energy stored in stretched rubber, being thermal in nature, may be felt by the hand.

## NORMAL STRESSES AND MOHR'S CIRCLE

Their own inertia being negligible, the thermal and viscous forces acting on the molecules are in balance. The reaction of the viscous forces upon the fluid transmit the molecular thermal forces to the bulk medium. Therefore, in reaction to the internal thermal forces, boundary stresses must arise on an infinitesimal fluid element; both shear stresses and nonisotropic pressures result. In order to calculate these boundary stresses (a tensor) in flow coordinates it becomes useful to introduce that most ubiquitous construction from the mechanics of deformable media — Mohr's circle (for simplicity we take a two-dimensional viewpoint here). Now quite clearly the boundary stresses in flow coordinates depend upon the orientation of the particles' principal strain axes relative to the flow as well as upon the magnitude of the principal strains themselves. As indicated at the top of Fig. 2, the orientation of the strain axes depends upon the product of the molecular relaxation time and the strain rate (note that the rotation time in shear is the inverse of the latter, i.e.  $(\partial u/\partial y)^{-1}$ ). Mohr's circle may easily be used to show how both normal and shearing stresses arise due to the presence of strained molecules, and how their magnitudes depend in general upon the molecular strains and on the molecular orientation. In the case of low shears (left side of Fig. 2), and therefore with  $|\sigma_{11}|$  slightly greater than  $|\sigma_{22}|$ , the normal stresses  $\sigma_{xx}$  and  $\sigma_{yy}$  are positive, almost equal ( $\sigma_{xx} > \sigma_{yy}$ ), and, we may show, proportional to the square of the strain rate — all in accord with the contentions of Marcus Reiner (Ref. 6, page 42). For high enough shears, however, the normal stresses may become of different sign, with  $\sigma_{xx}$  positive and much larger than  $\sigma_{yy}$  in magnitude — as has sometimes been contended by others. Perhaps some of these old controversies can be squared away within the confines of Mohr's circle.

## THE DYNAMICS OF MACROMOLECULES

It turns out to be somewhat easier to predict the effect upon the fluid stresses of molecular strains and orientations than it is to predict these latter quantities themselves — as a generation of rheologists can testify. Here I must particularly mention such physical chemists as Kirkwood and Riseman (7), Rouse (8), Bueche (9), Peterlin (10), Sadron (11), and Hermanns (12). Although they have endowed us with a great legacy, a really satisfactory treatment of flexible molecules in even simple flows cannot be said to exist, that is, free from restrictions on small strain, etc. Then, too, as hydrodynamicists we must be concerned with rather complicated flows — in comparison to simple shearing,



## Hydrodynamic Aspects of Macromolecular Solutions

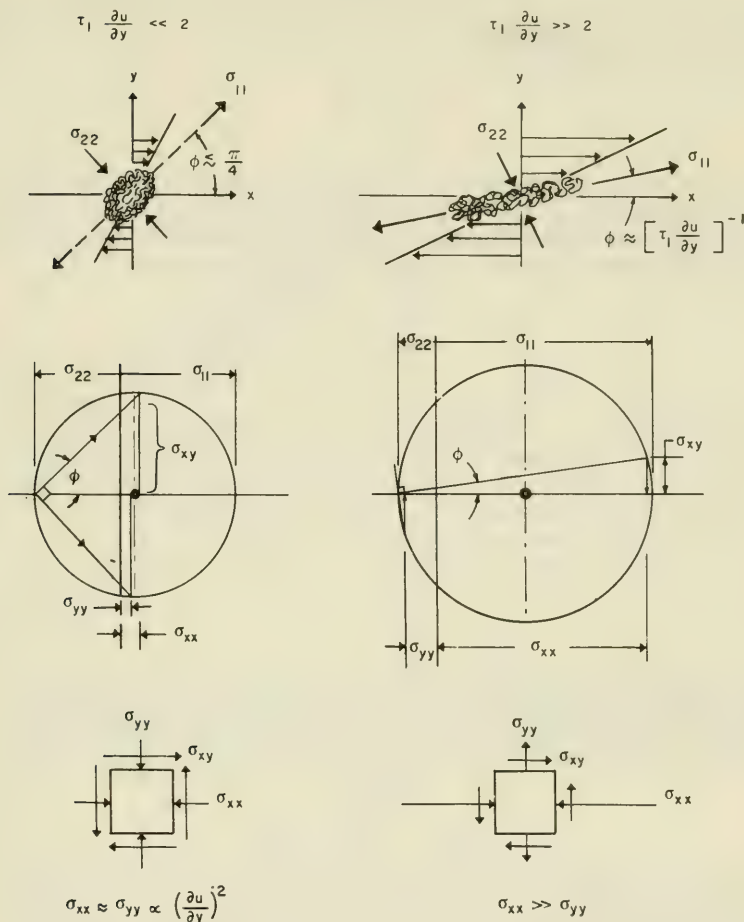


Fig. 2 - Mohr's circle and second order stresses

for instance. For both of these reasons we should be encouraged to consider for ourselves the response of flexible molecules being convected within flows.

We can adopt for this purpose the simplest model of the flexible molecule, according to which there exists no hydrodynamic interference between its different parts. This proves a limited but nevertheless highly fruitful model.

A large *flexible* molecule possesses many, say  $10^4$  links, virtually free to rotate at points of connection. Relative motion between the macromolecular links and the solvent in their immediate neighborhood results in dissipation of energy (because of the smallness of the Reynolds number, this dissipation,  $F$ , is quadratic in the velocities of the relative motion). In the absence of inertia,

changes in the strain energy must occur in equilibrium with viscous forces which may be described in terms of the dissipation:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}_i} = \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}_i}.$$

We may think of the molecular response in terms of the sum of the responses of independent groups of equal number of links, i.e., in modal terms. Accordingly, we may write dynamic equations, all based on the previous relation, for the molecular strains  $\alpha_i$  appropriate to each mode  $p$ :

$$\left( \alpha_i - \frac{1}{\alpha_i} \right)_p = \tau_p \left( \dot{\alpha}_i - u_i \sqrt{r_r^2/3} \right)_p.$$

These equations are nonlinear and of first order in the strain; notice the presence of a term due to the viscous stresses imposed by the flow velocities ( $u_i$ ) — which, itself, depends in part upon the stresses due to molecular strain. For a small strain disturbance  $(\Delta\alpha)_0$  imposed at time zero upon an equilibrium state, the strain  $(\Delta\alpha)$  decays exponentially in time:

$$\Delta\alpha_i = (\Delta\alpha_i)_0 e^{-2t/\tau_p}.$$

We call the characteristic time of decay the relaxation time  $\tau_p$ . Note its relation, in theory, to molecular size ( $r_r^2$ ), solvent viscosity ( $\eta$ ), temperature ( $T$ ), molecular dissipation function ( $C_{ip}$ ), and the mode number ( $p$ ):

$$\tau_p = \frac{\eta C_{ip} r_r^2}{3kT} = \frac{\tau_1}{p^2}.$$

We note how the higher modes have shorter relaxation times — in analogy to most mechanical systems.

The actual state of strain experienced by molecules in flows depends very much upon the relation between this molecular relaxation time and the characteristic times of the flow ( $(\partial u/\partial y)^{-1}$  in a shear flow); it is thus a quantity of great importance.

Twelve years ago, Prince Rouse (8) and others applied theory based on the same assumptions as we have indicated here to the prediction of viscosity and stiffness in unsteady Couette shearing. Provided that the fluid's relaxation time  $\tau_1$  is determined from the experiments themselves, the experimental data of Rouse and Sittel (13) are in wonderful agreement with the Rouse theory, as shown in Fig. 3. This gives us courage to pursue the theory utilizing the simple noninterference model of the molecule. Let us take this opportunity to notice in Fig. 3 how the energy storage ( $G'_r$ ) is enhanced at high frequencies, while the dissipation is depressed — in accord with the  $\lambda - \tau$  diagram presented as Fig. 1.

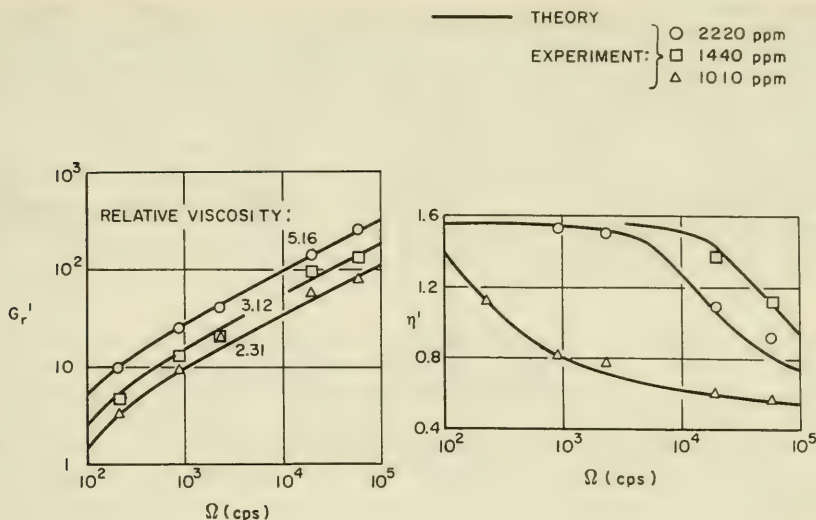


Fig. 3 - Experimental values (Rouse and Sittel) of dynamic stiffness and viscosity polystyrene ( $M = 6.2 \times 10^6$ ) in toluene at  $30.3^\circ\text{C}$

The considerations of Rouse and others were restricted to *small strains from the relaxed state*. However, the treatment of large strains is almost certainly essential for an understanding of general flows.

## LARGE STRAINS AND THE CRITICAL STRAIN RATE

Convecting fluid elements suffer shape distortion (strain) and rotation. The latter is partly associated with changes in streamline direction and partly with the vorticity; the effects of the latter are particularly important and subtle. The strains are due both to vorticity and to the fluid rates of strain associated with the nonvortical (potential) flow. In order easily to gain some understanding of molecular response without restrictions on the magnitude of strain, we consider the simplest case of all: simple extension. This corresponds roughly to the rectilinear potential flow approaching or leaving a symmetrical body along the stagnation streamlines, as indicated in Fig. 4. Assume a constant rate of strain  $A$ , as near the stagnation point on a circle or sphere. Then,

$$\left( \alpha_1 - \frac{1}{\alpha_1} \right)_p = \tau_p (\dot{\alpha}_1 - A \alpha_1)_p ; \quad \left( \alpha_2 - \frac{1}{\alpha_2} \right)_p = \tau_p (\dot{\alpha}_2 + A \alpha_2)_p .$$

An exact solution to these dynamic equations for the molecular strain along the streamline ( $\alpha_1$ ) and normal to it ( $\alpha_2$ ) may be found — with the remarkable result

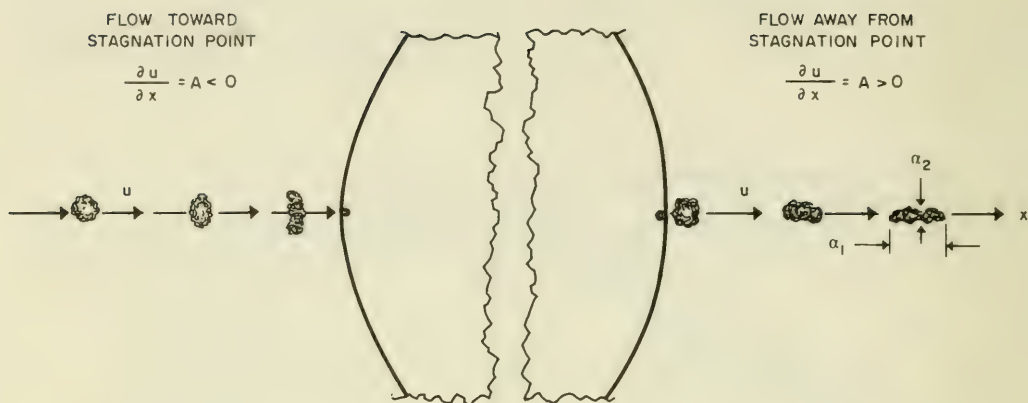


Fig. 4 - Schematic of molecular response in regions of constant rate of extension

$$\alpha_{1p}^2 = \frac{1 - A\tau_p e^{-(1-A\tau_p)2t/\tau_p}}{1 - A\tau_p} ; \quad \alpha_{2p}^2 = \frac{1 + A\tau_p e^{-(1+A\tau_p)2t/\tau_p}}{1 + A\tau_p} .$$

The behavior of the solution for large times is seen to depend on the value of the nondimensional strain rate ( $A\tau_p$ ) relative to unity, and we see that this simple theory predicts an unlimited strain  $\alpha_1$  for values of  $A\tau_p$  greater than unity:

$$A\tau_p < 1 \text{ (subcritical)}; \quad \alpha_{1p}^2 \rightarrow \frac{1}{1 - A\tau_p} ; \quad \alpha_{2p}^2 \rightarrow \frac{1}{1 + A\tau_p} ;$$

$$A\tau_p > 1 \text{ (supercritical)}; \quad \alpha_{1p}^2 \rightarrow \text{unbounded} .$$

Of course, the molecules cannot even in the case treated endure unlimited supercritical strains as indicated by theory, for its assumptions become invalidated first. Nevertheless, we may expect molecules to become greatly extended under certain flow conditions.

## SIMPLE SHEAR

In a simple shear flow, the radial strain rate is a maximum at an angle of  $45^\circ$  to the flow direction and there equals one-half of the maximum strain rate, say  $(\partial u / \partial y) / 2$ . We might thus expect the behavior of molecules in shear flows to be qualitatively different depending on whether  $\tau_1(\partial u / \partial y)$  is greater than or less than 2. Of course the molecule is rotating *through* the region of high shear, so that even in the supercritical case the extension of the molecule will be limited due to the finite time during which the high shear rates act. A careful discussion must involve the effects of rotary dispersion due to Brownian motion,



for it turns out that the characteristic angular dispersion time for a molecular mode is about equal to its relaxation time. This fact is of great importance in determining the strain for small shears. For supercritical shears, the elongated molecule is convected rapidly through the high shear region, and spends most of its lifetime within an angle to the horizontal which is of the order of  $1/(\tau_1 \partial u / \partial y)$ , as we earlier indicated in Fig. 2, top right side. Under *supercritical* conditions we find that the strain energy stored there is at first proportional to  $(\tau_1 \partial u / \partial y)^2$  and for higher shears to  $(\tau_1 \partial u / \partial y)^{1/2}$ . The change in stored energy is particularly rapid at the onset of supercritical conditions, where we estimate it increases by an order of magnitude as  $(\tau_1 \partial u / \partial y)$  increases from 2 to 4.

The kind of considerations and conclusions sketchily outlined here can lead to an improved understanding and prediction of rheological effects. For example, we are led to conclude that the unfolding and shearing over of molecules under supercritical shear conditions is almost certainly responsible for the so-called pseudoplastic behavior of polymer solutions in steady shearing. Beyond that, it suggests that the flow of a macromolecular solution be considered in terms of two separate but coupled flows of discrete fluids: the solvent and the imbedded molecules. A theory involving dual dynamic equations requiring simultaneous solution would result.

## TURBULENT FLOWS

Strain energy storage and release will effect turbulent flows. The magnitudes of elastic shear stresses associated with energy storage due to unsteady straining in a turbulent flow are easily estimated:

$$\frac{\text{elastic stresses}}{\text{Reynolds stresses}} \approx 2 \text{ strain } \left( \frac{c_s}{v_\ell} \right)^2 ;$$

$$\frac{\text{normal (second-order) stresses}}{\text{elastic stresses}} \approx \frac{\text{strain}}{2}$$

In ratio to the inertial or Reynolds stresses, we see that the elastic stresses depend upon the ratio of  $c_s$ , the speed of propagation of elastic shear waves, to the turbulent velocity  $v_\ell$ , where  $\ell$  is the characteristic eddy size.

The effective strain in a turbulent flow may be estimated as the product of the strain rate and the strain lifetime, where strain rate  $\approx v_\ell / \ell$  and strain lifetime  $\approx \ell / v_\ell$ , so that strain =  $O(1)$ .

In accordance with this rough order of magnitude estimate, the normalized elastic stresses (both shear and normal) are of the order of  $(c_s / v_\ell)^2$ .

The strain and therefore strain energy stored in elongated molecules subjected to a given imposed flow deformation is greater than for relaxed molecules. As a result, under certain conditions the bulk stiffness associated with the strain of molecules subjected to supercritical shears on a small scale can

be several orders of magnitude greater than if the molecules are relaxed, and as a result the potency of very small concentrations may become exaggerated. Of course, this does not mean to say that molecular unfolding or shear stiffening is *necessary* in order that macromolecules in solution affect turbulent flows. It does mean, however, that molecular unfolding may very well be responsible for the onset of those effects which begin to occur in the most dilute polymer solutions studied.

Some detailed considerations lead us to expect that this shear-stiffening effect will occur in a turbulent flow when the local shears associated with the dissipation scale are supercritical and when at the same time the local turbulence Reynolds number for the energy containing scale is less than about 100. The latter condition arises through the necessity that the characteristic time for local reorientation of the shear pattern on the smallest (dissipation) scale not be too much shorter than the duration of large scale strains due to shears on the largest (energy containing) scale. These conditions begin to occur in the turbulent shear flow closest to the wall itself when the rate of strain there becomes supercritical ( $\tau_1 \frac{\partial u}{\partial y} \lesssim 2$ ). The thickness of the viscous sublayer is, of course, determined in this region. The effects of macromolecules may thus be enhanced by unfolding in a region most sensitive from the point of view of wall friction.

In Table 1 are shown the theoretical dependencies (constants of proportionality are omitted) of bulk fluid stiffness upon molecular concentration  $c$ , molecular relaxation time  $\tau_1$ , and molecular weight  $M$  for the two cases involving the straining of relaxed and of unfolded molecules and for strain durations  $t_s$  both greater and less than  $\tau_1$ ; the case of relaxed molecules (first column) corresponds to that treated earlier by Rouse (8), etc.

Table 1  
Dependencies of Bulk Fluid Stiffness

|                                    | Subcritical:<br>$\tau_1 \frac{\partial u}{\partial y} < 2$ | Supercritical:<br>$\tau_1 \frac{\partial u}{\partial y} > 2$                         | Hypercritical:<br>$\tau_1 \frac{\partial u}{\partial y} \gg 10$                             |
|------------------------------------|------------------------------------------------------------|--------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------|
| $\frac{\tau_1}{t_s} < \frac{1}{2}$ | $\frac{c \tau_1^2}{M} t_s^{-2}$                            | $\frac{c \tau_1^4}{M} \left( \frac{1}{t_s} \frac{\partial u}{\partial y} \right)^2$  | *<br>$\frac{c \tau_1^{3/2}}{M} t_s^{-2} \left( \frac{\partial u}{\partial y} \right)^{1/2}$ |
| $\frac{\tau_1}{t_s} > 2$           | *<br>$\frac{c \sqrt{\tau_1}}{M} t_s^{-1/2}$                | $\frac{c \tau_1^{5/2}}{M} t_s^{-1/2} \left( \frac{\partial u}{\partial y} \right)^2$ | $\frac{c \tau_1}{M} \left( \frac{1}{t_s} \frac{\partial u}{\partial y} \right)^{1/2}$       |

\*Cannot occur in turbulent flow.

The case in the lower right corner (hypercritical) will tend to occur at the highest flow speeds and lowest concentrations; the conditions corresponding to

the upper left corner will tend to occur at the lowest flow speeds and largest concentrations. In the hypercritical case it may be shown that for large flexible molecules  $\tau_1 \propto M^{3/2}$ ; the stiffness therefore scales something like  $\text{conc.} \times M^{1/2}$ , a result in good agreement with the results of experiments with disks rotating at high speeds in polyethylene oxide (1).

## RADIATION DAMPING

What are the precise manifestations of the bulk fluid stiffness? There are no doubt many. For one, I have earlier offered the suggestion (14) that stiffness provides a mode for the extraction of turbulent energy from an eddy through the radiation of elastic shear waves; these, removed from the scene, later decay through the usual action of viscosity. In fact, this radiation may be calculated along the lines of the theory of the generation of sound by turbulence, pioneered by Lighthill (15). We are dealing with an incompressible fluid. The momentum balance is

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} T_{ij}^{(0)} + \frac{\partial}{\partial x_j} T_{ij}^{(1)} = 0,$$

where

$$T_{ij}^{(0)} = \left[ (u_i u_j) + \bar{p} \frac{\delta_{ij}}{\rho} - \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right],$$

which is the stress tensor for a normal viscous fluid, including the Reynolds stress  $(u_i u_j)$ , and where  $T_{ij}^{(1)}$  is the additional stress tensor due to the elastic properties of the fluid. At its very simplest, the latter takes the form:

$$T_{ij}^{(1)} = c_s^2 \left( \frac{\partial \epsilon_i}{\partial x_j} + \frac{\partial \epsilon_j}{\partial x_i} \right),$$

where  $\epsilon_i$  is a suitably defined displacement field.

Slow changes in displacement much less effectively generate elastic stresses because of the reduction in stiffness with increasing frequency, and we are not therefore concerned with the actual displacements, which become arbitrarily large with increasing time, but only those  $\epsilon_i$  whose time scale is sufficiently short. We thus make the short-time approximation

$$\frac{\partial \epsilon_i}{\partial t} = u_i$$

with the result that

$$\frac{\partial^2 u_i}{\partial t^2} - c_s^2 \frac{\partial^2 u_i}{\partial x_j^2} = c_s^2 \frac{\partial^2 s_{ij}^{(0)}}{\partial x_j^2},$$



where

$$u_i = u_i^{(0)} + u_i^{(1)},$$

$$\frac{\partial^2 u_i^{(0)}}{\partial t^2} = - \frac{\partial^2}{\partial t \partial x_j} T_{ij}^{(0)},$$

and

$$\dot{s}_{ij}^{(0)} = \frac{\partial u_i^{(0)}}{\partial x_j} + \frac{\partial u_j^{(0)}}{\partial x_i}.$$

The driven wave equation for the shear wave field  $u_i^{(1)}$  may be interpreted to show that propagating strain waves are generated by dipoles in the strain-rate  $\dot{s}_{ij}^{(0)}$  associated with the turbulent field. This wave equation may be solved, and the elastic strain field at any point in the flow calculated as an integral of the turbulent rate-of-strain field. The energy associated with the elastic strain field radiated from a turbulent eddy may thus be estimated. It turns out to be of the form

$$\frac{\text{radiated energy}}{\text{usual turbulent dissipation}} = f \left( \frac{c_s}{v_\ell} \right),$$

where the function  $f$  has a maximum value of about 2 when  $c_s/v_\ell \approx \pi$ , and where

$$\text{usual turbulent dissipation} \approx (C_D) \rho \frac{v_\ell^3}{\ell},$$

in which  $C_D$  is the coefficient of turbulent dissipation and usually has a value of about 0.8 (16).

These rather crude considerations thus lead us to the conclusion that the total dissipation, including radiation associated with a turbulent flow of given intensity in a suitably elastic fluid may be as much as tripled over the amount normally associated with inertial interactions alone. That is, the coefficient of turbulent dissipation  $C_D$ , which has the approximate value 0.8 in the previous formula, may take on values as large as 2.4 in the case of polymer solutions. In that case, the usual inertial transfer of the total energy production down through the eddy cascade to the dissipation range may become short-circuited, as indicated in Fig. 5.

## SUBLAYER THICKENING

Let us go further and speculate how the thickness of the viscous sublayer of a turbulent boundary may be affected by an increase in the coefficient of turbulent dissipation  $C_D$ . We may define (Fig. 6) the edge of the viscous sublayer as the position  $y^*$  where the total shear stress is equally compounded of laminar and Reynolds stresses:



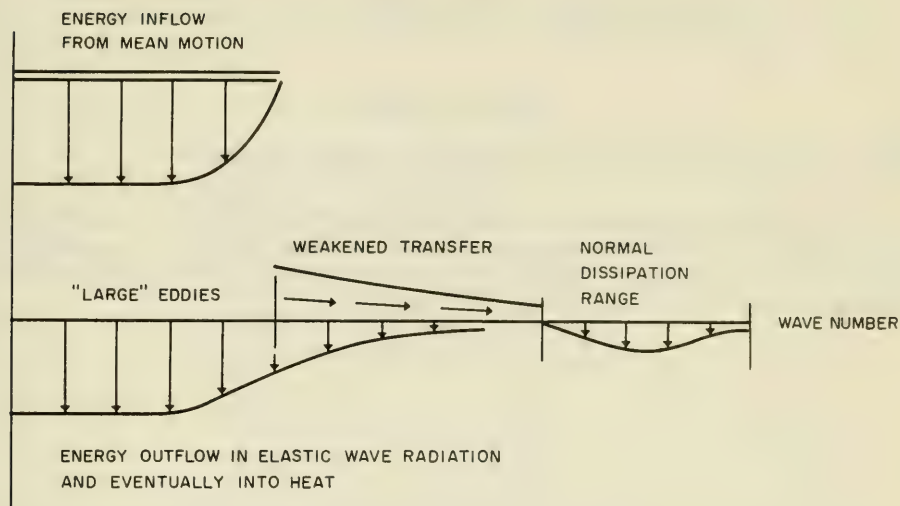


Fig. 5 - Schematic of turbulent energy flow in wave number space for a suitably stiff fluid

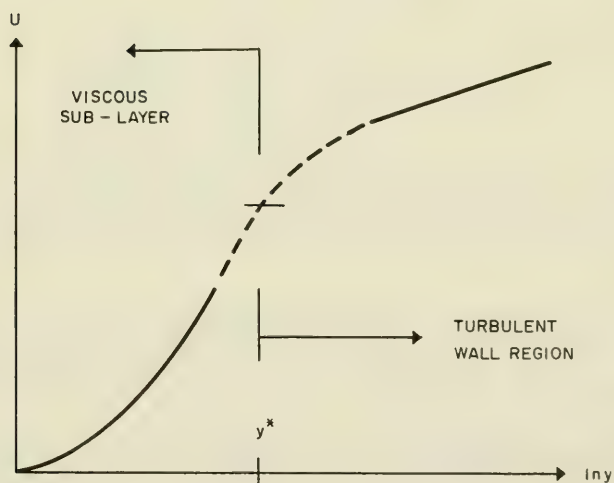


Fig. 6 - Schematic of velocity distribution ( $U$ ) versus distance from the wall ( $y$ )

$$\sigma_{\text{laminar}} \approx \sigma_{\text{turbulent}}, \quad y = y^*$$

or

$$\frac{\partial u}{\partial y}(y^*) \approx \frac{1}{2} \rho \frac{u^{*2}}{\mu},$$

where  $u^*$  is the usual friction velocity ( $\sigma = \rho u^{*2}$ ).

We may then argue that the local dissipation there due to the operation of the laminar and turbulent stresses must also be about equal (this argument depends upon the fact that turbulent production of energy there is approximately equal to turbulent dissipation (Ref. 17, Fig. 9.13):

$$(\text{dissipation})_L \approx (\text{dissipation})_T, \quad y = y^* ;$$

$$\mu \left[ \frac{\partial u}{\partial y}(y^*) \right]^2 \approx K \frac{\rho u^{*3}}{y^*}.$$

This specification leads to a requirement on the Reynolds number based on the viscous sublayer thickness:

$$\frac{u^* y^*}{\nu} \approx 4K.$$

For Newtonian fluids,

$$\frac{u^* y^*}{\nu} \approx 12,$$

but the important thing to notice is that a twofold or threefold increase in the coefficient of turbulent dissipation, to which  $K$  is proportional, will result in a proportionate increase in this Reynolds number and thus in the sublayer thickness:

$$\left[ \frac{u^* y^*}{\nu} \right]_{\text{max}} \approx 24 - 36$$

in polymer solutions.

This maximum thickening will occur when  $c_g/u^* \approx 0(5)$ , while larger and smaller concentrations of polymers than correspond to this last condition will result in a decrease in the polymer effectiveness. The sublayer thickening has been observed (2) and so has the plateau in the drag reduction as a function of polymer concentration (1-3).

## CONCLUSION

Presented here is only a partial and, of course, hurried view of some of our conclusions, the result of research supported by the Fluid Dynamics Branch

of the ONR under Contract Nonr-4181(00); a detailed description will be forthcoming. In the meantime let me reiterate in the following manner:

### 'LASTIC WAVES

Sticky fluids should not be  
treated with discourtesy.  
Strain them easy and you'll see,  
larger grows viscosity.  
Strain them quicker than fast time,  
relaxation shows no sign,  
Greater grows the molecule,  
which to shatter George Stokes' rule.

Small and quiet in the flow,  
waiting for the sign to grow.  
*Now* unfolding near the wall,  
where the shortest flow times fall.  
Greater grows the stiffness now,  
shear waves must the flow allow.  
Waves of size like eddies length,  
rob those eddies, half their strength.  
Viscous layer thickness gains,  
look now how the friction wanes!

Sticky fluids should not be  
treated with discourtesy.  
Flush them out from near the stem  
save yourself some rpm.  
Ship speeds' steady, oil it saves,  
Ah those lovely 'lastic waves.

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# TURBULENCE IN VISCOELASTIC FLUIDS

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## INTRODUCTION

Recent analyses and experimental studies dealing with turbulence and with unsteady flows of dilute polymeric solutions will be considered. Emphasis will be placed on those contributions which lead to proper descriptions of the relevant fluid properties and the interactions between these properties and the flow field, to yield mechanistic interpretations of the turbulent characteristics of these materials.

This paper will show that while the large-scale portion of the turbulent spectrum appears to be similar to that of Newtonian systems, major decreases in the energy levels are predicted at high wavenumbers. These predictions find some experimental support, but a major obstacle to research in this area appears to be a rather general inapplicability of the experimental devices normally used to study turbulent velocity fields. As a result only a few detailed measurements are available, and of these some are probably either in error or of restricted generality. Methods for overcoming these deficiencies will be considered, but no ready solutions of direct and easy applicability will be noted.

Interest in the area of turbulent flow of dilute polymeric solutions has been motivated by the important observation that the drag coefficients of such systems in turbulent flow may be significantly lower than those of Newtonian or other purely viscous fluids. Most available evidence suggests this effect to be due to the viscoelastic properties of these systems, though, as will be seen, few really definitive results are available with which to test this hypothesis rigorously. Few results are available owing to the difficulty of studying both the detailed turbulent motion in these systems and their physical properties; results obtained using conventional experimental techniques may be subject to serious error. Thus the hypothesis concerning the importance of viscoelasticity, while tentatively accepted and used herein, remains in need of much more rigorous tests than have been carried out to date. Comments concerning its limitations are available elsewhere (Refs. 45 and 51).

The format of this paper is as follows:

1. Continuum properties of polymeric solutions, and the relationships of these to the turbulent behavior of these systems.

2. Viscoelastic parameters, namely the Deborah and Weissenberg numbers, which are of importance for ordering purposes and scale-up.

3. Macroscopic (drag coefficient) results, reviewed in light of the above, and tentative analyses of the turbulent spectrum.

4. Experimental techniques for the more detailed study of viscoelastic fluids.

A large number of experimental reports in which drag coefficients have been reported but in which no attempt to either estimate or measure the physical properties of the polymeric solutions used are available. These are of value in distinguishing the systems and conditions to be used in maximizing the pragmatic effects desired. As they do not, however, usually lead to any insight into the mechanisms responsible for the observed results they will generally not be considered in the present report. The authors wish to emphasize that this omission is not intended to reflect a value judgment so much as a need for keeping the present paper within prescribed bounds, and a recognition of the fact that those directly involved in extensive test programs are more capable of surveying those activities.

## CONTINUUM PROPERTIES OF POLYMERIC SOLUTIONS

Five distinct mathematical approaches of some generality have been suggested for portraying the properties of viscoelastic fluids. All incorporate the intuitive notion that these systems possess a gradually fading memory for their previous deformation history; in order of historical development they are as follows.

1. The "convected Maxwellian" equations developed primarily by Oldroyd (42,43,62), though empirical in nature, appear to incorporate all of the real effects which are exhibited by materials of interest. Thus they appear to be generally applicable to all flow fields of interest, including the unsteady flows of importance in turbulence studies. While the complexity of these equations may be extended so that all real effects may be portrayed precisely, what appears to be more important is that even the simplest forms of the equations portray those unsteady effects studied to date as correctly as the experiments reveal them (20,27,32,58,60). For purposes of later application to turbulent processes the two-parameter equation\*

$$\tau' + \theta \frac{\delta \tau'}{\delta t} = \mu \mathbf{d} \quad (1)$$

will be used. The relaxation time  $\theta$  and the viscosity  $\mu$  are taken as arbitrary variables dependent on the invariants of  $\mathbf{d}$ .

Equation (1), with but two exceptions, precisely describes all effects which have been observed quantitatively. One exception is that real materials appear

\*A list of symbols follows the text.

to possess a spectrum of relaxation times in the limit of infinitesimal deformations, not just a single value (24). The implications of this fact have been considered lucidly by Slattery (55). The second exception is that in steady laminar shearing flows one of the two normal stress functions is incorrectly predicted to be zero. It is not zero (27,28,30,35), but in fact appears to be only a small fraction of the other stresses; hence it is properly negligible in many problems.

2. The simple fluid formulations of Coleman and Noll (10-12,15) possess both a conceptual and mathematical elegance not found in empirical equations such as Eq. (1). Unfortunately, this is lost almost immediately as one applies them to problems of interest, as only simple approximations to the behavior of these fluids have been worked out in detail. In the important case of steady laminar shearing flows the Rivlin-Ericksen approximations are applicable and give (10,11,15,39):

$$\tau = -p \mathbf{I} + \bar{\mu} \mathbf{d} + \bar{\omega}_1 \mathbf{d}^2 + \bar{\omega}_2 \mathbf{d}_2, \quad (2)$$

in which the three material property functions  $\bar{\mu}$ ,  $\bar{\omega}_1$ , and  $\bar{\omega}_2$  are all arbitrary functions of the invariants of  $\mathbf{d}$  and  $\mathbf{d}_2$ . It is possible to show that in the limiting case of very low deformation rates these functions become constants (15,39). The resulting "second-order-fluid" approximation, being conceptually attractive and mathematically simple has been extensively employed (15,19,26,54,59,61). However, if it is used, a good approximation to the behavior of real materials is obtained only at such small deformation rates as to make even the experimental determination of the coefficients very difficult (28,35). For the slow flows of interest to a number of workers (26,59) it thus represents the logical choice; for the rapid flows of interest in turbulent fields reference to experimental results (28,35,51,53)\* shows that the second-order approximation is clearly invalid. However, Eq. (2) with the coefficients  $\bar{\omega}_1$  and  $\bar{\omega}_2$  chosen as variables so as to portray correctly the approximate linearity of the stresses, would appear to be a sound choice, provided the flows are nearly steady.

In the case of unsteady flows integral expansions representing first- and second-order approximations to the behavior of simple fluids have been developed and applied to several engineering problems (2,39,40,60). The complexity of these suggests that full approximations, not limited to specific asymptotic flow conditions, are not likely to be forthcoming. However, in rapidly accelerating velocity fields such as those encountered in turbulent studies, asymptotic approximations may be of value.

3. The theory of anisotropic fluids advanced and developed by Ericksen (21) possesses a natural conceptual attractiveness not found in either of the earlier approaches, as the mathematical basis appears to be the only one developed which possesses a counterpart in the actual molecular structure of the fluid. Unfortunately its applicability is at present restricted to flows in which the fluid properties are described adequately and completely by Eq. (2); hence use of such anisotropic fluid theories in engineering analyses would appear to introduce unnecessary analytic complications (16).

\*These have recently been extended by Oliver (44) into the range of concentrations of interest in turbulent drag reduction.



4. Recent papers by Bernstein, Kearsley, and Zapas (8) and by Bogue (9) develop promising constitutive relationships which, though yet largely unapplied to problems of interest in the context of the present paper, may well serve to extend the asymptotic validity of simple fluid analyses into ranges of greater engineering interest, as well as having unique regions of applicability directly.

5. Although molecular theories have been used in several instances, their ability to portray effects at finite deformation rates remains untested. Fluid relaxation times are portrayed as strong functions of concentration, which appears unrealistic (44), except possibly at very low concentrations. In the low concentration range they may work, but as no data are yet available in that range, it does not appear possible to draw useful conclusions at this time.

In summary Eq. (2) represents a valid approximation to the behavior of simple fluids which also portrays correctly all effects noted experimentally under the asymptotic steady flow conditions under which it applies. As such it represents the logical choice for analyses of steady flows. The same is not true, however, of the "second-order" approximation to Eq. (2).

In the case of unsteady flows Eq. (1) appears to be of better than first-approximation utility and will be employed herein. This is the area of primary interest in studies of turbulence, of course, but definitive comments concerning the choice of applicable equations do not yet appear possible. As Eq. (1) also portrays the major effects of interest in steady flows it appears to be of rather general utility.

## VISCOELASTIC PARAMETERS

As a further guide in formulating an understanding to problems of interest, and to aid in choosing which constitutive approximation is most useful, it is of interest to formalize our intuitive ideas of what a viscoelastic material is. Generally a viscoelastic material may be viewed as a material possessing properties that are partly those of a purely viscous material and partly those of an elastic solid. Whether either or both of these characteristics are emphasized depends not only on the material but also on the flow process: if the deformation is an extremely sudden one, it is intuitively reasonable that the time scale of the process may be such that the viscous modes of the material response do not have time to be operative. Thus, for ordering purposes dimensionless groups which consider the deformational processes in question (i.e., the velocity field) as well as the fluid properties, will evidently be necessary.

Considering the fluid properties first, application of the pi theorem to any flow process in which Eqs. (1) or (2) are used in place of the Newtonian expression\* to describe the fluid properties requires the inclusion of additional dimensionless groups in view of the additional variables involved. Thus, if all the

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\*If Newtonian fluids are considered, then Eq. (1) with  $\theta = 0$  and  $\mu = \text{constant}$  will describe the fluid properties. Alternately, Eq. (2) with  $\bar{\mu} = \text{constant}$  and  $\bar{\omega}_1 = \bar{\omega}_2 = 0$  may be used.



physical property parameters are taken as constants, the one additional variable appearing in Eq. (1) ( $\theta$ ) requires one additional dimensionless group, and the two additional variables of Eq. (2) ( $\bar{\omega}_1$  and  $\bar{\omega}_2$ ) require two. Physically these are necessary to consider the additional forces arising in deformation of viscoelastic materials. They have been considered in some detail in laminar boundary layer analyses (39,61), where it is shown that they may be expressed as

$$N_{Ws} = \frac{\bar{\omega}_1}{\bar{\mu}} \frac{V}{D}$$

$$= \frac{\text{elastic forces}}{\text{viscous forces}} \quad (3a)$$

and

$$N_{Vr} = \frac{\bar{\omega}_1}{\bar{\omega}_2}$$

$$= \text{ratio of several elastic forces} \quad (3b)$$

if Eq. (2) is employed.\* Correspondingly if Eq. (1) were used, the Weissenberg number  $N_{Ws}$  may be expressed as

$$N_{Ws} = \frac{\theta V}{D} \quad (3c)$$

and represents the only additional dimensionless group.

If the physical property terms  $\bar{\omega}_1$  and  $\bar{\omega}_2$ , or  $\theta$ , are chosen as variables, then correspondingly additional dimensionless groups as required to consider the parameters which govern the dependency of these variables on the deformation rates must be introduced.

In the case of laminar boundary layer in two-dimensional flows it may be shown that the viscoelastic ratio term of Eq. (3b) does not appear and that while the Weissenberg number (Eq. (3a) or (3c)) does, its influence on the flow field will frequently be too small to be measured experimentally unless the ratio of the physical property coefficients  $\bar{\omega}_1/\bar{\mu}$  or  $\theta$  is greater than found experimentally for most real fluids (61). This conclusion arises because of the high and predominating level of the inertial effects, a factor which effectively introduces consideration of the velocity field as well as the fluid properties. Conversely in low Reynolds number flows the elastic effects are likely to exert greater influences and must be considered. This is brought out clearly in the case of slow flows about rotating spheres as studied by Giesekeus (26) and Walters and Savins (59), for example.

Considering now those more sudden deformations in which the effects of fluid elasticity may intuitively be expected to be more dramatic than in rapid

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\*These same dimensionless groups were also employed by Elata and Poreh (19).

steady flows, the experiments recorded by Johnson (31) using high-speed photography are of interest. If a flat object is dropped, slowly, into a puddle of a dilute polymeric solution it simply moves aside like a conventional fluid would. If, on the other hand, the impact is intense and rapid the sheet of fluid created does not "splash" but instead stretches and recoils just as a suddenly stretched elastic solid would. On the time scale involved (a few milliseconds) only the elastic response is felt. Quantitative analyses of this effect have been published (39,40,60) in which it is shown that Eq. (1) as well as the correct asymptotic forms from simple fluid theory (though not Eq. (2)) predict such purely elastic responses for the short time scales involved.

These considerations suggest that the kind of material response observed, and the correct choice of asymptotic approximations such as Eq. (2) which may be used to describe the material properties, are dependent on an additional dimensionless parameter in which the time scales of the material and the deformational process are considered. This dimensionless parameter, the Deborah number, may be defined (40) by

$$N_{\text{Deb}} = \frac{\text{characteristic time of fluid}}{\text{characteristic time of process}} \quad (4)$$

For flows that are steady in a Lagrangian sense, for example steady laminar shearing flow in a tube, the Deborah number is identically zero. It is in this limit (and only in this limit) that Eq. (2) may be used to portray the material properties of simple fluids. In this asymptotic, limiting flow process it may be shown that for ducted flows through round tubes neither the velocity profile nor the viscous drag are influenced by the presence or absence of elastic forces. That is, the fluid is behaving as a purely viscous material even though one may have present large elastic forces as described by the Weissenberg number in Eq. (3). (Experimental results discussed in the next section show the elastic forces may be 10 to 40 times larger than viscous forces at the shear rate levels of interest.)

On the other extreme if we consider fluid elements subject to deformations of a duration smaller than the relaxation time of the fluid, as in the case of the Johnson experiments in which a puddle of fluid is suddenly deformed, the Deborah number is large and only an elastic response is observed. Intermediate cases, and typical magnitudes of the Deborah number in flows of pragmatic interest, have been described elsewhere (40,48,51).

The intuitive notions embodied in Eq. (4) have been formalized mathematically by Astarita (2) as

$$N_{\text{Deb}} = \theta \sqrt{\frac{D\sqrt{\Pi}}{Dt}} \quad (5)$$

Equation (5) represents a properly invariant definition of the Deborah number and gives a process time scale that is a measure of how long an element of fluid is subjected to a particular state of deformation.

## MACROSCOPIC PHENOMENA, TURBULENT STRUCTURE

Measurements of the physical properties of viscoelastic fluids more concentrated than those of general interest in drag reduction have appeared in the rheological literature for some time, though the majority of these do not extend into the high shear rate levels of interest. However, several such studies have appeared (28,51,53), the data of Figs. 1 and 2 being typical. The Weissenberg number (Eq. (3a)) representing a ratio of the elastic to viscous forces is given by the ratio of the ordinates of Fig. 1; the fluid relaxation times  $\theta$  (Eq. (1)) calculated from similar measurements are shown in Fig. 2.

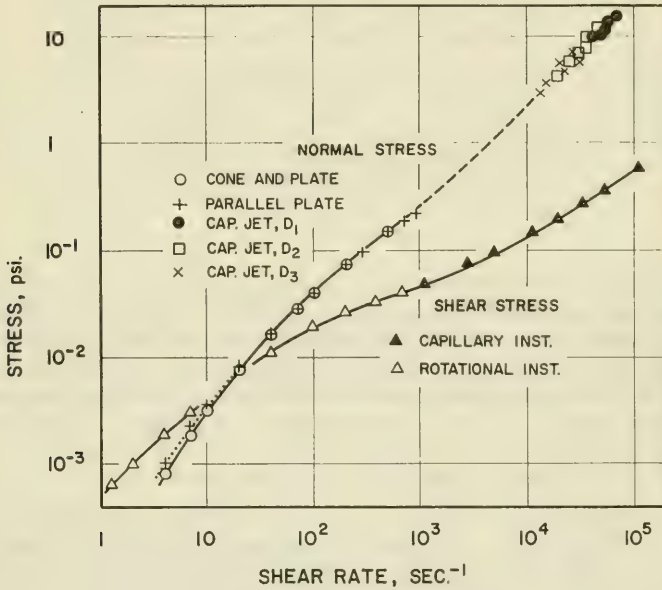


Fig. 1 - Rheological properties of viscoelastic fluids: normal and shearing stress measurements under steady laminar flow conditions depicted as a function of shear rate. The data are from Ginn and Metzner (28) for 5% solution of polyisobutylene in decalin.

Several important observations may be made. First the Weissenberg number, hence the importance of elastic effects, may be very great and is an increasing function of shear rate. Secondly, relaxation times are typically about  $10^{-2}$  or  $10^{-3}$  second at the shear rate levels of interest and do not seem to be very dependent upon the concentration level of the fluid used in obtaining the results depicted in Fig. 2. Thirdly, and most importantly, such measurements have been recently extended by Oliver (44) to concentrations as low as 100 ppm — i.e., well into the range of direct interest — and the same conclusions appear



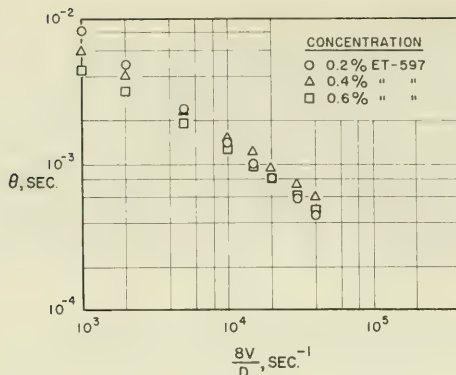


Fig. 2 - Rheological properties of viscoelastic fluids: relaxation time measurements on aqueous solutions. The data are from Seyer and Metzner (51).

applicable at these concentration levels also. Finally it is clear that while the second-order approximation to simple fluid behavior (Eq. (2) with all parameters taken as constants) may be a valid approximation at the lowest shear rate levels studied, at which the shearing stress appears to be linearly dependent on shear rate and the normal stress term quadratically, such an approximation is clearly not very useful at shear rates beyond 1 to 10  $\text{sec}^{-1}$ . The shear rate levels of turbulent fields are generally in the range  $10^3$  to  $10^5 \text{ sec}^{-1}$ , and herein the normal stresses depend nearly linearly upon shear rate.\*

Figure 3 depicts typical drag coefficient vs Reynolds number relationships; these are for the polymeric solutions having the time constants depicted in Fig. 2. Also shown in Fig. 3 are the turbulent drag coefficients curves for purely viscous fluids ( $0.4 \leq n' \leq 1.0$ ) having flow behavior indices in the range of interest.†

In contrast to the turbulent behavior shown by purely viscous fluids the experimental drag coefficients shown are all very low and show a pronounced diameter effect. This latter is expected, since at a given Reynolds number the fluid velocities are greatest in the smallest tubes. This means that the shear rates are greatest here; hence any effects of either the Weissenberg or Deborah

\*Less directly but possibly more sensitively, Pruitt and Crawford (46) have shown that in laminar ducted flow fields entry region effects are large in dilute solutions and may be used to infer fluid properties. Analysis of their results (5) yields elastic effects, for several dilute (250 ppm) polymeric solutions, which also appear to be nearly linearly dependent on the shear rate of the fluid.

†Obviously purely viscous fluids with shear-dependent viscosities ( $n' \neq 1.0$ ) do not behave in the manner indicated by drag reducing materials. This has been clear for nearly a decade (17), but the contrary illusion still reappears with disturbing frequency in the fluid mechanics literature.



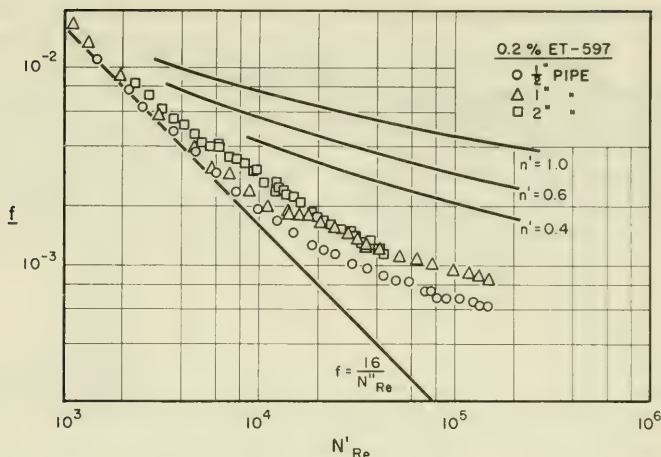


Fig. 3 - Typical drag coefficient vs Reynolds number curves for viscoelastic fluids. The laminar curve and purely viscous turbulent curves are shown for reference purposes (51).

numbers (Eqs. (3a) and (5)) will, in view of Fig. 1, be most pronounced in the smallest tubes. The result of a correlation of the data depicted in Fig. 3, applying these concepts, is given in Fig. 4. The drag reduction parameter  $F$  defined as

$$F = \frac{f_{pv} - f}{f_{pv} - f_e}$$

is normalized with respect to the differences between the purely viscous and the laminar curves. Full details are available elsewhere (51); similar studies have also been performed by Astarita (3) and by Rodriguez, Zakin, and Patterson (43).

Somewhat more detailed analyses of the underlying mechanisms leading to the observed reductions in the drag coefficient may be developed by means of several distinct approaches. Perhaps most directly, an extension of the Reynolds equations for the turbulent motion may be carried out using not the usual Newtonian equation but rather an expression such as Eq. (1) to describe the fluid properties. In fact the convected derivative term of Eq. (1) is in attractive mathematically; hence the much simpler second-order expression (Eq. (2) with all coefficients taken as constants) commends itself mathematically. As noted earlier the only available experimental measurements on dilute solutions, those of Oliver (44) and of Pruitt and Crawford (46), support the general results shown in Fig. 1, viz., that the elastic stress terms increase linearly, not quadratically, with deformation rate under the high deformation rates of interest. Thus, use of the second-order approximation will apparently in general tend to overpredict the magnitude of any real effects. Secondly, it must be noted that Eq. (2), representing as it does a steady flow asymptotic approximation to the behavior of

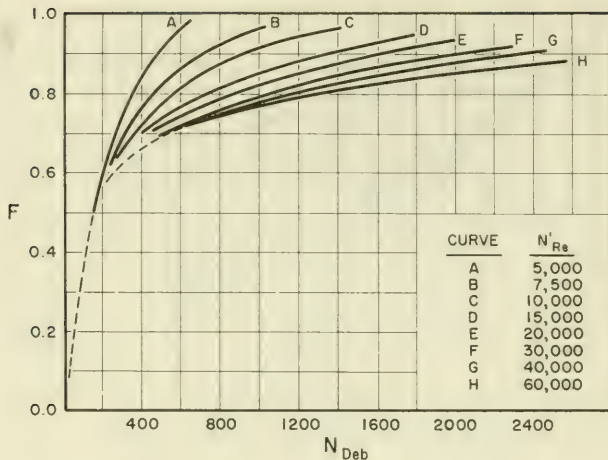


Fig. 4 - Dependency of normalized drag reduction upon Reynolds and Deborah numbers (51)

real fluids, can only be used if the Deborah number of the fluid elements to which it is applied (Eq. (5)) is small. This restricts analyses based on Eq. (2) to the small-wavenumber end of the spectrum — in view of Fig. 2 and the comparable Oliver data to wavenumbers below, say,  $10^2$ .

Two such analyses based on Eq. (2), and both correctly either noting or implying the above restrictions, have been reported (19,54), that of Singh being more detailed.\* Singh finds what appear to be very small terms leading to decreased energy levels in this small-wavenumber portion of the spectrum. Such a conclusion is in agreement with the intuitive notion expressed by Eqs. (4) and (5) that major elastic effects are to be expected only in the high-wavenumber region. Elata and Poreh (19) argue that at high Reynolds numbers the terms arising from the Weissenberg and viscoelastic ratio terms (Eqs. (3a) and (3b)) may be large and override in importance the linear viscous terms, but in fact this result stems solely from the evident discrepancy between the quadratic predictions of Eq. (2) when the coefficients of these terms are taken as constants and the actual fluid behavior noted in Fig. 1 and by Oliver, Pruitt and Crawford. If, as suggested by these physical property measurements, the elastic stresses vary linearly with deformation rate, then both the viscous and elastic terms in the Elata-Poreh analysis decrease in magnitude in the same way with increasing Reynolds numbers and the effects of elasticity would appear to be small ones, in this sense in agreement with the small effects noted by Singh at these wavenumbers. In this case the results also show no dependence on diameter,

\*The complete inapplicability of Eq. (2) under flow conditions described by a large Deborah number is perhaps most clearly emphasized in the Coleman-Duffin-Mizel paradox and its resolution (14,40).

as noted in Fig. 3. Evidently the major effects of interest do not occur in this low-wavenumber region.

At high wavenumbers, a much more difficult analysis is necessary in view of the much greater mathematical complexity of Eq. (1) as compared to Eq. (2) or other expressions of this kind. However, a beginning may be made in the following manner. It has been noted (7,47,56) that the predominant kinematic process at large wavenumbers is the "stretching" of fluid elements. This process may be characterized by defining, at a given wavenumber, a stretch rate as

$$\xi = \frac{v}{x} \quad (6)$$

in which  $v$  denotes the relative elongational velocity and  $x$  the length of the fluid element under consideration. Batchelor (7) has shown that the inverse of the characteristic stretch rate may be identified with a time scale characteristic of the dissipative eddies in a turbulent flow, so that one may define a Deborah number characterizing the dissipative process as

$$N_{\text{Deb}} = \theta \xi. \quad (7)$$

Separately it has been shown analytically (2,13,34) and in part confirmed experimentally (6) that the resistance offered to stretching by a fluid element of viscoelastic material is much larger than that of a Newtonian fluid and is dependent exponentially on the dimensionless grouping defined in Eq. (7). For example, in the case of a deformation of a two-dimensional area element described by Eq. (1) it may be shown readily that the ratio of stresses developed in a viscoelastic fluid to those in a Newtonian fluid is of the form

$$\frac{(\tau_{xx})_{\text{viscoelastic}}}{(\tau_{xx})_{\text{Newtonian}}} = \frac{1}{1 - (2\theta\xi)^2}. \quad (8)$$

Thus, it is noted viscoelastic materials possess a maximum stretch rate at which the tensile stresses would increase without limit. This conclusion has been shown to be independent of the form of the constitutive equation used to portray the fluid properties (2).

If we now consider a given turbulent flow field, the energy levels available to deform a fluid element are fixed at some finite value by inertial effects, from which it follows, in view of Eq. (8), that the rates of stretching will be far lower than in a Newtonian fluid. Equivalently, since the stretching processes for a Newtonian fluid are primarily associated with the dissipative eddies, one may predict the increased resistance to stretching to inhibit much of the small-scale high-frequency turbulence. In view of the form of Eq. (8) it would appear that the elastic effects would serve to cut off rather suddenly the high-wavenumber end of the spectrum, starting at eddies with time scales of the order of the fluid relaxation time. Several qualitative observations based on observing the spread of a dye filament (25,50,52) have shown that the structure of the turbulent flow is grossly altered in drag reducing systems in just this way: the dye filaments spread as large discrete "lumps" of material, whereas in a Newtonian fluid the



spread appears continuous owing to the many small-scale high-frequency oscillations which do not appear to be present in the viscoelastic systems. In agreement with the above are some hot-wire data reported by Merrill (36), although, as will be shown in the next section, it is not clear that these measurements are correct quantitatively.

## EXPERIMENTAL TECHNIQUES

This section considers briefly the major tools or methods available for studying the flow fields of interest and in so doing points to some of the significant results, limitations of techniques, and areas in need of further work.

### Heated Probes

Limitations, owing to viscoelastic properties, on the use of hot probes have been recently discussed (38) and will not be considered in detail. Of interest is the fact that for many cases of practical interest the flow near the probe tip is characterized by large values of the Deborah number (Eq. (5)), which is predicted to result in "thick" boundary layers or even stagnant regions of fluid near the leading edge of the probe. Practically, these results suggest that the frequency response of the probe will be significantly reduced and that the probe may become insensitive to velocity at sufficiently high values of the Deborah number. These predictions find some support in the experimental results of Acosta and James (1) and of Leathrum (33); as the cutoff velocities are frequently of the order of 1 fps or even less, the restriction is a very serious one.

It is not clear at this point how the above problems can be alleviated, as reductions in velocity or increases in probe size in order to circumvent the Deborah number limitation increase the uncertainty of the measurements due to natural convection effects, and take the measurements out of the ranges of primary interest.

### Impact Probes

In the case of impact probes several effects arise which may, in principle, serve to alter or limit the utility of the probe. First, the boundary-layer considerations noted in the previous section suggest that here too the response of the probe may be significantly reduced, and more importantly Metzner and Astarita (38) have shown that a boundary layer of the same order of thickness as the probe size may exist at the leading edge of the probe. This problem is particularly acute in the case of small probes and imposes a severe restriction on use of the probes for measurements near the wall.

Secondly, in laminar flows the impact reading is directly dependent on the magnitude of the deviatoric normal stress terms, as well as on the fluid momentum (4,49). Consequently, unless the magnitudes of the deviatoric stresses are negligible or known beforehand one cannot obtain velocity information. Similar problems arise under turbulent conditions with an additional contribution



to the impact reading due to time averaging of the fluctuating stresses. The paper by Astarita and Nicodemo (4) includes a careful consideration of this effect.

At this point it is appropriate to consider the recent velocity profile measurements, reported by Elata et al. (18), Ernst (22,23), and Meyer (41), in terms of these probe limitations. The data obtained by Ernst and analyzed by Meyer in terms of an altered velocity profile equation appear to suffer from a 7% discrepancy between the integrated flow rate as computed from Pitot tube measurements and the recorder flow rate, the recorder flow rate being the lower of the two. Data are analyzed by using a velocity correlation of the type

$$\frac{u}{u^*} = A \log \frac{yu^*}{\nu} + B. \quad (9)$$

The data suggest the non-Newtonian effect appears as a change in the numerical value of B, while A remains the same. In Eq. (9) the minor term contributing to the value of  $u/u^*$  is B, and over most of the range of the data it seems that the noted changes in B could possibly be attributed to the flow rate discrepancy. Thus, it is not clear if the apparent shifts in the velocity curves are significant or not.

Qualitative considerations of the curves predicted by the Meyer equation (41) show that gross changes in the shape of the curves, as compared to those of Newtonian fluids, must take place in the sublayer regions. At equal flow rates the velocity profile in the turbulent core is predicted to be flatter in viscoelastic fluids than in Newtonian, since  $u^*$  is smaller. At the wall, since the shearing stresses are lower, the drag-reducing fluid must also have the flatter profile. The only way to join these two flatter curves is by means of a very sharp curve in the outer sublayer or buffer zones, implying very sharp changes in the profile in the buffer zone. This would appear to be unlikely on physical grounds, as it implies very sudden changes in the radial momentum transport rates. As it is doubtful whether useful probe measurements may be made in this region, other measurements may be necessary to resolve these questions and to infer the detailed shapes of the velocity profiles. One possibility would appear to be the use of heat transfer measurements employing drag-reducing fluids of low Prandtl number.

Notably the solutions used in the above studies were very dilute and exhibited only slight drag reduction effects. Other data (4), using solutions showing major drag reduction effects, indicate that if the impact readings are interpreted directly as velocity, the apparent flowrate may be as much as 30% lower than the true flowrate. As these results appear to have obtained using systems similar to those used by Elata et al. (18), the absence of similar effects in the latter study is surprising. Clearly it would be premature to suppose any validity for reported turbulent velocity profile measurements on systems which show significant viscoelastic effects, as there is at present no rigorous method for differentiating between the inertial velocity contributions desired and the effects arising due to viscoelastic (normal stress) terms.

## Tracer Studies

Studies using small particles or bubbles suspended in liquids, although representing tedious experiments, may be of special merit owing to the restrictions applicable to the use of conventional probes. A recent study by Uebler (57) has shown that tracer particles closely follow the flow field for flows in which only moderate accelerations are present (entry flow into a tube) and for very small tracer particles (bubbles  $\approx 0.05$  cm in diameter). However, for large bubbles and high flow rates spurious effects were noted. A tentative analysis of this phenomenon (37) shows these effects could also be significant in the high-wavenumber region of a turbulent field; nevertheless the confidence level of such data would clearly be superior to that of impact probe measurements at the present time.

## Heat Transfer

Recent heat transfer results (29), for drag reducing system, have shown that in conjunction with the decrease in drag coefficient there is a corresponding but much larger relative decrease in the heat transfer coefficient. The accompanying detailed analysis shows no matter what the change in the bulk velocity profile might be, it alone is insufficient to explain the low heat transfer rates. Therefore, one may conclude that the wall region is thicker or that the turbulence levels near the wall (where the greatest resistance to heat transfer is) are much weaker, or both. It is interesting to note that the same conclusion has been drawn elsewhere (18) on the basis of velocity profile measurements which, as noted earlier, could have been expected to be incorrect.

## ACKNOWLEDGMENT

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## LIST OF SYMBOLS

- A Slope of universal velocity correlation, Eq. (9)
- B Constant in universal velocity correlation, Eq. (9)
- $d$  Rate-of-strain tensor, having components  $d^{ij}$
- $d_2$  Second Rivlin-Ericksen tensor (15)
- D Tube diameter
- $f$  Observed friction factor  $= \tau_w / \frac{1}{2} \rho V^2$
- $f_1$  Laminar friction factor  $= 16 / N'_{Re}$

- $f_{pv}$  Purely viscous friction factor (17)
- $F$  Friction factor ratio, Fig. 4
- $\mathbf{I}$  Unit tensor
- $n'$  Flow behavior index (17)
- $N_{Deb}$  Deborah number, Eq. (4)
- $N'_{Re}$  Generalized Reynolds number (17)
- $N_{Vr}$  Viscoelastic ratio number, Eq. (3b)
- $N_{Ws}$  Weissenberg number, Eq. (3a) or (3c)
- $p$  Isotropic pressure
- $u$  Local velocity in pipe
- $u^*$  Friction velocity  $\sqrt{\tau_w/\rho}$
- $v$  Tube velocity, Eq. (3a); stretching velocity, Eq. (6)
- $x$  Length (or Cartesian coordinate direction in Eq. (8))
- $y$  Distance from pipe wall
- $\delta/\delta t$  Convected time derivative (42)
- $D/Dt$  Material time derivative
- $II$  Second invariant of rate of strain tensor
- $\theta$  Fluid relaxation time, Eq. (1)
- $\mu$  Variable viscosity coefficient, Eq. (1)
- $\bar{\mu}$  Fluid viscosity coefficient, Eq. (2)
- $\nu$  Kinematic viscosity
- $\xi$  Fluid stretch rate, defined by Eq. (6)
- $\rho$  Fluid density
- $\bar{\omega}_1, \bar{\omega}_2$  Fluid material properties, Eq. (2)
- $\boldsymbol{\tau}'$  Stress tensor, components  $\tau'^{ij}$ , relative to arbitrary hydrostatic pressure  $\alpha$  (61):  $\boldsymbol{\tau} = -\alpha \mathbf{I} + \boldsymbol{\tau}'$



- $\tau$  Total stress tensor
- $\tau_w$  Shear stress at tube wall
- $\tau_{xx}$  Normal stress component of stress tensor.

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\* \* \*



## DISCUSSION

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The authors imply that the drag reduction associated with these dilute polymeric solutions would be due to a suppression of the high-frequency or small eddies. Is it not correct to believe that the majority of the energy is first picked up by relatively larger eddies through interaction of the bulk stream and the wall layer and then successfully transferred to smaller and smaller eddies where it is more rapidly consumed? If this is so and the suppression of small eddies is due to a more rapid absorption of the energy by these small eddies, I do not see that this would result in any large change in the drag characteristics per se.

I would presume that for some non-Newtonian fluids, at least, relaxation of stress in a current direction of flow or shear would change the fluids resistance to flow in a new direction by a different amount and that the change might well be of opposite sign. That is, while the resistance to flow for shear in the  $x$  direction was decreased because of flow in the  $x$  direction, resistance to flow or shear in the  $y$  or  $z$  direction could be increased relative to that of a stationary randomly oriented fluid. Since common turbulence requires movements not only in the primary direction of flow, but also in all other directions such behaviour would suppress the conversion of energy into eddies of any size in the wall region where the flow is most predominantly unidirectional.

The possible behaviour for the polymeric solutions might be inferred from microscopic physical models of the system. These will depend very much upon the relative interactions between the low-molecular-weight liquid and the high-molecular-weight polymer. If the interactions with the polymer unit were sufficiently high, each polymer molecule could be dispersed through the liquid as some sort of thread, possibly tangled. This form would be very strongly aligned by flow shear to give a reduced drag in the direction of flow and considerably increased drag in other directions. If the interaction between liquid and polymer were less favourable, polymer would tend to be forced out of the liquid (or pull itself out, it does not matter which) to form small globules, but in this case there would be almost as strong a tendency to collect large numbers of these molecules to produce fairly sizable micelles as in soap solutions. As noted in the preceding paper these globules themselves tend to become deformed and elongated in the direction of shear, which results in reduced resistance in this direction and increased resistance in others, but to a lesser extent the fragmentation of these globules could result in a reduced resistance in all directions. Other possibilities exist depending on the liquid, the polymer units, and the polymer end groups, including the setting up of local or extended gels which are more or less fibrous networks through the liquid. To what extent are these arrangements known and how do they affect drag reduction?

I am not very familiar with the equations of fluid dynamics, having recently entered the field to apply a new measurement technique we have developed. With this technique we have thus far measured, I believe for the first time, the instantaneous thickness and velocity profile for the laminar region at the wall in turbulent pipe flow. I would judge that the equations presented in this and the preceding paper are largely either empirical or semiempirical (the form specified by some theory, but the constants fitted to experiment) and that, therefore, accurate time-continuous experimental data on the stress and rate of strain relationships for these fluids is required — possibly to verify the theoretical form of the equations and certainly to provide the constants which would determine the detailed behaviour of the fluid. This paper indicates that the standard hot wire or hot probe results for fluid velocity are highly questionable due to expected but unknown effects of the non-Newtonian fluid on the probe. When one further considers that in Newtonian flow corrections are frequently necessary which were not originally expected, such as, for example, the effect of small-scale-turbulence intensity on the heat transfer velocity relationship. Furthermore all probes tend to disturb the flow for measurements near a wall, which may be the region where drag reduction occurs with these unique solutions. Our new technique involves the instantaneous introduction of a very fine dye tracer at any angle to the flow, including perpendicular, without disturbing the flow. It will permit continuous and accurate determinations of the liquid strain rate (velocity profiles). I would welcome suggestions as to experimental conditions.

\* \* \*

## REPLY TO DISCUSSION

A. B. Metzner

Professor Hummel correctly emphasizes the inconsistencies between the several approaches available at the present time for interpretation of the turbulent drag reduction effects noted in viscoelastic fluids. The suggestion that local strain rate measurements in such turbulent systems would be of much interest is obviously a good one, and he is to be encouraged to both carry out such studies, if he has an interest in them, or make the details of his experimental technique available to others who may have an interest in it.

\* \* \*

# AN EXPERIMENTAL STUDY OF GRID TURBULENCE IN DILUTE HIGH-POLYMER SOLUTIONS

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## ABSTRACT

Longitudinal, one-dimensional turbulence energy spectra were measured behind a towed grid in a tank filled with dilute aqueous high-polymer solutions. Several types of hot-film flowmeters with constant-temperature bridges were used, with emphasis on quartz-coated, platinum-film sensors of conical shape. Spectral measurements in water were consistent with wind-tunnel measurements. Cylindrical sensors were found unsuitable because of evidence of coating of the hot-film leading edge. The additives were poly(ethylene oxides) (Polyox WSR-301 and coagulant), polyacrylamides (Separan AP-30 and J100) and guar gum. Concentrations of up to 137 ppm were used. The guar gum produced temporary suspensions which were not suitable for turbulence measurements because of high noise level. With the other additives, the sensor signal near the grid included a high-frequency component (called "raggedness") which was attributed to temporary (or temporarily altered) inhomogeneity caused by the action of the grid bars. With extreme raggedness, the interpretation of the signal as a turbulent velocity fluctuation would have implied impossible viscous dissipation rates.

The raggedness was insensitive to the range of solution preparation techniques used, which included demineralization of the water and 40 hours of gentle stirring of concentrated "master" solutions. When there was a decrease of raggedness (at a fixed distance from the towed grid) with solution age, there was also a significant decrease of specific viscosity.

Three types of spectral effects were the spectral distortion due to raggedness, a greater experimental scatter of spectral levels near the grid in newly prepared solutions, and the predictable spectral shift due to viscosity increase. No non-Newtonian spectral effects were detected under conditions of nonragged signals or at frequencies below the spectral distortion due to raggedness.

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Note: This paper is based on a thesis listed as Ref. 1.



The absence of non-Newtonian, viscoelastic effects is attributed to the largest viscoelastic relaxation times (according to molecular theory) being necessarily much smaller than the smallest inertial spectral transfer time scales of the turbulence. The raggedness phenomenon prevented operation under conditions for which a significant viscoelastic effect on the energy spectrum would have been likely.

## INTRODUCTION

The mechanism of high-polymer-additive friction reduction in turbulent pipe and boundary-layer liquid flows is currently of great research interest. It is clear that turbulence measurements in the shear flows with friction reduction are of interest. However, in such work it would be desirable to be able to resolve the smallest turbulence scales which are known to be important. This is difficult to do in liquid-flow facilities of ordinary size with available turbulence sensors. Thus other types of turbulent flow were also considered in planning this investigation.

In choosing the type of flow for study, the following requirements were considered essential:

1. The experiment should contribute to eventual understanding of the friction-reduction phenomenon in wall-turbulence shear flows.
2. The available turbulence sensors for use in liquids should have adequate spatial resolution and frequency response for the flow to be studied.
3. The experiment should be free as far as possible from bias due to effects of degradation of solution properties by mechanical cleavage of polymer chains.

With these considerations in mind, it was decided to study a turbulent flow without mean shear, specifically a grid-turbulence flow, which offers a useful approximation to isotropic turbulence. This hypothetical type of turbulence is more amenable to theoretical treatment and still is of practical value in understanding actual turbulence, in which the fine-scale structure is known to be nearly isotropic.

Requirements 2 and 3 led to the selection of a towing-tank experiment in which the scale of the turbulence could be made large enough relative to the available sensors, and in which the test fluid is sheared only by the moving test grid. If the same experiment were performed in a water tunnel, the test fluid would also be sheared by the pump and in the tunnel-wall boundary layers.

Many investigators of the friction-reduction phenomenon have suggested that the mechanism of friction reduction involves the viscoelastic properties of the dilute solutions. The specific aim of this work was to determine the differences, between solution and solvent, in the longitudinal one-dimensional grid-turbulence energy spectra, and to relate the differences to the viscoelastic properties of the solutions. A concurrent theoretical investigation was made by



Singh (2), and the intention was to obtain a comparison of theory and experiment. Due to the nature of the results, this comparison cannot be made yet, and no theoretical discussion is needed in this work beyond that given in the later discussion of the experimental results.

Several high-molecular-weight industrial polymers which are known to be effective friction reducers were used as additives:

| <u>Polymer</u>       |                    | <u>Supplier</u>                           |
|----------------------|--------------------|-------------------------------------------|
| Poly(ethylene oxide) | (Polyox WSR-301)   | Union Carbide Chem. Co.                   |
| Poly(ethylene oxide) | (Polyox Coagulant) | Union Carbide Chem. Co.                   |
| Guar gum             | (J2FP)             | Western Company                           |
| Polyacrylamide       | (Separan AP-30)    | Dow Chemical Company                      |
| Polyacrylamide       | (J100)             | Dowell Division,<br>Dow Chemical Company. |

The polymers were used as received, without drying, screening, or fractionation. Most of the work was done with a 50-pound barrel of Polyox WSR-301, within several months of being shipped "from new stock." This polymer will be indicated by P301B. The last three additives were used only briefly, since the results indicated no advantage in their use. Because of the nature of the results no detailed comparison of the additive molecular properties is needed in this work.

It is clear that the test solutions would have produced significant friction reduction in turbulent pipe flow at typical wall shear stresses, say of the order of  $10^3$  dynes/cm<sup>2</sup>. In two cases, friction reduction tests with solutions made from Polyox WSR-301 and from J100 showed the expected high degree of effectiveness (e.g., 70% pressure-drop reduction) at low concentrations (e.g., at 20 ppm — parts per million by weight). Maximum concentrations in the towing-tank tests were about 140 ppm. All the towing-tank solutions are "dilute" (3) in that the dimensionless concentration  $c'[\eta]$ , where  $c'$  is the polymer concentration in g/100 cc or g/dl and  $[\eta]$  is the intrinsic viscosity in dl/g, is at most about 0.3. In general, local tap water was used as the solvent. As discussed later, there was no evidence that the results were dependent on the solvent impurities.

## PROCEDURE

A Turbulence Research Towing Tank, 48 feet long with a cross section of about 4 by 4 feet, was constructed for this work. The tank is lined with a smooth vinyl film which was hosed clean between tests. A grid-turbulence experiment is shown in progress in Fig. 1. The grid carriage, carrying the vertical grid, is pulled along by cables from constant-speed winches. The sensor carriage, generally towed by a cable from the grid carriage, follows at a fixed distance  $x$  from the grid (i.e., at a constant value of  $x/M$ , where  $M$  is the grid mesh width). The duration of useful run time is about 25 sec for the standard towing speed of 36 cm/sec.

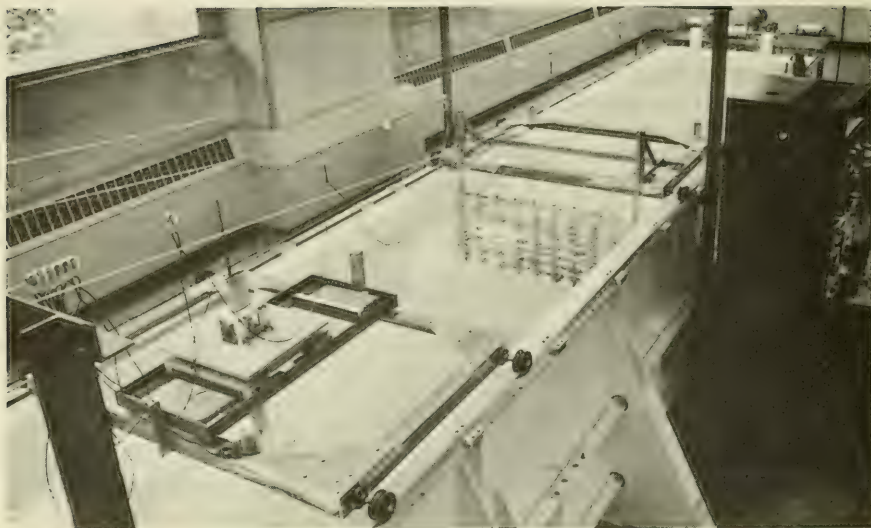


Fig. 1 - Towing tank during a grid-turbulence run

A better view of the sensor carriage with the sensor-carrying, surface-piercing strut, is given in Fig. 2. The strut is faired to reduce vortex-shedding excitation and is mounted on a simple vibration-isolation mounting of very low natural frequency. The sensor is connected to its constant-temperature bridge by the long cable seen in Fig. 1.

Two types of hot-film flowmeters were used with constant-temperature bridges. The thermistor type developed by Lumley (4) was initially emphasized. An advantage of this type is its small size, with the diameter of the sensitive area being about 0.006 in. In water, the sensor must have a coating to keep water out of the thermistor film, but the glass coatings used for this purpose have been unreliable.\* Thus most of the work was done with quartz-coated platinum film sensors manufactured by Thermo-Systems, Inc., of Minneapolis, Minnesota. Figure 3 shows the form and relative size of the sensor types used in this work. Names were usually assigned to the various sensors as a convenient means of identification.

Due to the low frequency range in this work, the frequency response of the bridges used is flat, and only the frequency response of the sensors is of concern. Their comparative frequency response in water was evaluated by measurements of a standard grid-turbulence spectrum. Significant frequency response drop-off was found for circular-cylindrical sensors and for a glass-coated thermistor. For the circular cylinders, Lighthill's analysis (5) was used to

\*Modifications made since the completion of this work have eliminated this difficulty.

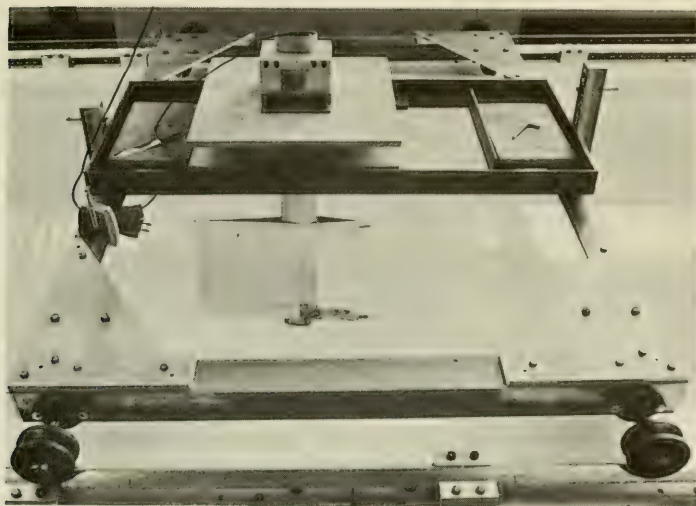


Fig. 2 - Sensor carriage

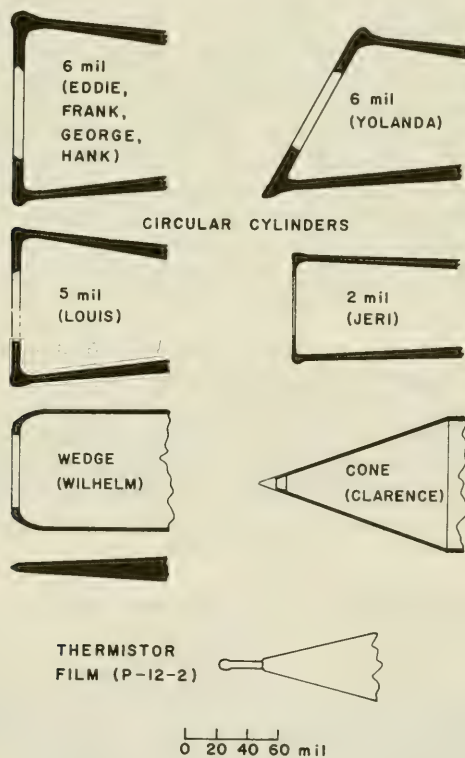


Fig. 3 - Relative size and shape  
of hot-film sensors



show that the drop-off was not a boundary layer effect, and the drop-off was attributed to the separated flow which occurs at the Reynolds numbers in this work. The circular-cylindrical sensors were also found to produce a false turbulence signal in the form of "wake ripple" and "wake noise" starting at sensor Reynolds number of 40 to 50. The drop-off for the thermistor-film sensor was verified by "interface-penetration" tests and was attributed to an extra-thick glass coating. It was found that the wedge sensor gave abnormal calibration curves in Polyox solutions, presumably due to coating of the hot film by gelatinous material collecting on the leading edge. Thus most of the solution work was done with conical sensors, which displayed only minor calibration-curve shifts due to the additives. The improvement may be attributed to the better shape and the film location. In other work (6), the frequency response of similar conical sensors was found to be essentially flat out to 200 cps for 50 cm/sec which is close to the mean flow speed in this work. Because the upper frequency of the useful spectral measurements in this work was usually about 200 cps, the conical sensors were assumed to have flat response in both water and in the dilute solutions. The results do not deny this assumption, but when most concentrated solutions are used in future work, the frequency-response question should be re-examined.

In order to provide for comparison of the towing-tank results for water with measurements in air by Comte-Bellot and Corrsin (7), a square-bar, square-mesh biplane type of grid was used with the ratio of mesh width to bar thickness taken as 5.3. Thus the nominal grid solidity is 0.34. The grid Reynolds number,  $R_M = U_0 N/\nu$ , where  $U_0$  is the mean speed and  $\nu$  is the kinematic viscosity, was about 40,000 in the towing-tank tests and from 17,000 to 135,000 in the wind-tunnel tests. The mesh width was chosen as 11.8 cm on the basis of several considerations. It had to be sufficiently small compared to the test section dimensions to encourage lateral homogeneity of the turbulence. On the other hand a large value was desirable so that the available sensors would accurately resolve the smallest turbulence scale of interest. This scale can be assigned as follows. The characteristic dissipation wavenumber is  $k_d$  where  $1/k_d$  is the Kolmogoroff microscale,  $(\nu^3/\epsilon)^{1/4}$ , with  $\epsilon$  the time rate of decay of turbulence energy per unit mass. It is desirable to resolve eddies of the corresponding scale, and so  $\ell_s$ , the longest scale of the sensor hot film, should be less than about the half wavelength corresponding to  $k_d$ , i.e.,  $\pi/k_d$ . The corresponding frequency seen by the sensor,  $k_d U_0/2\pi$ , should be well below the sensor cut-off frequency  $f_c$ .  $k_d$  decreases with distance from the grid, and in this work a typical high value (for  $x/M \approx 10$ ) is  $30 \text{ cm}^{-1}$ . Thus the sensor size condition is

$$\ell_s < 0.1 \text{ cm}, \quad (1)$$

and the cut-off-frequency condition is

$$f_c \gg 170 \text{ cps}. \quad (2)$$

These conditions were satisfied in this work with the conical sensors and marginally satisfied with the wedge sensor. A large value of  $M$  was also desirable to make the grid Reynolds number as high as possible.



The towing speed was chosen from considerations of sensor frequency response, towing-tank length and test run duration, grid towing power, and pressure drop across the grid. Sensor frequency response and test run duration have been discussed already. The grid towing power is about 1/3 horsepower. The hydrostatic head corresponding to the pressure drop through the grid is about 1/4 in. There was no need to restrain the free surface in this work to reduce wave generation, since spectral measurements showed no wave components. Waves moving relative to the grid were observed only ahead of the grid.

A common problem with water velocity measurements by hot-element sensors is contamination. In this work the problem was unimportant, as illustrated by the fact that one particular quartz-coated sensor was used at intervals over periods of weeks without cleaning and with only minor calibration shift. The most important step in obtaining this reliability was tank water conditioning by filtering and deaeration. It was also found to be important to stir the tank by several passes of the grid before beginning calibration tests, to eliminate any temperature gradients.

The tank solution preparation problem was sizable, since the 6000 gallons of solution had to be homogeneous but not appreciably degraded by the method of preparation. Two methods were used. In the first method the polymer particles were dispersed directly into the tank filled with conditioned tap water, using an aspirator type of disperser. The towing-tank fluid was then mixed by towing the grid back and forth, sometimes with an off-center board on the grid to produce larger-scale mixing. The grid turbulence helped to dissolve the gelatinous blobs which the polymer particles become after being wetted. The results with the first Polyox solutions prepared by the "direct-dispersal" technique, led to work with the polyacrylamides. Prolonged grid stirring or stirring by air bubbling did not dissolve the gelatinous blobs of Separan AP-30 and J100. Thus it was necessary to prepare a master solution at a relatively high concentration (e.g., 0.8%) and mix it for a long period of time to encourage homogeneity. This solution was then diluted to the desired concentration with conditioned water and further mixing.

Sensor calibration was done by towing the sensor carriage along the tank and measuring bridge output voltage as a function of speed for a fixed bridge operating condition. The slope of the bridge voltage curve at the mean speed behind the grid was taken as the sensor sensitivity. Each sensor was calibrated before and after each series of grid-turbulence tests. Sensor deterioration was detected as a significant loss of sensitivity and as an upward drift of cold resistance. When the drift of sensitivity in 2 hours was as much as 4%, the sensor was replaced.

Turbulence samples were obtained using the equipment indicated in Fig. 4. Each set of grid-turbulence tests for various  $x/M$  and amplifier and filter settings was paralleled by a corresponding set of noise tests. For these tests the tank was allowed to become quiet, as for the regular tests, and only the sensor carriage was run along the tank. All signal generation, conditioning, and recording conditions were kept the same as in the grid-turbulence tests. Usually for each test condition, one or more tests were made to record the unfiltered signal, and one or more additional tests were made to record the filtered,

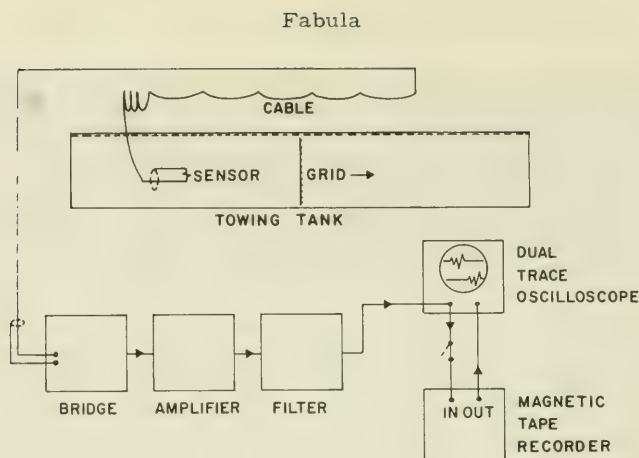


Fig. 4 - Equipment for towing-tank grid-turbulence recordings

approximately prewhitened signal in order to record at a suitable signal-to-noise ratio over the entire frequency range of interest.

Ordinary methods were used to obtain the power spectral density functions of the grid-turbulence samples. The noise-test recordings were analyzed in the same way, and noise corrections were made to the grid-turbulence spectra, by assuming that the "signal" and "noise" were uncorrelated. Because of the nature of the results, it was generally inappropriate to convert the frequency spectra of recorded bridge voltage fluctuation to turbulent velocity spectra. Thus the results will be presented as curves of "relative spectral level" in db units versus frequency, after correcting for signal conditioning, but not for sensor sensitivity.

## RESULTS

### Signal Waveforms

It was important in this work to show that certain spectral differences were not due to a distortion of the turbulent eddy structure but rather to a fairly uniformly distributed "raggedness" of the signal. This was done by photographs of the bridge output oscilloscope trace. Many oscilloscope sweeps were put on each picture by shifting the film between sweeps while the run was in process. The sweeps are thus spaced at roughly equal intervals along the length of each run. The time scale is 10 ms per major division, so that a major division corresponds to 0.36 cm of sensor travel, and each total sweep covers about one quarter of a mesh length. A positive deflection of bridge output voltage (increased heat transfer) produces a downward deflection on the photograph. The variation of the vertical scale nominal velocity calibration, due to variation in sensor sensitivity, is not large and is unimportant to the discussion.

Figure 5 shows a typical family of signal waveforms in water, when the sensor is towed at  $x/M = 10.7, 12.6$ , and  $20.2$ . Also included is the case where all conditions are the same except that the sensor is being towed without the grid along the tank, which is completely quiet. This noise test case will be labeled  $x/M = \infty$ .

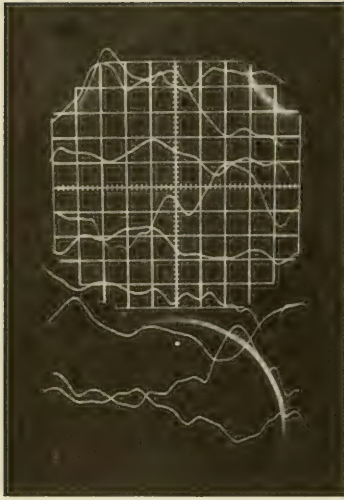
The absence or very low level of fine-scale structure in the waveforms of Fig. 5 is due to the dominant action of viscosity which begins at  $k/k_d \approx 0.1$ . For  $x/M = 10.7$  the wavelength corresponding to  $k_d$  is roughly  $0.2$  cm or half of one major division of the time scale.

For all conditions the same except for tank fluid and a negligible change in sensor sensitivity, Fig. 6 shows the character of the signal at  $x/M = 10.7, 20.2$ , and  $\infty$  in Solution F-5 (P301B; 137 ppm; prepared by the master-solution method) on the third day after dispersal, and at  $x/M = 10.7$ , 11 hours later. This is a typical case of significant raggedness near the grid, negligible raggedness farther away, and (only for Polyox) a decrease of raggedness with solution age at a given  $x/M$ . The lower group of sweeps in Fig. 6c is for the ordinary speed, but the upper group of sweeps is for the sensor at rest. Since the noise with the quartz-coated sensor at rest in quiet fluid was independent of the test fluid, the upper group also applies to the case of Fig. 5d for water. Thus there was a decrease in noise level when the sensor was set in motion in water but a slight increase in the solution. With variation of the lateral,  $y$  and  $z$ , coordinates, there were no notable changes in signal character, but some further evidence for a low degree of intermittency of the raggedness.

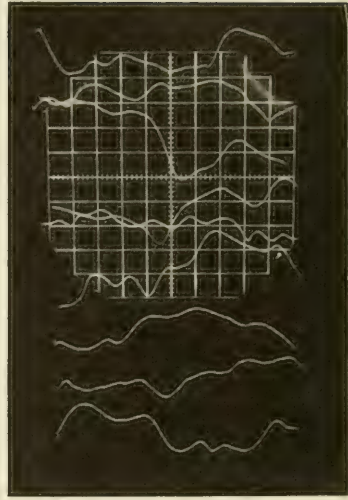
The decay of raggedness with solution age is illustrated for Solution F-10 in Figs. 7-9. At 71 hours of age, raggedness is seen at  $x/M = 20.2$ , but by 89 hours it is essentially gone at the same  $x/M$  but still evident at  $x/M = 12.6$ . By six days of age, raggedness is still detectable at  $x/M = 6$ . The waveforms for  $x/M = \infty$  in Fig. 9c show distinctive blips (corresponding to depressed heat transfer) which are not seen for  $x/M = \infty$  in Fig. 7d. This phenomenon was found in several cases of the concentrated, older Polyox solutions, and visible blobs could be detected when none had been found before.

With other additives used in this work, there was more intense and age-independent signal raggedness (for Separan AP-30 and J100), and there was a higher level of solution noise which interfered more with spectral measurements (for guar gum, Separan AP-30, and J100). Some typical waveforms for J100 are given in Fig. 10 for Solution F-6 with a concentration of 46 ppm, prepared by the master-solution method. The waveforms in Fig. 10 can be compared with waveforms in water for corresponding  $x/M$  values in Fig. 5, for the same sensor and only a minor change in sensitivity. The more intense raggedness in this case seems to be related to the higher solution noise. In Fig. 10d the waveforms for  $x/M = \infty$  are for two sensor towing speeds of about 20 and 45 cm/sec for the upper and lower groups respectively. A reasonable hypothesis is that the cause of the higher noise level is solution inhomogeneity, and it implies that the main result of changing speed from  $U_1$  to  $U_2$  should be a distortion of the characteristic times by the factor  $U_1/U_2$ . The waveforms in Fig. 10d are at least superficially consistent with this requirement.

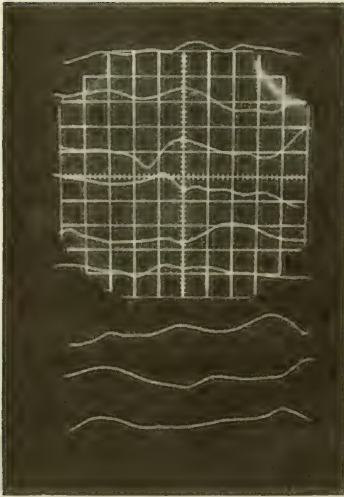




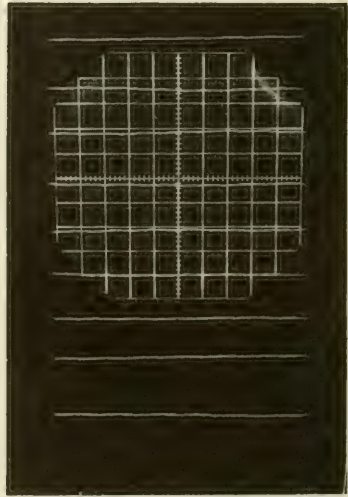
(a)  $X/M = 10.7$



(b)  $X/M = 12.6$



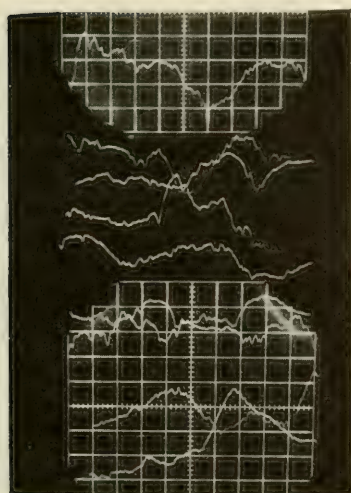
(c)  $X/M = 20.2$



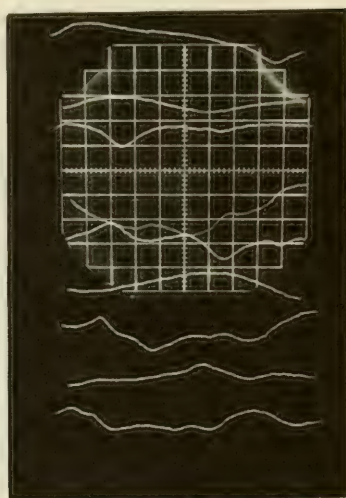
(d)  $X/M = \infty$

Fig. 5 - Signal waveforms with sensor Constance  
(see Fig. 3) in water

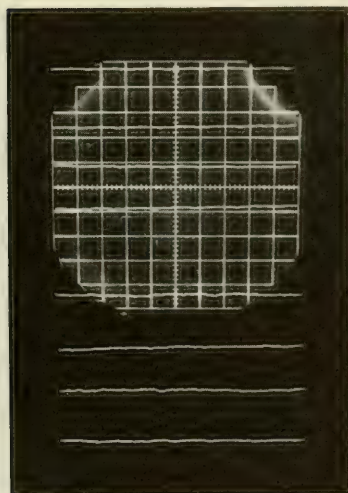
# Grid Turbulence in Dilute High-Polymer Solutions



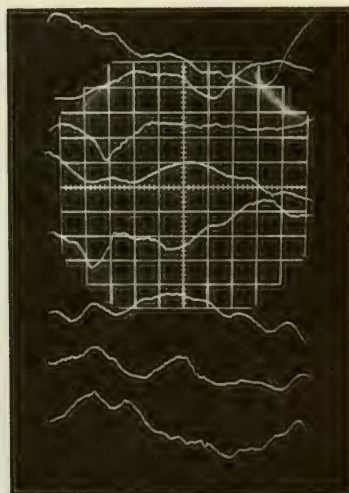
(a)  $X/M = 10.7$ , AGE 72 hrs



(b)  $X/M = 20.2$ , AGE 72 hrs

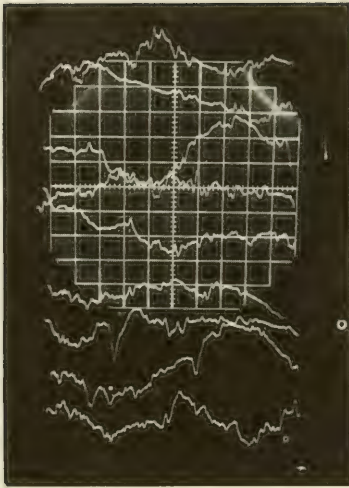


(c)  $X/M = \infty$ , AGE 72 hrs

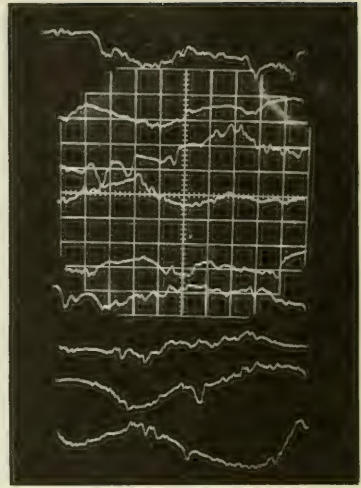


(d)  $X/M = 10.7$ , AGE 83 hrs

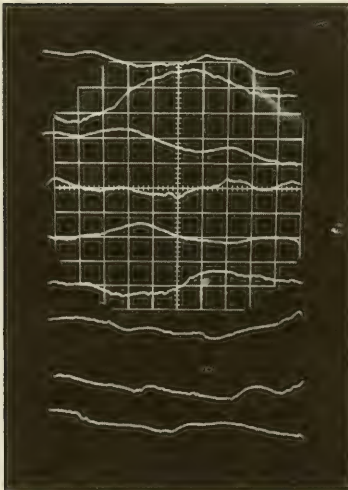
Fig. 6 - Signal waveforms with  
sensor Constance in Solution F-5



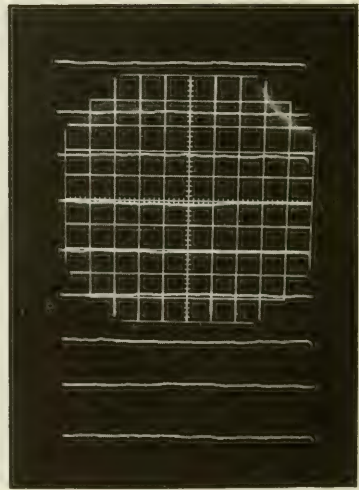
(a)  $X/M = 10.7$



(b)  $X/M = 12.6$



(c)  $X/M = 20.2$

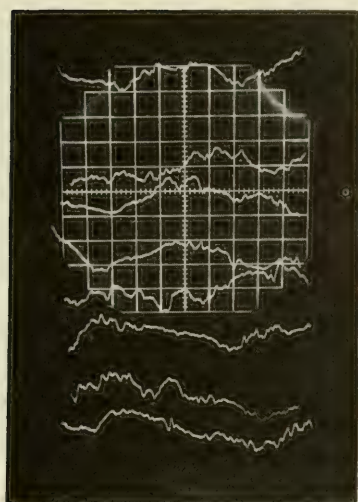


(d)  $X/M = \infty$

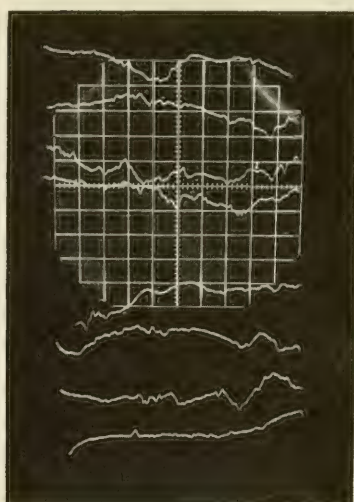
Fig. 7 - Signal waveforms with sensor Constance  
in Solution F-10; age 71 hours



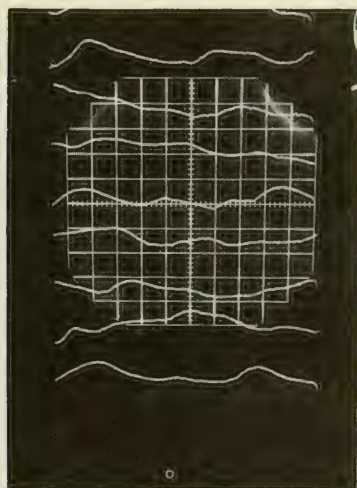
# Grid Turbulence in Dilute High-Polymer Solutions



(a)  $X/M = 10.7$

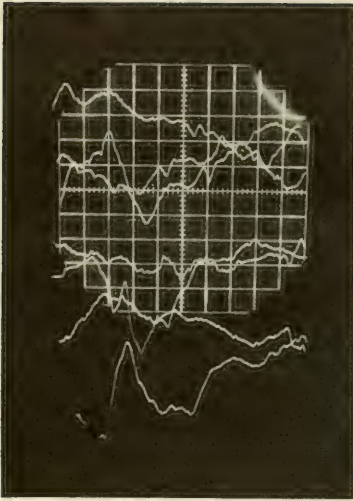


(b)  $X/M = 12.6$

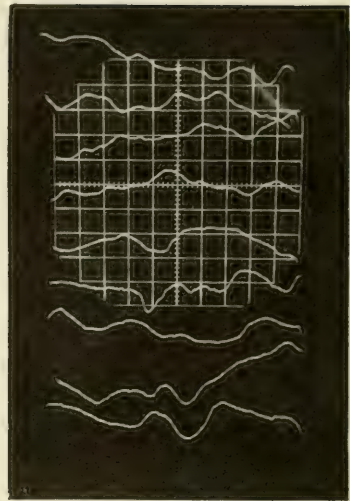


(c)  $X/M = 20.2$

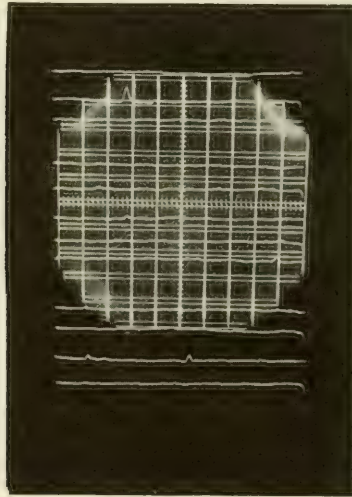
Fig. 8 - Signal waveforms with sensor Constance  
in Solution F-10; age 89 hours



(a)  $X/M = 5$



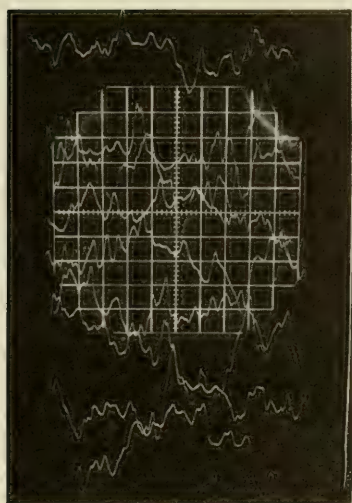
(b)  $X/M = 10.7$



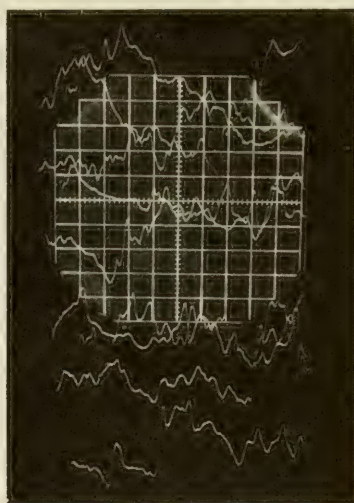
(c)  $X/M = \infty$

Fig. 9 - Signal waveforms with sensor Constance  
in Solution F-10; age 6 days

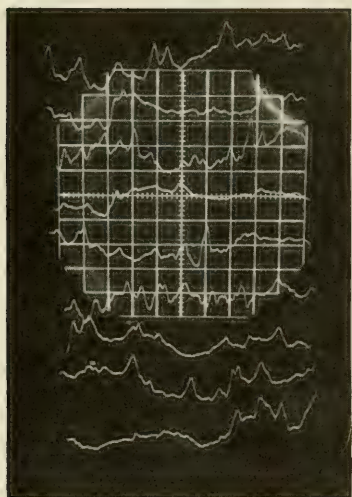
Grid Turbulence in Dilute High-Polymer Solutions



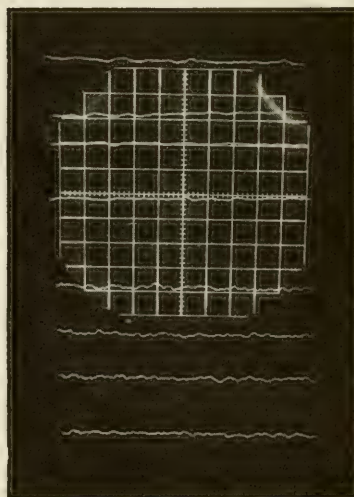
(a)  $X/M = 10.7$



(b)  $X/M = 12.6$



(c)  $X/M = 20.2$



(d)  $X/M = \infty$  (TWO SPEEDS)

Fig. 10 - Signal waveforms with sensor Constance  
in Solution F-6; age 22 hours



At the time of the photographs of Fig. 10, 22 hours had passed since the dispersal of the J100 into mixing barrels. The master solution had been gently stirred for 15 hours. In the other 7 hours, the master solutions were mixed with the towing-tank water by a continuous-flow technique giving a 50:1 dilution, with grid stirring to make the tank fluid homogeneous. After the photographs, the tank was stirred again for 4-1/2 hours by continuous back and forth towing of the grid by automatic switching. No changes in raggedness and noise resulted from this stirring. The solution was left in the tank over a 7-day period and then checked. No changes were found. Thus there is an extreme difference of behavior of Polyox and J100 solutions with respect to dependence of the signal raggedness upon solution age. This difference is similar to their difference with respect to the dependence of viscosity upon solution age.

The variation of towing-tank solution viscosity with age was studied with a Caraher, low-shear-rate viscometer (8). Figure 11 gives the relative viscosity ( $\eta_r$ ) measurements\* of towing-tank solutions of P301B. The data are in three groups, according to whether the total amount of polymer added is 2, 4, or 6

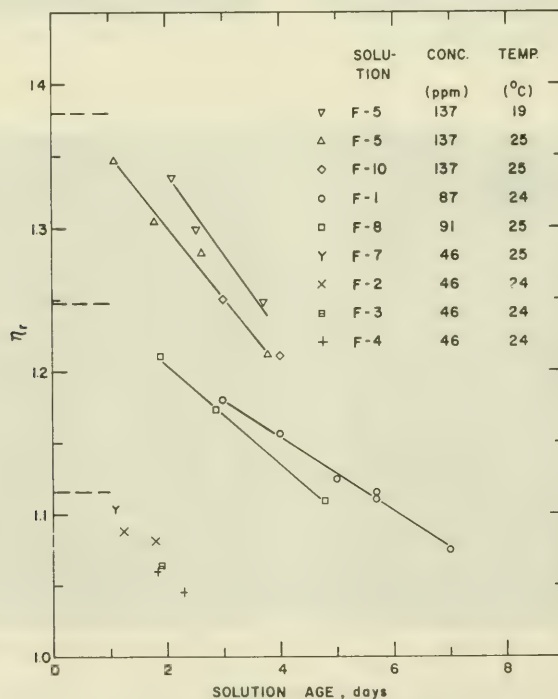


Fig. 11 - Collected data of relative viscosity versus solution age for P301B solutions in the towing tank

\*The measurements are for wall shear rates of about  $5 \text{ sec}^{-1}$  and the values are believed to be near to the zero-shear-rate values (8).

pounds. (Solution F-1 has a slightly lower concentration than normal due to spillage loss.) The results for Solution F-5 at 19° and 25°C show that relative viscosity is only slightly temperature dependent, so that the difference between temperatures of 24° and 25°C can be ignored. There is fair agreement between the backward extrapolations of the towing-tank data to one day of age with the predicted values (based on tests of solutions prepared in small quantities) given by the horizontal dashed lines. Polymer chain cleavage has been indicated as the cause of viscosity loss of Polyox solutions (9).

By comparing the data in Fig. 11 with the preparation methods for the various solutions in Table 1, some notable relations are found. Comparing Solutions F-8 and F-1, the change from the direct-dispersal technique to the master-solution technique apparently produced no major change in the loss of specific viscosity  $\eta_{sp} \equiv \eta_r - 1$ . Comparing Solution F-5 and F-10, the major increase in master-solution stirring time, from about 20 to 40 hours, produced no changes in the  $\eta_{sp}$  decrease.

Table 1  
Preparation Methods for Towing-Tank P301B  
Solutions Tested with the Caraher Viscometer

| Direct-Dispersal Technique                                                                                                                              |                              |                                  |                    |                                                      |
|---------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------|----------------------------------|--------------------|------------------------------------------------------|
| Solution F-8 was prepared by dispersing 4 lb of polymer directly into the tank, using intermittent stirring by the grid fitted with the stirring board. |                              |                                  |                    |                                                      |
| Master-Solution Technique                                                                                                                               |                              |                                  |                    |                                                      |
| Solution                                                                                                                                                | M.S.<br>Concentration<br>(%) | M.S.<br>Stirring<br>Time<br>(hr) | Dilution<br>Method | Notes                                                |
| F-1                                                                                                                                                     | 0.76                         | 10                               | Mixer barrel       | Approximate 16:1 dilution, 10-minute stirring        |
| F-2                                                                                                                                                     | 0.8                          | 15                               | Mixer barrel       | Precise 16:1 dilution, 20-minute stirring            |
| F-3                                                                                                                                                     | 0.4                          | 21.5                             | Mixer barrel       | 8:1 dilution, 20-minute stirring                     |
| F-4                                                                                                                                                     | 0.4                          | 17.5                             | Food mixer         | 8:1 dilution, 1-minute stirring                      |
| F-5                                                                                                                                                     | 0.8                          | 21                               | Mixer barrel       | 8:1 dilution, 1-minute stirring, demineralized water |
| F-7                                                                                                                                                     | 0.27                         | 11                               | Mixer barrel       | 20:1 dilution, continuous flow                       |
| F-10                                                                                                                                                    | 0.8                          | 40                               | Mixer barrel       | 50:1 dilution, continuous flow                       |

The time for raggedness of Polyox solutions to disappear was not noticeably affected by the range of solution preparation technique. The time depended, of course, upon the value of  $x/M$  of interest. Roughly speaking, the times for raggedness to disappear at  $x/M = 10.7$  were about the same as the times for a 50% loss of specific viscosity, according to Fig. 11, independent of the solution preparation technique.

With J100 solutions prepared in the same way as the P301B solutions, no significant decrease in  $\eta_r$  was found over the longest period tested, viz., one week, and as has been mentioned, there was no decrease of signal raggedness for the same period.

An effort was made to eliminate the raggedness by purifying the solvent. "Softened" and demineralized tap water were tried with P301B, with no detectable effects.

No dependence of raggedness upon sensor overheat or angle of attack could be detected. Raggedness was only found when the moving sensor reached liquid disturbed by the grid or by other obstacles moving at similar speeds, such as a circular cylinder or the sensor strut itself.

### Spectral Effects of Signal Raggedness and Solution Noise

The spectral effects of raggedness and solution noise are illustrated in Figs. 12 to 16. Relative spectral levels are given for  $x/M = 20.2$  in water and in Solution P301-3 at various solution ages in Solution P301-3 (WSR-301; direct dispersal; 18 ppm). The conversion from relative levels versus frequency to absolute spectral levels versus wavenumber is not needed. There is a common reference for the relative levels in all five figures, and the change in sensor sensitivity corresponds to only a 0.7-db shift. This is less than the standard error of the single-run spectral measurements of about  $\pm 1$  db. In each figure the data of three runs are shown: (a) a grid-turbulence test with recording of the unfiltered signal (e.g., Run C6 in Fig. 12), (b) a grid-turbulence test with recording of the filtered signal in order to determine the high-frequency end of the spectrum (e.g., Run C9), and (c) a noise test with recording for the same filter setting as in (b) (e.g., Run C10N). Noise tests without filter are not shown because the resultant noise corrections are negligible. The plotted spectral-level data points are always the values without noise correction, and the associated fairing curves (the solid lines) are with noise corrections. Thus the noise correction produces the departure of the fairing curves from the data points. In later figures only the points for the signal-plus-noise spectral levels and the noise-corrected fairings will be shown. The latter are always ended when the noise correction becomes about 4 db, which occurs when the noise level is 2 db below the signal-plus-noise level.

In Figs. 13 to 16 the water noise level from Fig. 12 is shown by the curve labeled W.N.L. The increase in noise in the solution is about 5 to 10 db at the frequencies shown and is fairly independent of solution age. Figure 12 shows that in water the noise correction would have reached 4 db at about 250 cps. In the 4-hour-old solution, the noise correction is negligible up to above 250 cps



# Grid Turbulence in Dilute High-Polymer Solutions

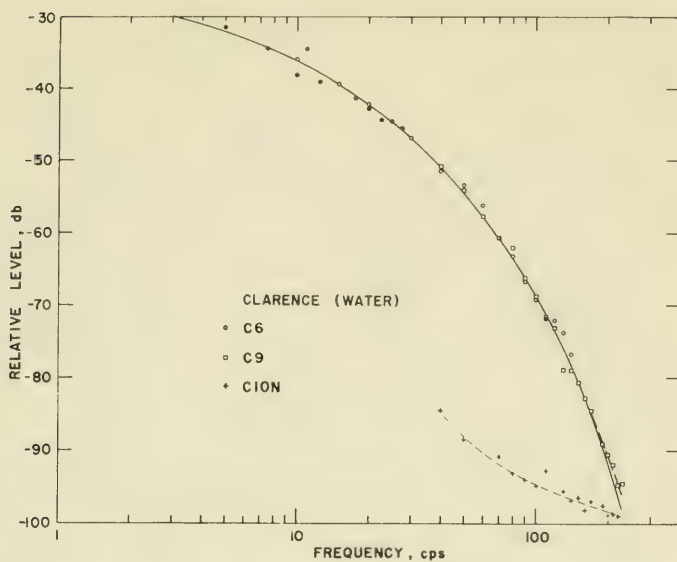


Fig. 12 - Relative spectral levels in water at  $x/M = 20.2$  and  $\infty$

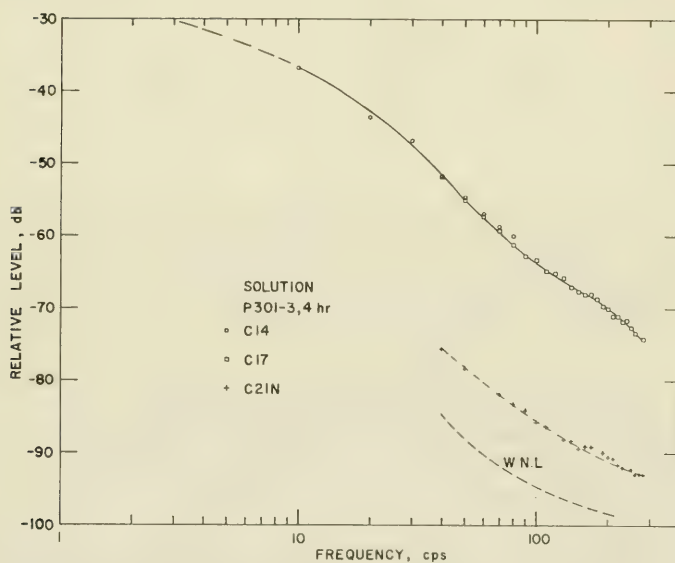


Fig. 13 - Relative spectral levels in Solution P301-3 at  $x/M = 20.2$  and  $\infty$ ; solution age 4 hours

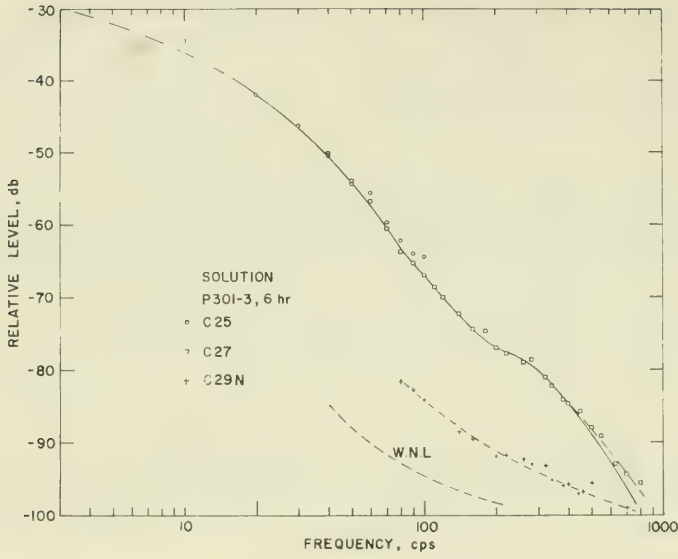


Fig. 14 - Relative spectral levels in Solution P301-3 at  $x/M = 20.2$  and  $\infty$ ; age 6 hours

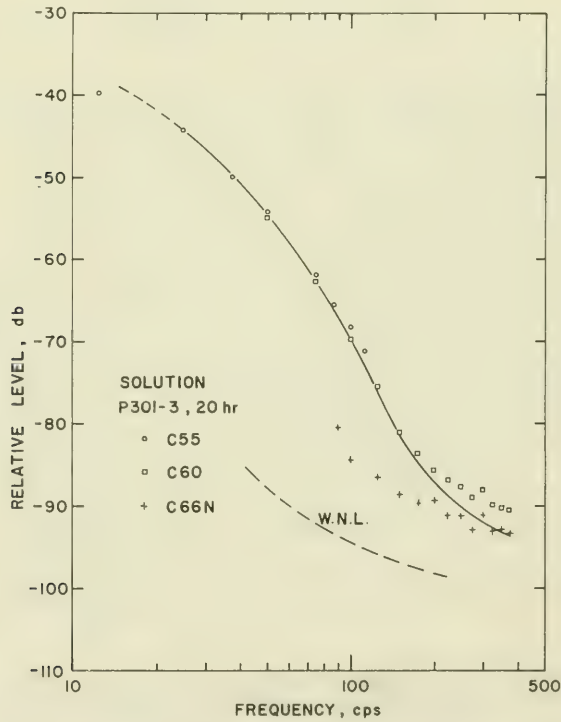


Fig. 15 - Relative spectral levels in Solution P301-3 at  $x/M = 20.2$  and  $\infty$ ; age 20 hours

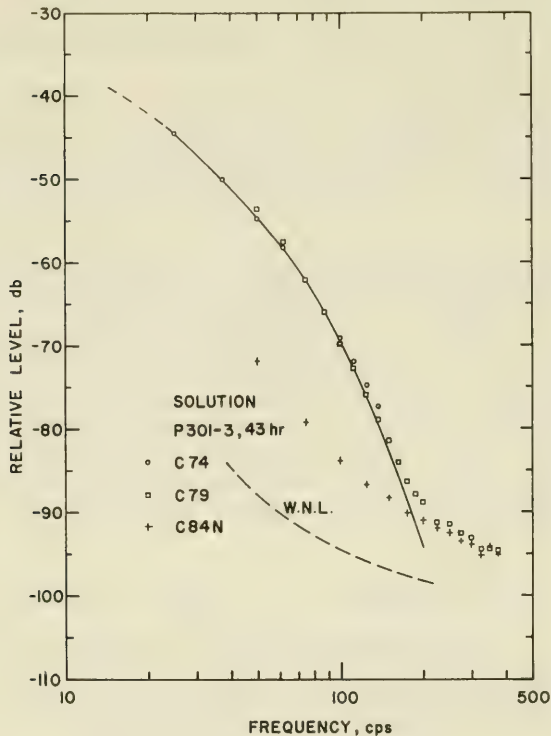


Fig. 16 - Relative spectral levels in Solution P301-3 at  $x/M = 20.2$  and  $\infty$ ; age 43 hours

because of raggedness. As raggedness decreases with solution age, the fixed noise levels become more important, and at 43 hours, the noise correction is about 4 db at 200 cps. Raggedness can be said to have disappeared at  $x/M = 20.2$  by 43 hours because the noise-corrected spectral level curve shows no departure from the behavior in water of a monotonic decrease of logarithmic slope.

All the solutions prepared in this work showed some degree of increased noise level relative to water. The lowest increases were obtained with the Polyox solutions. Only a limited amount of spectral analysis was done with other additives than Polyox. With guar gum, there was a high level of noise, due to visible particles in the suspensions, which greatly restricted the range of  $x/M$  in which accurate spectral data could be obtained for the standard sample lengths. With Separan AP-30 and J100, no spectral analysis was done because the signal raggedness near the grid did not decrease in time and because the higher noise levels with these additives would have interfered with tests far from the grid.

The decay of raggedness with distance from the grid is seen in the higher frequency portions of the curves of Fig. 17. In these tests, Solution PC-4,



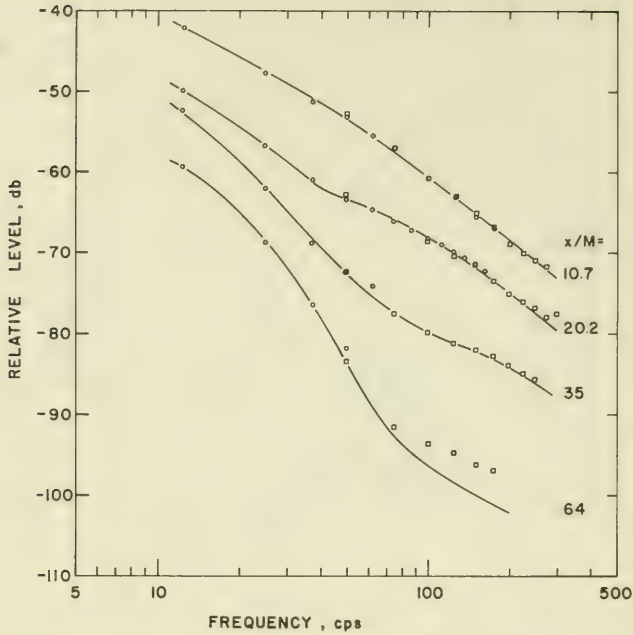


Fig. 17 - Relative spectral levels in Solution PC-4 at  $x/M = 10.7, 20.2, 35$ , and  $64$

containing 83 ppm of Polyox coagulant, was only 8 hours old, and there was extreme raggedness at both  $x/M = 10.7$  and  $20.2$ .

The falseness of a velocity interpretation of the raggedness can be seen as follows. Under conditions of extreme raggedness, such as seen in Fig. 17 for  $x/M = 10.7$  and  $20.2$ , the relative spectral level curves generally had a slope of about  $-20$  db per decade over the frequency range of roughly  $10$  cps to the upper limit of frequency analysis, typically about  $200$  or  $300$  cps. If such a frequency spectrum were interpreted as a true velocity spectrum and converted to a wavenumber spectrum  $\phi(k)$ , the latter would behave as  $k^{-2}$  in the corresponding wavenumber range. Thus the viscous dissipation spectrum,  $k^2 \phi(k)$ , would be nearly flat over that range, while the water dissipation spectrum drops to very low levels. Therefore a dissipation increase by a factor of perhaps  $10$  could be implied. Such an increase in dissipation is seen to be completely out of the question, as follows. For water, the decrease in turbulence energy due to viscous dissipation between  $x/M = 10.7$  and  $20.2$  is roughly  $50\%$ . Since the raggedness spectral distortion can be severe enough to imply the factor of  $10$  increase in dissipation at both  $x/M = 10.7$  and  $20.2$ , and, therefore, over the interval in between, the decrease in turbulence energy implied by the spectra would be an impossible  $500\%$ .

## Spectral Shift Due to Viscosity Increase

The comparison of the  $\phi(k)$  spectrum for the solution with the corresponding spectrum for water must be made with allowance for viscosity difference. Non-Newtonian behavior is indicated only if the spectral changes cannot be correlated with the predictions of the Newtonian-fluid scaling laws for the effects of a simple viscosity increase.

As shown in Ref. 1, the good agreement of the normalized dissipation spectra measured in air and water, with  $R_M$  varying from 40,000 to 68,000, indicates that the effect upon  $\phi(k)$  in the dissipation range of a change in  $\nu$  should be predicted by assuming invariance of the function\*  $\phi(k)/(\epsilon\nu^5)^{1/4}$  measured at a particular  $x/M$  for water and considered as a function of  $k/k_d$ . Thus the spectral shift in  $\phi(k)$  due to a change in  $\nu$  arises only from the changes in  $k_d$  and  $(\epsilon\nu^5)^{1/4}$ . Furthermore because the wind-tunnel data also imply that  $\epsilon M/U_0^3$  for a given  $x/M$  is essentially independent of  $R_M$ , then  $\epsilon$  will be assumed to be fixed for a given  $x/M$ . This assumption should be valid as long as the energy-containing scales are largely separate from the dissipation scales, as discussed in Ref. 1. With

$$\tilde{\phi}(k/k_d) \equiv \phi(k, x/M, \nu)/(\epsilon\nu^5)^{1/4},$$

where  $\phi(k, x/M, \nu)$  has been measured for a particular grid,  $x/M$  and  $\nu$ , the effect of changing  $\nu$  to  $\nu'$  in the dissipation range should be given by

$$\phi'(k, x/M, \nu') = \tilde{\phi}(k/k'_d)(\epsilon\nu'^5)^{1/4}, \quad (3)$$

where  $k'_d \equiv (\epsilon/\nu'^3)^{1/4}$  and where  $\epsilon$  remains constant. For conciseness, this scaling law will be referred to as Kolmogoroff similarity with constant dissipation. If  $m$  is the logarithmic derivative of  $\tilde{\phi}(k/k_d)$  with respect to  $k/k_d$ , it can be shown that

$$\phi'/\phi \approx (\nu'/\nu)^{(3m+5)/4}. \quad (4)$$

Then the spectral level shift in db is approximately

$$\frac{15}{2} \left( m + \frac{5}{3} \right) \log (\nu'/\nu). \quad (5)$$

Thus if there is an inertial subrange with  $m = -5/3$ , the spectral level shift due to viscosity in that inertially dominated range is zero as expected. Of course, viscosity increase would not affect the even lower wavenumber range which is also inertially dominated. Thus Eq. (5) applies only where  $m < -5/3$ . The upper wavenumber for which spectral measurements were made was typically of the order of  $k_d$ . Since  $m \approx -7$  for  $k/k_d = 1$ , the factor  $(m + 5/3)$  varies from 0 to roughly  $-16/3$  over the applicable range in this work. Thus for a 10% increase in  $\nu$ , the spectral shift will be about -1.6 db. Since the viscosity increases due

\*Because of equipment limitations,  $\epsilon$  was not determined but was replaced by the isotropic viscous dissipation rate calculated from  $\phi(k)$ .

to the additives were often 25%, nonnegligible spectral shifts just due to Newtonian-fluid effects are to be expected.

In order to test the similarity law, spectral measurements were made in water at temperatures of 20.7°C and 11.7°C, giving a viscosity increase of 30%. The results for  $\phi(k)/(\epsilon\nu^5)^{1/4}$  versus  $k/k_d$  are shown in the log-log plot of Fig. 18 for  $x/M = 10.7$ . The magnitude of the maximum separation between the two curves is about 0.1 in terms of  $\log [\phi(k)/(\epsilon\nu^5)^{1/4}]$ , or in other words 1 db; thus the expected similarity is found to within the typical measurement scatter. Similar agreement was found for  $x/M = 12.6$  and 20.2.

The magnitudes of the spectral shifts are not evident in Fig. 18 but are seen in Fig. 19 where the spectral level ratio (cold water compared with warm water) is shown for  $x/M = 10.7$ . Also included is the approximate prediction of Eq. (5) (dashed curve), based on values of  $m$  determined graphically from Fig. 18. Despite the approximation made in deriving Eq. (5) it can be seen that Fig. 19 is quite consistent with Fig. 18, so that the approximate relation can be used in considering viscosity increase in the polymer solutions.

### Spectral Shifts in Polymer Solutions

Some general characteristics of the spectral measurements in Polyox solutions can be seen in the data already presented. For example in the P301-3 tests of Figs. 13 to 16 the distortion of the spectra due to signal raggedness decreased with solution age until finally any remaining signal raggedness effect

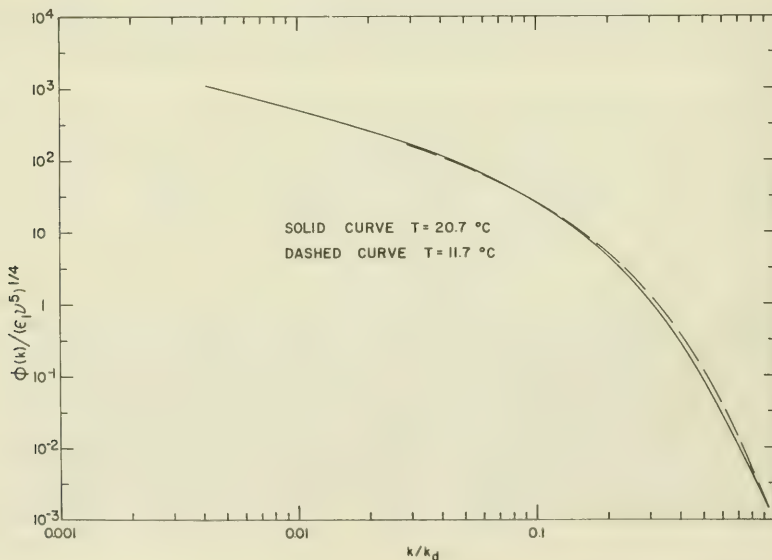


Fig. 18 - Comparison of normalized spectra in water for  $x/M = 10.7$  for a 30% viscosity difference

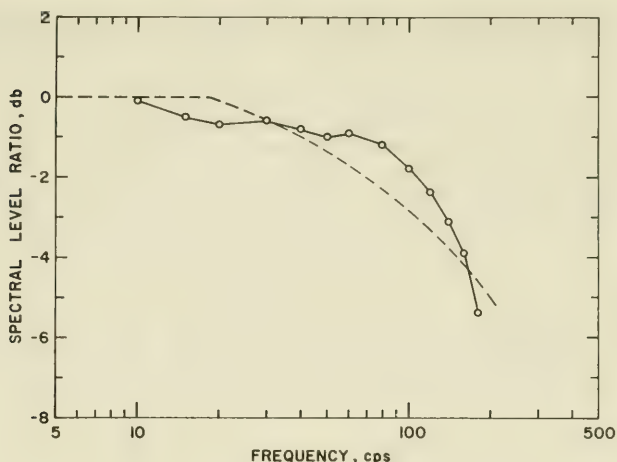


Fig. 19 - Spectral level shift for  $x/M = 10.7$  due to a 30% viscosity increase, compared with the predicted shift (dashed curve) from Eq. (5) using values of  $m$  determined from Fig. 18

was lost in the noise. A detailed comparison shows that there are no significant spectral changes other than those which can be attributed to raggedness. The estimated viscosity increase was so small that the expected spectral shift due to viscosity increase was insignificant.

For higher concentrations of Polyox, two other types of spectral shift were observed. The first type was a greater scatter in spectral measurements near the grid. A possible cause of such behavior could be temporary sensor sensitivity loss, due to intermittent coating. However, no abrupt changes of signal level were ever observed. Another possible cause is poorer mixing of the wakes of the grid bars, since the diffusion of turbulent wakes has been shown to be reduced even by low Polyox concentrations (10). The grid-turbulence experiment is not appropriate for a study of the wake mixing, for which a large increase in the number of runs would be necessary. With older solutions or greater  $x/M$  the scatter of spectral measurements was the same as the scatter in water.

The final type of spectral shift in the more concentrated Polyox solutions was that due to the viscosity increase. An example of this type of shift can be seen with Fig. 20, which gives the relative spectral curves for water and Solution F-5 at 66 hours, both for  $x/M = 39$ . Considering Fig. 6 which shows waveforms for  $x/M = 20.2$  and 72 hours, and noting the similar effects of grid distance and solution age, we can assume that the waveforms in Fig. 6 for  $x/M = 20.2$  would have also applied, except in amplitude, to  $x/M = 39$  at 66 hours. Thus no spectral effects due to raggedness are expected. In Fig. 20, the curve for the solution departs from the data points due to the usual noise correction, and the curve is stopped when the correction becomes 4 db. When correction of



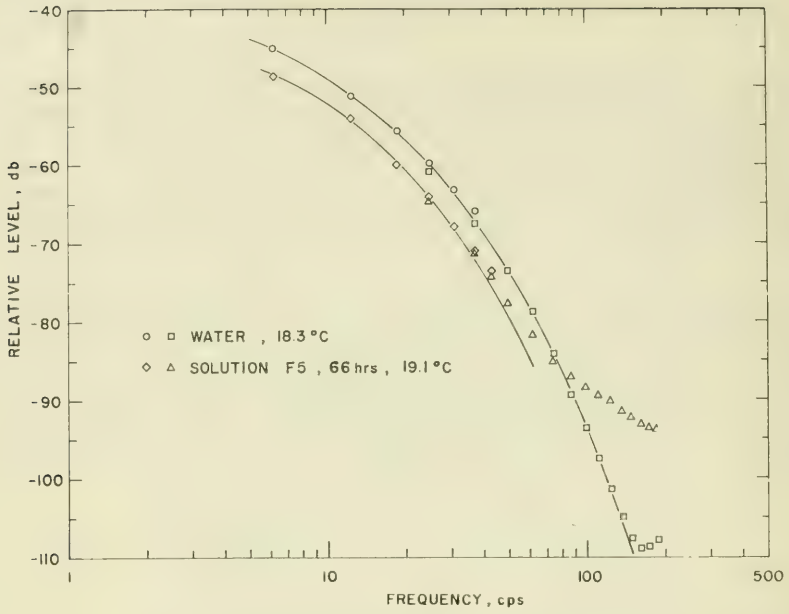


Fig. 20 - Relative spectral levels in water and in Solution F-5 for  $x/M = 39$

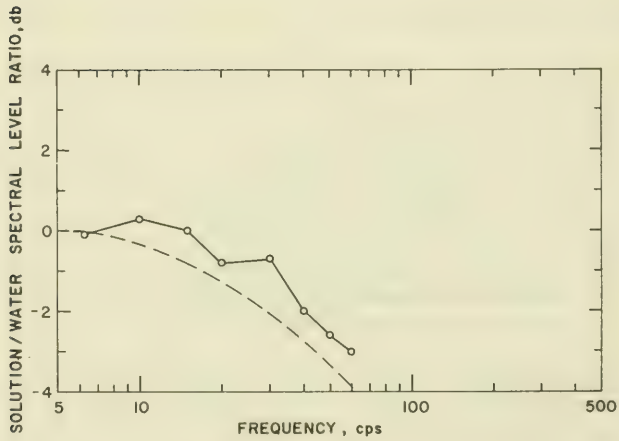


Fig. 21 - Spectral level shift in Solution F-5 compared with water for  $x/M = 39$

+3.6 db to the solution curve (due to sensor replacement by one of much lower sensitivity) is made, the spectral level shift data in Fig. 21 are obtained.

The amount of the spectral shift can be compared with the predicted shift due to simple viscosity increase according to Kolmogoroff similarity with constant dissipation. Using the relative viscosity measurements on Solution F-5 discussed earlier, the viscosity ratio  $\nu'/\nu$  for 66 hours can be estimated as 1.27. Thus the dashed curve in Fig. 21 results from the viscosity spectral shift relation of Eq. (5). The agreement is as good as in the case of the cold-water tests. Thus there is no evidence of any non-Newtonian effects.

Spectral shift curves for solution F-5 for various solution ages and  $x/M = 10.7$  are given in Fig. 22. The theoretical dashed curve is for 73 hours, but applies for all three solution ages, since the decrease of viscosity ratio from 1.27 to 1.23 for 66.5 to 85 hours produces only a 4% decrease in  $\log(\nu'/\nu)$ .

The changes in signal waveforms in Fig. 6 for  $x/M = 10.7$  between 72 and 83 hours can be compared with the changes in spectral shift for the same  $x/M$  in Fig. 22 for roughly the same period. It is clear that as the solution ages the decrease of raggedness reveals more and more of the spectral shift due to the

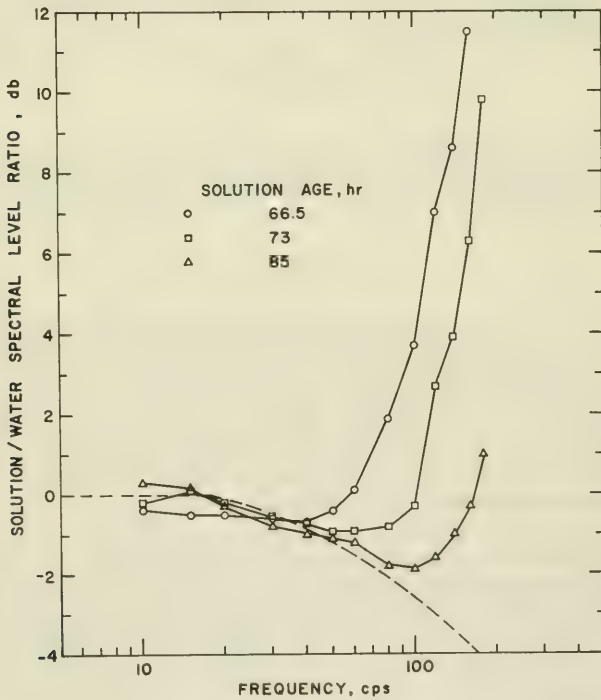


Fig. 22 - Spectral level shift in Solution F-5 compared with water for  $x/M = 10.7$  and various solution ages

viscosity increase. Similar behavior was found for the lower P301B concentrations, except that the viscosity spectral shifts was not so large. The theoretical prediction for the absence of any non-Newtonian effects is given in the following section.

## DISCUSSION

A theoretical estimate of the requirements for viscoelastic effects on the grid-turbulence spectrum can be obtained as follows. With the molecular theories of viscoelasticity of dilute polymer solutions, a spectrum of relaxation times is associated with the solvated macromolecular coils. When the theoretical results for infinitesimal oscillatory shear are generalized in order to obtain materially objective forms, as in the work of Spriggs and Bird (11), the same characteristic times appear, as long as one assumes that the coils are subject only to small perturbations. We will use the molecular theory value of the terminal (largest) molecular-coil relaxation time  $t_m$  as the best available estimate for the largest viscoelastic time of the dilute solutions.

A variety of characteristic times can be associated with turbulent flows. With respect to the effects of viscoelasticity upon grid turbulence, the time characteristic of the spectral transfer of energy from larger to smaller eddies would seem to be appropriate. This is the Onsager time of inertial spectral transfer, as used by Corrsin (12):

$$t_a \equiv [\kappa^3 E(\kappa)]^{-1/2} \quad (6)$$

where  $E(\kappa)$  is the three-dimensional energy spectrum function and  $\kappa$  is the wavenumber vector magnitude. Nondimensionalizing with dissipation variables, we obtain, for  $k_d = (\epsilon/\nu^3)^{1/4}$ ,

$$t_a(\epsilon/\nu)^{1/2} \equiv [k/k_d]^3 E(\kappa)/(\epsilon\nu^5)^{1/4}]^{-1/2}, \quad (7)$$

where  $(\epsilon/\nu)^{1/2}$  is the rms vorticity in isotropic turbulence. Figure 23 gives the plot of Eq. (7) using a Heisenberg spectrum for  $E(\kappa)$ . The curve is taken from Corrsin's report (12). Also shown are points based on measurements of  $E(\kappa)$  in grid turbulence by Uberoi (13). The Lagrangian integral time scale of strain-rate fluctuations would seem to be necessarily of the same order or larger than  $t_a$ , which can be interpreted as the time for energy associated with wavenumber  $\kappa$  to be transferred to wavenumber  $2\kappa$ .

Corrsin also considers two other characteristic times:  $t_b$  for energy exchange due to pressure-velocity correlations and  $t_c$  for viscous dissipation.  $t_b$  has to be assumed to be approximately equal to  $t_a$ .  $t_c$  becomes appreciably smaller than  $t_a$  only for  $k/k_d > 1$ . In this region, we can expect that the effects of viscoelasticity to be difficult to separate experimentally from purely viscous non-Newtonian effects.

Thus as the viscoelastic relaxation time,  $t_m$ , is increased from values far smaller than the minimum characteristic times associated with the turbulence,

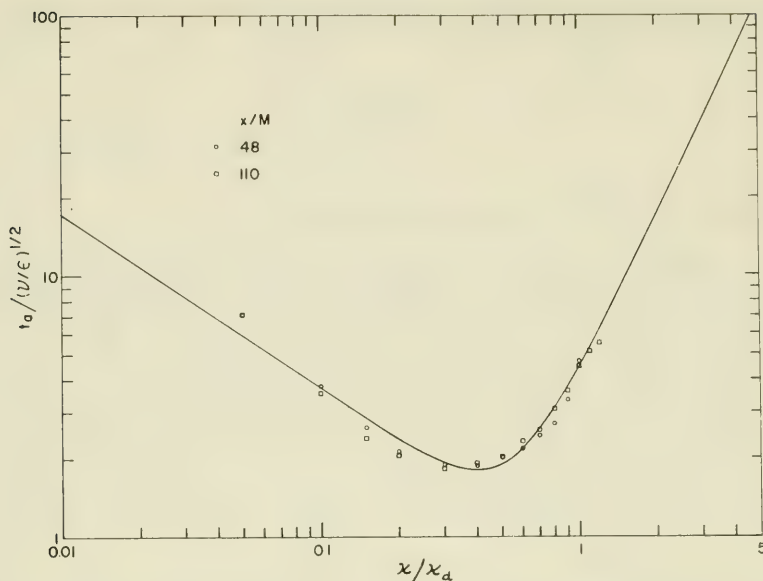


Fig. 23 - Inertial spectral transfer time, calculated (12) from a Heisenberg spectrum. The points are from grid-turbulence measurements by Uberoi (13).

we may expect experimentally significant viscoelastic effects upon the grid turbulence to first appear at  $\kappa/\kappa_d \approx 1/2$ , and we presume the effect would become significant when

$$t_m(\epsilon/\nu)^{1/2}/2 \approx 1. \quad (8)$$

The molecular theories (14-16) which predict the value of  $t_m$  are for sufficiently dilute solutions that the macromolecular coils are well separated. Thus

$$t_m = a \frac{100 \eta_0 [\eta] \bar{m}}{N_A k_B T}, \quad (9)$$

where  $\eta_0$  is the solvent viscosity;  $[\eta]$  is the intrinsic viscosity for vanishing steady shear rate or vanishing frequency of infinitesimal oscillatory shear;  $\bar{m}$  is the molecular weight;  $N_A$  is the Avogadro constant;  $k_B$  is the Boltzman constant; and  $T$  is the absolute temperature. The numerical constant  $a$  varies somewhat with the theory, but can be taken as  $1/2$  for this work. The factor of 100 appears when  $[\eta]$  is given in the usual units of dl/g.

Comparison of the molecular theory predictions with experimental measurements have only been made for solutions which are not "dilute" in the sense  $c'[\eta] \ll 1$ . Reasonable agreement for well-characterized solutions, i.e., of narrow or known molecular weight distributions, has been obtained by many



investigators using an empirical concentration correction to the  $t_m$  formula, in which  $[\eta]$  is replaced by  $(\eta - \eta_0)/c'\eta_0$ , so that

$$t_m = a \frac{100 (\eta - \eta_0) \bar{M}}{c' N_A k_B T} \quad (10)$$

The concentration dependence is not very large for  $c'[\eta] \ll 1$ , as in this work, and its effect on Eq. (7) is reduced by the factor  $\nu^{-1/2}$ .

We can make only a crude prediction of  $t_m$  with Eqs. (9) and (10), because of the unknown molecular weight distributions of the industrial Polyox polymers used. Thus for P301B with  $[\eta] \approx 20$  dl/g and  $\bar{M} \approx 4 \times 10^6$ , we obtain  $t_m \approx 0.002$  sec from Eq. (9). With this value and  $\nu = 0.01$  cm<sup>2</sup>/sec, Eq. (8) gives a required  $\epsilon$  of about  $10^4$  cm<sup>2</sup>/sec<sup>3</sup>. This value is extremely high compared with values in the present grid-turbulence flow. With  $M/U_0^3 \approx 12/34$  sec<sup>3</sup>/cm<sup>2</sup>, the required value of  $\epsilon M/U_0^3$  is about 3, while from data in Ref. 1 the extrapolated value for  $x/M = 5$  is about 0.01. Five mesh widths is as close to the grid as one might hope to find some degree of turbulence homogeneity. Thus the dissipation rates in the grid-turbulence experiment are too low by a factor of 300 to expect any effect on the turbulence spectrum. If we assume that the ratio of times in Eq. (8) needs to be only about 1/3 instead of 1, the dissipation rates are still too low by a factor of about 30. Thus no viscoelastic effect upon the turbulence spectrum in this work is expected.

The need for matching the viscoelastic relaxation time to the minimum turbulence characteristic time became clear early in this work. It was thought that polymer concentration could be increased to produce the desired magnitude of  $t_m$ . The task of solution preparation became formidable when the master-solution technique was introduced in an attempt to solve the raggedness problem. Nevertheless, much higher concentrations than 140 ppm could be used in the towing tank. However, the present results indicate that this is not worthwhile until the signal raggedness can be eliminated or circumvented.

The raggedness problem also prevents a useful increase in  $(\epsilon/\nu)^{1/2}$  which might more nearly satisfy Eq. (8). This is seen when possible changes in the grid-experiment parameters are considered. While energy decay rates can be increased by going to higher grid solidity, problems of flow instability (17) limit the increase in solidity. Thus  $\epsilon M/U_0^3$  versus  $x/M$  for the present grid can be assumed to apply. As discussed in Ref. 1, there is little dependence on  $U_0 M/\nu$ . Thus at a given  $x/M$ ,  $\epsilon$  can be increased in proportion to the increase in  $U_0^3 M$  which is consistent with other requirements, namely avoidance of raggedness, adequate sensor frequency response and spatial resolution, and a suitable turbulence Reynolds number. The raggedness problem imposes the most severe restriction, since the time from grid passage for a given  $x/M$  is proportional to  $M/U_0$ . If the decay of raggedness is a process of disentanglement or relaxation of clusters formed or changed in character by the bars of the grid, as discussed next, then if  $U_0/M$  is increased, it may be presumed that even greater solution aging will be required before signal raggedness disappears at a given  $x/M$ . Since major solution degradation was suggested by the loss of specific viscosity when signal raggedness disappeared at  $x/M = 10.7$  in the present experiment,

and since concurrent degradation of elastic properties can be expected to occur (18), an increase in  $U_0/M$  is undesirable. If  $U_0/M$  is kept fixed, an increase in  $\epsilon$  by a factor of 30 requires increases in  $U_0$  and  $M$  by a factor of  $\sqrt{30}$ . This is completely impracticable in the present towing tank. Thus it is clear that  $U_0/M$  must be increased to increase  $\epsilon$ , and therefore the signal raggedness problem must be solved before the desired near match of characteristic times can be achieved in the grid-turbulence experiment.

Solution inhomogeneity, perhaps with polymer impurity as a contributing factor, would seem to be the most likely cause of signal raggedness. For example, if the solution ahead of the grid contains large clusters of macromolecules, then a simple mechanism for the production of signal raggedness can be hypothesized. During the short period of high shear rate which the fluid undergoes in the grid-bar boundary layers, these clusters might be stretched out into long "fibers." Such fibers may be expected to have a stiffness of form and a long relaxation time, because a high degree of alignment has been given to the polymer chains in the cluster and some solvent has been forced out of the interior of the cluster. The difference between signal raggedness and solution noise may be presumed to represent the greater disturbance to the shear flow over the hot film by the passage of "fibers" as compared with the passage of soft clusters.

Another mechanism by which the bars of the grid might produce raggedness is by accumulating smaller clusters on their stagnation lines. Presumably the rate of accumulation would be balanced by a washoff of temporary agglomerates of a much larger size than the constituent clusters, so that the sensor might detect only the agglomerates.

None of these hypotheses could be tested in this work. By working with a smaller volume of liquid in a small flow facility or in a miniature tank, solution purification techniques such as filtering and centrifugation could be used to see if signal raggedness could be eliminated in that way. Also solutions with controlled inhomogeneity could be tested to determine the dependence of signal raggedness on existing clusters, and optical means might be used to determine if washoff of coating material is involved.

There is very little other work on hot-film flowmeter measurements with which this work can be compared. However several concurrent investigations should answer the natural question of whether or not there is any connection between the signal-raggedness phenomenon found in this grid-turbulence study and the friction-reduction phenomenon in pipe and boundary-layer flows.

#### ACKNOWLEDGMENT

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## DISCUSSION

G. E. Gadd  
*National Physical Laboratory  
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Dr. Fabula's suggested explanation of "raggedness," that it might be due to molecular clusters stretched out into long fibers, could also perhaps be relevant to the following phenomena:

1. In experiments\* on eddy shedding behind a small circular cylinder, Polyox in concentrations greater than 20 ppm had been found to halve the frequency, whereas below 10 ppm it had no effect.

2. Similarly, when Polyox solutions of greater than 20 ppm were poured, they showed "stringiness," the formation of long hairlike filaments, which did not seem to occur for less than 10 ppm concentration.

3. Further, a laminar jet of coloured Polyox solution of greater than 20 ppm concentration emerging into uncoloured Polyox solution showed, when it turned turbulent, a quite different kind of turbulence than for the similar flow of water, with a complete suppression of the small eddies. No such eddy suppression occurred, however, below 10 ppm concentration, as White of Hendon had showed, even though these weaker solutions are still very effective<sup>†</sup> in reducing turbulent drag.

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\*G.E. Gadd, Nature 211:169 (1966).

†J.W. Hoyt and A.G. Fabula, "The Effect of Additives on Fluid Friction," pp. 947-974 in "Fifth Symposium on Naval Hydrodynamics: Ship Motions and Drag Reduction," ONR Report ACR-112, 1964.



Perhaps, therefore, small-eddy suppression is essentially irrelevant to turbulent drag reduction but is due rather to Dr. Fabula's "raggedness," which might be absent in very weak concentrations.

\* \* \*

## DISCUSSION

Richard L. Hummel  
*University of Toronto*  
*Toronto, Canada*

This work was well conceived and carefully conducted, and the "signal raggedness" which seems to have frustrated the original aims of determining a frequency distribution curve for comparison with that predicted theoretically by certain models may suggest important aspects in the behaviour of these solutions which had not been sufficiently expected. Ideally, possibly one-half of our experimental work should come out in the form we are prepared for, or possibly expect, so that it may be tied up in a neat package for the scientific community, and the remainder, which at first does not seem as satisfactory, may eventually lead us to new experiments and insights.

The author's suggestion that clusters of macromolecules may exist and contribute substantially to the raggedness is certainly justified from the high attraction of water molecules for each other relative to their interaction with ether linkages as would be present in the Polyox WSR-301. Thus the water molecules would squeeze the polymer molecules together into clusters. It is well known that much smaller molecules, such as soaps and detergents will cluster under similar conditions into relatively large micelles. The ratio of polar bonds to hydrocarbon segment is high and, therefore, more favourable for these macromolecules than for most soaps and detergents, and the larger molecule considerably reduces the opposing entropy effect of molecular association. Thus, larger molecules are statistically more likely to be associated with each other.

Whether or not raggedness is occasioned by the sensor meeting sufficiently large agglomerates or not, another effect should also be considered. We have conducted tests with a fluidized bed where a Newtonian liquid flows vertically upward, suspending to some degree of porosity hard spheres of uniform diameter, and we have optically chosen planes of these spheres for study.\* Far from obtaining the random movement postulated for spheres not in contact, we have found that to a large extent substantial numbers of these spheres move in consort. For example, these spheres tend to move upward in planes and downward

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\*E.J. Ryan, J.W. Smith, and R.L. Hummel, "Dynamically Preferred Particle Orientations in Liquid Fluidized Beds," submitted to International Symposium on Fluidization, Eindhoven, 1967.

in vertical streamers, and the planes will tend to rotate first clockwise and then counterclockwise in the tube. Another more unexpected observation, resulting from the fact that our optical system was similar to that for a Tyndall beam (which is very sensitive to small suspended material), was that very small colloidal suspended material was separated into regions of high and low concentration on passing through the fluidized bed. That is, as tracers they did not truly follow the streamlines. This observation has led to the thought that inherent in the absence of density differences larger particles of micron or submicron size simply cannot follow the consorted movement of molecules precisely, and that furthermore if such larger entities are present in appreciable concentration they would tend to stabilize flow patterns such as eddies or vortices. Even in the absence of elastic behaviour larger entities must move as a unit from, for example, one eddy to a new eddy, while with eddies in collision, the fluid itself can gradually be sheared off into, for example, a new smaller eddy. A macromolecule or larger entity would tend to be carried with the fluid unit in which most of its mass resides, and any remnant in an opposing flow would be pulled at least to the interface between the two. The shearing down of an eddy in size would become more difficult, although loss of momentum would not necessarily be prevented.

With regard to the raggedness, Prof. Metzner in the previous paper has already suggested that the probe will have a thickened stabilized boundary layer at the leading edge as well as further along the probe due to the viscoelastic behaviour of the fluid and that this layer is sensitive to factors other than strictly the fluid velocity. It is also known for Newtonian fluids that a towed grid such as used in this study will create not only general turbulence but also pronounced vortices near the grid bars and particularly the crossings. These normally lose their identity or die out a short distance behind the grid, but in a solution of macromolecules these vortices might retain their identity for a considerably longer distance behind the grid. The periodic interaction of these vortices with the probe boundary layer would contribute to the raggedness observed.

\* \* \*

## REPLY TO DISCUSSION

A. G. Fabula

Dr. Gadd's suggestion that the hypothesized macromolecular entanglements might be absent in very weak concentrations seems reasonable in view of his experimental evidence. In the grid turbulence tests of Polyox WSR-301 (presumably the same type as in the experiments reported by Dr. Gadd) the minimum concentration was 18 ppm. Only much higher concentrations were of major interest, so that the critical concentration range of about 10 and 20 ppm suggested by Dr. Gadd's evidence was not checked in this work. It is not clear, however, that only concentration determines whether or not entanglements occur. It seems notable that 10 to 20 ppm is very much lower than the concentration for

incipient overlap of the macromolecular coils in the solution at rest, since approximate theoretical calculations give that value as roughly a few hundred ppm. Thus, if entanglements are formed at much lower concentrations, the relative motion of the coils and their deformation due to flow may also be involved.

In connection with Dr. Gadd's suggestion that small eddy suppression is perhaps irrelevant to turbulent drag reduction, Tu and Willmarth remark\* that "small eddies near the wall may become larger when they distend into the more central region." Thus, perhaps the small-eddy suppression phenomenon in free jets discovered by Dr. Gadd is relevant but depends on flow parameters as well as concentration.

Professor Hummel's note of the probable importance of the high polarity of water molecules to entanglement is very important, and it may be desirable to use other solvents for that or other reasons. In some work under Professor J. L. Zakin at the University of Missouri, Rolla, wedge hot-film velocity sensors in high-polymer organic solvent solutions have not shown the abnormal calibration behavior found in Polyox aqueous solutions in this work.

Professor Hummel's suggestion that macromolecules could act to stabilize vortices is intriguing, since such a mechanism could be important to the turbulent friction reduction effect by reducing turbulent energy production.<sup>†</sup> However, his suggestion that such stabilized vortices could contribute to the signal raggedness in the grid turbulence experiment does not seem to be supported by the observations. One can predict from vortex shedding observations the order of magnitude of the expected spacing for those vortices, and one can detect the major vortices as individual dimples on the water surface behind the grid. In both ways a spacing of the order of several centimeters is obtained for these tests. If the signal raggedness were to be associated mainly with such vortices, the raggedness would be expected to come in random bursts with an average spacing corresponding to several centimeters of sensor travel. Since the sensor travel during a typical sweep in the waveform photographs is about 3.6 cm, only one or two bursts would be expected per sweep. In fact, however, the raggedness is quite uniformly distributed over most sweeps. Thus, there seems to be no discernible evidence of the suggested behavior in this work.

\* \* \*

\*B.-J. Tu and W.W. Willmarth, "An Experimental Study of the Structure of Turbulence Near the Wall Through Correlation Measurements in a Thick Turbulent Boundary Layer," University of Michigan, Dept. of Aerospace Eng. Tech. Rep. 02930-T, 1966.

<sup>†</sup>A.G. Fabula, J.L. Lumley, and W.D. Taylor, "Some Interpretations of the Toms Effect," in "Modern Developments in the Mechanics of Continua," Academic Press, New York, 1966.



Thursday, September 29, 1966

Afternoon Session

## PHYSICS OF FLUIDS

Chairman: A. Silverleaf

National Physical Laboratory  
Feltham, Middlesex, England

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# CAVITATION, TENSILE STRENGTH, AND THE SURFACE FILMS OF GAS NUCLEI

Leslie H. Bernd

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Surfaces were sought that did not promote cavitation when in contact with water. These surfaces were used to encapsulate sonar transducers. With these transducers, appreciable tensile strengths were found to exist in water; i.e., power levels higher than normal were obtained without cavitation.

The tensile strength of water is set by the gas nuclei present in the water. Using encapsulated transducers, the behavior of gas nuclei was followed by measuring tensile strength. Surface films were found to form about the nuclei, retarding the acquisition of tensile strength. To acquire an understanding of these surface films, various waters were tested. Also, surface films were created about gas nuclei from solutions of hydrocarbons and proteins in water.

## SUMMARY

This paper summarizes some of the results of experimental investigations into the basic factors that determine the tensile strength of water. The stress range covered was 0-100 psi. These factors in effect determine the "cavitation properties" of water, since they can set the level of inception of cavitation, and to a certain extent affect its intensity and violence.

The underlying purpose was to learn how to achieve two opposite applications; the prevention of tensile strength where considered undesirable, and the obtaining of tensile strength where useful. In addition, the action of gas nuclei and other factors related to tensile strength are in many ways something of a mystery. Knowledge of these factors should be important to our basic understanding of cavitation phenomena.

For instance tensile strength has occasionally been noted in water tunnels (1,2). The inception of cavitation has occurred at higher stress levels than customarily expected. Higher flow velocities or lower tunnel pressures than normal have been needed to produce cavitation about a body. The tensile strength acts as if an additional static head were present. In this case, tensile strength is undesirable in order to make for a uniformity of test results, and duplicate "prototype" conditions.

Alternately, the creation or utilization of tensile strength is a desirable goal in order to improve the performance of devices normally limited in maximum output by cavitation, such as ship sonar, pumps, turbines, and propellers. The gains, the prevention of cavitation damage, obtaining a greater output from a given size or weight of equipment, i.e., an increase in return for a given economic investment, are tempting.

Hence, a specific objective was to learn how to operate sonar transducers at power levels above the commonly accepted limit at which cavitation occurs, but without cavitation.

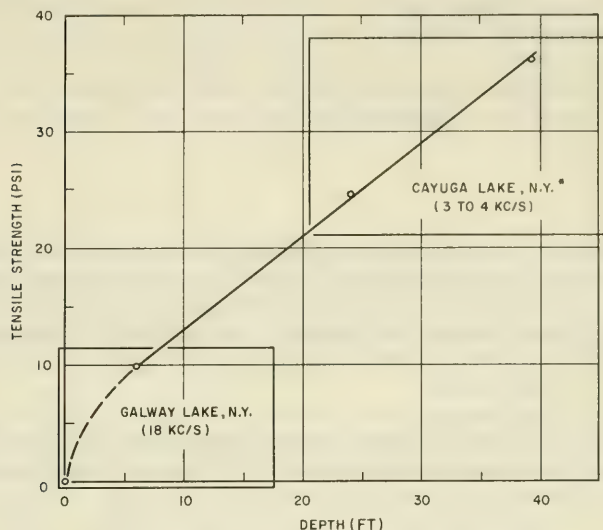
In ordinary water, the maximum tensile strength attainable should be limited by the strength of the weakest link in the system consisting of water, dissolved materials, undissolved gas, solids, and possibly other liquids that may comprise the water and its immediate environs. The weakest links found to be important in the 0 to 100 psi stress range were small gas bubbles (gas nuclei) in the water, gas bubbles on the surface of a solid, gas nucleation sites in cracks or pores in the surface of the solid, and the strength of the molecular bond between water and solid.

Various solid materials were therefore categorized and tested to determine the minimum tensile strength at which cavitation would ensue as a consequence of the conditions at the water/solid interface. "High strength" materials were selected so that the water/solid interface would not be the weakest link in the system. Sonar transducers covered with these high strength materials were then used to stress water to the point of rupture, i.e., the inception of cavitation.

The tensile strength of the water — once weak water/solid interfaces were removed — was limited by the presence of gas nuclei in the water. Gas nuclei are small bubbles containing air. When cavitating, gas nuclei are the end products of cavitation voids produced in a low pressure area. Waves and splashing at the surface of water ingest small air bubbles to produce nuclei.

In a body of water, the tensile strength should increase with depth. This is due to several actions. Generally speaking, the tensile strength of the water is inversely proportional to the maximum size of gas nuclei present. Large bubbles that prevent tensile strength from occurring rise rapidly to the surface; small bubbles rise slowly and so remain behind. However, small bubbles tend to dissolve. A high ratio of interface surface to gas volume, and internal pressure created within the bubble by surface tension, favor the dissolving of small bubbles. In addition, dissolving is promoted by increasing depth. Hence bubble size decreases with increasing depth because of the relative rates of rise, and because of dissolving.

Tensile strength tests were made to see if this picture were so. Figure 1 gives the tensile strength obtained versus depth in two fresh water lakes, using the specially constructed sonar transducers to stress the water. The tensile strength did indeed increase with depth. A 6.7-db increase in power level above the "normal" inception of cavitation was obtained at a 39.4-foot depth before cavitation took place. Thus a 36.1-psi tensile strength was obtained at a 39.4-foot depth, and the transducer cavitated as if it were at a 123.0-foot depth



\* THESE TESTS WERE PERFORMED BY TED MADISON, NOW WITH INTERNATIONAL TRANSDUCER CORP., SANTA BARBARA, CALIFORNIA.

Fig. 1 - Tensile strength of water at different depths in two fresh water lakes

instead of 39.4 feet. Thus usable tensile strength was found to be available in a natural water. These tests were done during the summer.

Nuclei should dissolve ideally within seconds to a few minutes but, as shown by tests, often do not. This retards the acquisition of tensile strength, and limits the amount of tensile strength available in the water. This phenomenon is a consequence of surface films that gather about the nuclei from small amounts of organic materials dissolved in the water. If surface films did not exist, much higher tensile strengths would be found.

Various waters tested varied widely in their tendency to form these surface films, as evidenced by variations of more than 100:1 in the dissolving rate of the nuclei contained in the water. Waters that did not produce surface films dissolved nuclei quickly. For example, water from the David Taylor Model Basin's test tunnels acquired tensile strength quickly after being cavitiated. Here, one would attribute the lack of surface films to the fact that chlorinated water is used in the tunnels. In comparison, the "prototype" sea water and lake water tested were approximately an order of magnitude slower in dissolving. When heavy organic growths were present in water, much slower dissolving — or none at all — resulted.

Surface films were deliberately produced about nuclei by dissolving small quantities of hydrocarbons or proteins in water. It was found possible to obtain



faster or slower dissolving than the "prototype" norm, and largely duplicate the action of various natural waters.

The body of this paper reports upon the basic investigations of surfaces, water, and surface films of gas nuclei.

## CONCLUSIONS

It was found that usable tensile strength can exist in water in nature; laboratory conditions and extreme purity of water are not prerequisites. However, this tensile strength may not be realized because of the nature of the solid materials in contact with the water. Also the maximum tensile strength obtainable may be limited by surface films retarding or preventing the dissolving of gas nuclei entrained in the water.

On the other hand, fast dissolving water with little film formation can also exist, and should be a partial explanation of the occurrence of tensile strength in water tunnels.

Surface films about gas nuclei are produced by trace organic materials possessing solubilities in the order of 100 to 10 parts per million. These materials are surfactant; i.e., they are able to migrate to the surface of the nucleus and form a monolayer. Thus a low concentration in solution is brought to a high concentration at the surface. A monolayer is a layer one molecule thick. The monolayer may be surrounded by additional structure-building materials.

Continued cavitation can remove film-forming materials from action. Unless they are replenished, a slow dissolving water will ultimately become a fast dissolving water.

As a result of these investigations, it was found possible to achieve high or negligible tensile strength at will in a small test system. This required the use of suitable solid materials in contact with the water and control of the formation of surface films about gas nuclei. One can therefore foresee the possibilities of the use of such techniques in water tunnel operation or in obtaining higher outputs from devices normally limited by cavitation.

## WATER/SOLID INTERFACE

A container, pipe, or some form of solid surface is usually to be found in contact with water. Solids are often present as debris. Therefore the inception of cavitation was measured for different solid materials immersed in "strong" water possessing tensile strength. Thus, the water/solid interface was made the weak link under test. Materials were cemented to the face of a glass-housed acoustic transducer used to produce cavitation. Glass is a "high strength," noncavitating material. Rough screening tests were also performed, using supersaturated carbon dioxide water solutions in contact with the material to stress the water/solid bond.

Gas on the surface of solid materials in contact with water was found to be responsible for cavitation at very low stress levels. In fact, most solid materials were unable to achieve tensile strength with water unless air was first removed from the surface of the material. Poor molecular bonding started to be important in the 40 to 50 psi tensile stress range.

## Gas

If only a few air bubbles were present on a surface in contact with the water, cavitation was brief at high stress levels, ceasing as soon as the bubbles were hurled away from the surface of the material by the violence of the cavitation action. Operation at high stress levels thus provided a ready means of removing gas from a surface to make it suitable for noncavitating duty.

On the other hand, numerous nucleation sites may exist on a surface, seeding cavitation for extended periods of time. Nucleation sites are usually small pores or surface imperfections a few microns in size containing gas (17). Depending upon the characteristics of the material at hand, prolonged soaking may often (but not always) dissolve gas to remove sites from action.

Figure 2a shows the effect of a surface containing many nucleation sites on the severity of cavitation. Both the water and water/solid interface were weak, so that each could cause cavitation independently. The severity of the cavitation was measured by the harmonics picked up in the water by a hydrophone. The fundamental frequency of the signal fed the transducer was filtered out from the hydrophone output. Increasingly violent cavitation increases the amplitude and frequency range of the harmonics produced (25). As can be seen, the use of a high strength surface that limited the source of cavitation to the water greatly reduced the severity of the cavitation.

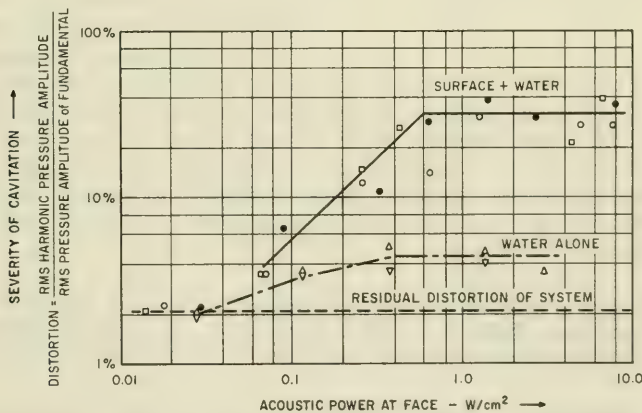


Fig. 2a - Effect of a solid surface possessing nucleation sites on the severity of cavitation

Certain types of nucleation sites were shown to be troublesome and so were avoided in sonar transducer construction. These were:

1. Fine cracks and joints.
2. Pores created in metals by electric arc welding.
3. Pores in ceramics.
4. Nucleation sites in permeable materials such as rubberlike polymers. Appreciable amounts of air may be contained within such materials. The air diffuses through the material to reach the nucleation sites at the surface, keeping them in action almost indefinitely.

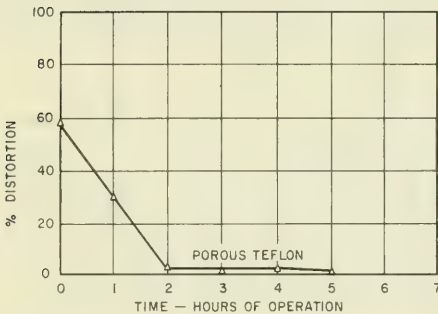


Fig. 2b - Cavitation caused by a porous Teflon surface

It was found feasible to determine the suitability of a polymer for non-cavitating duty by counting the number of nucleation sites and measuring the amount of gas contained in the material. If only a small amount of gas was in the material, and if it possessed only a few nucleation sites, the resultant cavitation was not severe. The air feeding the sites was soon exhausted, stopping the cavitation without noticeable damage within a minute or less. Figure 2b shows this process taking place in a porous Teflon surface not considered usable. Here the duration of cavitation was several hours.

An annoying source of trouble was the small bubbles that adhered tenaciously to the surface of materials that were rough and possessed wetting

angles larger than  $45^\circ$ . (The wetting angle is the angle of the gas/liquid interface of the bubble to the surface of the solid, measured through the liquid as shown in Fig. 3.) Many bubbles would adhere to the surface when first submerged in water. These materials, even when wet, also picked up bubbles from the water. Such bubbles may take hours to dissolve. Smooth, low wetting angle materials were therefore found desirable in order to achieve high tensile strengths without difficulty.

A rating of a few materials for noncavitating duty in the light of these experiences is given in Table 1. Galvanic action is included as being undesirable; low carbon steel was found unable to achieve high tensile strength because of it.

Glass in the form of Pyrex beakers was used to face the acoustic transducers used in the laboratory. The transducers (18 kc/s) were small, and so could readily be encapsulated in a glass beaker. Wet, clean glass was outstanding in being nonporous, without nucleation sites, while possessing a wetting angle approaching zero. Polyurethane was used for large prototype sonar



# Cavitation, Tensile Strength, and the Surface Films of Gas Nuclei

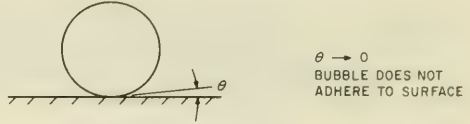


Fig. 3 - Bubble adherence to a surface

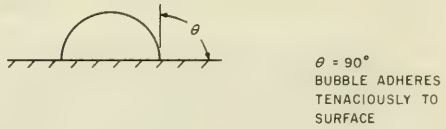
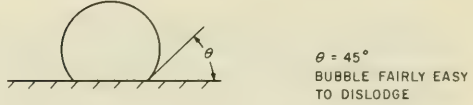


Table 1  
Rating of Materials for Noncavitating Duty

| Ranking    | Type of Material or Construction | Ratings* of Requisite Characteristics |                                           |                                       |                       |                             |
|------------|----------------------------------|---------------------------------------|-------------------------------------------|---------------------------------------|-----------------------|-----------------------------|
|            |                                  | High Bond Strength to Water           | Lacks Internal Gas to Nucleate Cavitation | Lacks Retention of Bubbles on Surface | Lacks Galvanic Action | Cavitation Damage-Resistant |
| Usable     | Glass                            | 3                                     | 3                                         | 3                                     | 3                     | 1                           |
|            | Polyurethane                     | 3                                     | 2                                         | 2                                     | 3                     | 3                           |
|            | Stainless steel                  | 3                                     | 3                                         | 1                                     | 3                     | 3                           |
| Not Usable | Carbon steel                     | 0                                     | 3                                         | 1                                     | 0                     | 2                           |
|            | Ceramics                         | 3                                     | 0                                         | 2                                     | 3                     | 0                           |
|            | pc Rubber†                       | 1                                     | 0                                         | 0                                     | 3                     | 2                           |
|            | Hydrocarbons (paints, rubbers)   | 1                                     | -                                         | 0                                     | 3                     | -                           |
| Not Usable | Cracks, joints                   | -                                     | 0                                         | 0                                     | -                     | -                           |
|            | Arc welds                        | -                                     | 0                                         | 1                                     | -                     | 1                           |

\*Excellent - 3; good - 2; fair - 1; bad - 0.

†pc rubber is a general designation for rubberlike materials possessing the same acoustic properties as water.



transducers (3 to 4 kc/s) because it could be cast in large sizes, contained few nucleations sites and little gas, and stood up well under cavitation. Stainless steel, if smooth and degreased, was also satisfactory. All of these materials have high bond strength to water.

### Bond Strength Between Water and Solid

Solid materials were treated to remove gas from the surface. Physical surface contaminants foreign to the material (oil, grease, dirt) were removed. Thus the weak link under observation was the bond between water and solid. Materials were selected over as wide a spectrum as possible. Correlations were sought between known physical properties and bond strength so that entire general categories of materials could be classified as being "cavitating" or "noncavitating" on the basis of a reasonable number of tests.

It was assumed that the tensile strength of the bond between water and solid was obtained when increasingly severe degassing treatments did not raise the tensile strength any further. An initial treatment was soaking in ethyl alcohol or water and then if necessary subjecting the solid immersed in these liquids to higher and higher pressures. This forced gas to dissolve. Assuming adverse conditions of geometry and wetting angle, the pore size filled in overcoming the surface tension of water in this way is approximately as given in Table 2. Little change was noticed above 100 psi, implying that pore sizes much less than a micron were not present as a rule.

Table 2  
Pore Size Filled in a Solid Immersed at Pressure

| Pressure Applied (psi) | Pore Size               |                           |
|------------------------|-------------------------|---------------------------|
|                        | Microns ( $10^{-4}$ cm) | Angstroms ( $10^{-8}$ cm) |
| 100                    | 0.42                    | —                         |
| 5,000                  | 0.021                   | —                         |
| 100,000                | 0.00042                 | 4.2                       |

In the supersaturated gas tests used for rapid screening, dissolved carbon dioxide gas in the water was used to stress the bond until rupture occurred, as evidenced by gas evolution. These tests placed the tensile strength of the bond into groupings; i.e., 0-24, 24-48, 48-72, 72-96, 96-120, 120-135, or greater than 135 psi. The material under test was submerged in water, and the water was pressurized with carbon dioxide until saturated. This application of pressure also dissolved free air. The pressure was then decreased in uniform increments until gas evolved from the surface of the material. The duration of the test was kept limited in time so that negligible amounts of gas would escape by diffusion through the gas/liquid interface at the surface of the water, thereby keeping the stress (degree of supersaturation) constant for each pressure increment.

The ability of a material to adhere to water should be determined by (a) the strength of its molecular forces, (b) the fraction of these forces available at a surface, and (c) the fraction of the surface forces that are actually able to provide a bond with the water. One might liken the situation to a riveted joint between metals. The strength of the joint is proportional to the basic strength of the metal times a joint efficiency expressing the fraction of material that is used effectively.

The relative strength of molecular forces can be obtained in various ways, for instance by comparing boiling points, melting points, heat of evaporation, modulus of elasticity, etc. A readily available measure of the forces available at a surface is surface tension or surface energy.\* In similar types of materials, one would expect an increasing adhesion to water as the numerical values of these properties increase.

One can roughly categorize materials by their type of atomic bond: covalent or ionic. Covalent materials do not bond as well as ionic materials. Langmuir showed this by a comparison of the heat of evaporation of alcohols possessing different ratios of covalent (hydrocarbon) groups to ionic (OH) groups in the alcohol molecule (3). Many materials are not strictly covalent or ionic but a simultaneous combination of the two. Water is highly ionic (polar), and bonds to adjacent molecules via hydrogen bonds. Thus there should be an increasing adhesion to water as a material contains more ionic groups, or if the groups themselves are more highly ionic.

One would therefore expect the poorest adhesion to water to result with low-surface-energy, covalent compounds. If the tensile strength of such a poor bond were appreciable in magnitude, then tensile strength with water as a general case should be readily obtainable, since all other materials would be likely to exhibit higher tensile strengths.

Teflon is the lowest-surface-energy material known (6 ergs/cm<sup>2</sup>). Common with other low-surface-energy materials, its surface is slippery and difficult to cement or bond to. A common group of low-surface-energy, covalent materials are waxes, paraffins, and similar hydrocarbons. Table 3 shows that the lowest tensile strength obtained with covalent, low-surface-energy materials is about 40 psi. *This is therefore the minimum tensile strength to be expected irrespective of the materials used.*

Tests on quite a few materials substantiated this. To give some examples, Table 4 lists the tensile strengths obtained with metallic oxides. Higher tensile strengths were generally obtained with increasing molecular forces. Table 5 lists tensile strengths obtained with various polymers. Increasing tensile strength resulted as covalent compounds contained polar groups.

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\*Surface energy (by definition) is the energy expended to create ideally a surface where none existed before. For liquids, a related property is surface tension. Water has a surface tension of 73 dynes/cm. This is numerically equal to the surface energy of water. That is, 73 ergs would be expended to create a square centimeter of water surface in contact with air, using water originating from the interior of the liquid where no surface exists.

Table 3  
Tensile Strength with Water of Low-Surface-Energy,  
Covalent Materials

| Material     | Tensile Strength     |                       | Comments                                             |
|--------------|----------------------|-----------------------|------------------------------------------------------|
|              | Screening Test (psi) | Transducer Test (psi) |                                                      |
| Teflon       | 24/48                | 40                    | Lowest known surface energy - 6 ergs/cm <sup>2</sup> |
| Paraffin     | 24/48                | 48                    | Surface energy - 20/30 ergs/cm <sup>2</sup>          |
| Polyethylene | 24/48                | —                     | —                                                    |
| Sulfur       | 24/48                | —                     | —                                                    |

Table 4  
Tensile Strength Between Water and Solid Oxides

| Material*                      | Tensile Strength Range (psi) |       |       |       |        |         |       | Melting Point (°C) | Surface Energy (ergs/cm <sup>2</sup> ) |
|--------------------------------|------------------------------|-------|-------|-------|--------|---------|-------|--------------------|----------------------------------------|
|                                | 0-24                         | 24-48 | 48-72 | 72-96 | 96-120 | 120-135 | > 135 |                    |                                        |
| PbO <sub>2</sub>               |                              |       |       | x     | x      |         |       | 290                | —                                      |
| Ag <sub>2</sub> O              |                              |       |       | x     |        |         |       | 300                | —                                      |
| PbO                            |                              |       |       | x     |        |         |       | 888                | 250                                    |
| CuO                            |                              |       |       |       | x      |         |       | 1026               | —                                      |
| Cu <sub>2</sub> O              |                              |       |       | x     |        |         |       | 1235               | —                                      |
| WO <sub>3</sub>                |                              |       |       |       |        |         | x     | 1473               | —                                      |
| Fe <sub>2</sub> O <sub>4</sub> |                              |       |       |       |        | x       |       | 1538               | —                                      |
| Fe <sub>2</sub> O <sub>3</sub> |                              |       |       |       |        | x       |       | 1565               | —                                      |
| SiO <sub>2</sub>               |                              |       |       |       |        |         | x     | 1728               | 600/316                                |
| MnO <sub>2</sub>               |                              |       |       |       | x      |         |       | 1780               | 1010                                   |
| TiO <sub>2</sub>               |                              |       |       |       |        |         | x     | 1840               | 520/300                                |
| SnO <sub>2</sub>               |                              |       |       |       |        | x       |       | 1900               | 680/272                                |
| NiO                            |                              |       |       |       |        |         | x     | 1950               | —                                      |
| ZnO                            |                              |       |       |       |        |         | x     | 1975               | 600                                    |
| Cr <sub>2</sub> O <sub>3</sub> |                              |       |       |       |        |         | x     | 1990               | —                                      |
| Al <sub>2</sub> O <sub>3</sub> |                              |       |       |       |        |         | x     | 2015               | —                                      |
| BeO                            |                              |       |       |       |        |         | x     | 2550               | 1420                                   |
| ZrO <sub>2</sub>               |                              |       |       |       |        |         | x     | 2677               | 800/600                                |
| MgO                            |                              |       |       |       |        | x       |       | 2800               | 1090                                   |

\*All materials were pressurized in alcohol at 5000 psi to remove gas.



Table 5  
Tensile Strength Between Water and Polymers

| Material                            | Polar Group             | Tensile Strength Range (psi) | Wetting Angle (deg) |
|-------------------------------------|-------------------------|------------------------------|---------------------|
| Teflon                              | None                    | 40                           | 95                  |
| Polyethylene                        | None                    | 24/48                        | 45/50               |
| Paraffin                            | None                    | 48                           | 100                 |
| Latex (NH <sub>4</sub> OH solution) | Ionic, proteins, sugars | 72/96                        | 30                  |
| Neoprene                            | Cl                      | 96/120                       | 40                  |
| Polyvinyl chloride (Saran Wrap)     | Cl                      | 96/120                       | 40                  |
| Epoxy                               | (OH)                    | 96/120                       | 60                  |
| Cellulose (cotton, linen, paper)    | (OH), O                 | 72/120                       | 0/30                |
| Polyvinyl alcohol                   | OH                      | 120/135                      | —                   |
| Gelatins                            | Amides, carboxyls       | 96/135                       | 0                   |
| Silk                                | Amides                  | 96/120                       | —                   |
| Rayon                               |                         | 135                          | 0                   |
| Cellulose nitrate                   | NO <sub>3</sub>         | —                            | —                   |
| Cellulose acetate                   | acetate                 | —                            | —                   |
| <u>Transducer Materials</u>         |                         |                              |                     |
| pc rubber                           | —                       | 48/72                        | 45/90               |
| polyurethane                        | —                       | 120/135                      | 30/45               |

Should tensile strengths greater than 40 psi be desired, not many types of materials need to be prohibited from use, since high-strength bonds (70 psi and greater) were obtained with many common materials, including metals, metallic oxides, ceramics, clays, and silicates. In practice, obtaining a high tensile strength would generally mean avoiding materials of a hydrocarbon nature such as rubbers, plastics and paints, should they be shown to possess a low tensile strength with water.

It is also to be noted that low tensile strength materials possess large wetting angles, and so are undesirable in any event because of their ability to retain bubbles on the surface of the material.

## STRENGTH OF SEA WATER

From the preceding discussion of solid materials, one can conclude that it is unlikely that solids causing low bond strength are to be found as natural contaminants in ocean or inland waters. Tests were also performed to show that dissolved salts and impurities did not weaken water enough to promote cavitation in the 100-psi range of interest, for concentrations likely to be encountered



in practice. Moreover, foreign materials in liquid form were found not to cause cavitation where a similar material in solid form did. Thus a hydrocarbon oil did not limit tensile strength, whereas a hydrocarbon solid surface did.

A few seaweed samples in sea water were tested. The weakest caused cavitation at a 16.2-psi tensile stress; others were appreciably higher. The cause of rupture appeared to be gas in pores on the surface of the seaweed. Thus it may be necessary to prevent marine growths from accumulating in order to maintain a "high-strength" surface.

In conclusion, high purity, or removal of organic or inorganic debris, were found not to be necessary for high tensile strength.\* It was found impossible to cavitate a sea-water sample in a glass container. The sample had previously stood long enough so that gas nuclei were not present to weaken the water.

## GAS NUCLEI AND SURFACE FILMS

Gas nuclei are generally invisible to the unaided eye. Because of this, their existence is largely an abstraction. Their presence is inferred from acoustic absorption measurements, or the increase in tensile strength obtained by applying pressure to dissolve the nuclei or by allowing water to stand so that nuclei rise to the surface. Investigations associated with these approaches are Harvey et al. (4), Knapp (5), Strassberg (6), and Iyengar and Richardson (7). In an investigation of nuclei aspect, one must be certain that nuclei are the sole variable under observation.

Because of their small size, and the internal pressure created as a consequence of surface tension, gas nuclei should dissolve within a few minutes in still water (8), but paradoxically they do not appear to do so. It has been suggested that this is due to organic surface films forming a shell about the gas nuclei (9), creating a barrier that prevents diffusion of the gas from the high-pressure area within the bubble into the surrounding water. To determine the extent to which dissolving might thus be retarded, different waters were cavitated to create gas nuclei; and the behavior of the nuclei was followed by measuring the tensile strength of the water.

### Tensile Strength

The tensile strength of water containing nuclei is

$$P = \frac{4\sigma}{D \times 69,000} ,$$

where  $\sigma$  is the effective surface tension (dynes/cm),  $D$  is the diameter of the nucleus (cm), and  $P$  is the tensile strength (psi). If no surface films are

---

\*The above tests were done at room temperature. No measurements were made close to 0°C, where Briggs found the inherent tensile strength of water to be 300 psi as contrasted to a maximum of about 4000 psi at 10°C (27). Briggs' measurements were made in a glass tube, presumably without nuclei present.

present at the interface between gas and water, the surface tension acting is that of water, 73 dynes/cm. The surface tension acts to pressurize the gas within the nucleus and also to oppose the growth of the nucleus when a tensile stress is applied to the water. When the tensile stress in the water is greater than the pressure within the nucleus, the nucleus expands and the water is ruptured to cause cavitation. The equilibrium stress for the incidence of rupture (versus nucleus size) is calculated to be as follows:

Table 6  
Limiting Tensile Strength of Water Due to Nuclei  
(Surface Films Absent)

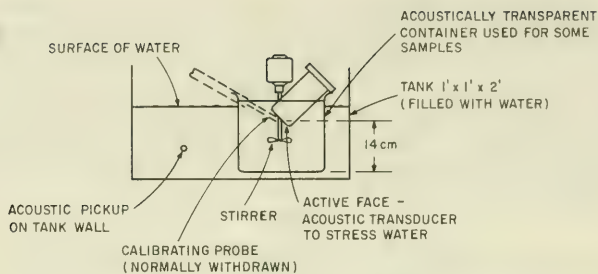
| Nucleus Diameter<br>( $\mu$ , or $10^{-4}$ cm) | Tensile Strength<br>(psi) |
|------------------------------------------------|---------------------------|
| 1000                                           | 0.0423                    |
| 100                                            | 0.423                     |
| 10                                             | 4.23                      |
| 1                                              | 42.3                      |

Although one cannot precisely relate tensile strength to nuclei size because the surface tension is not constant when a surface film is present, the important effect — the presence of tensile strength — can be measured directly.\*

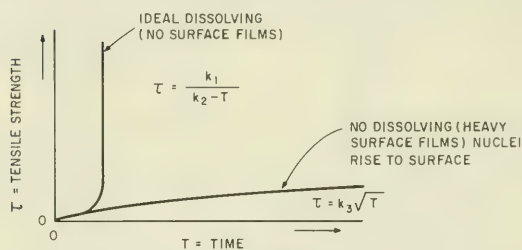
#### Test Technique for Observing the Effect of Surface Films

The test setup for observing the effect of surface films on gas nuclei dissolving is shown schematically in Fig. 4a. The test consists of cavitating water to create nuclei, then measuring the tensile strength after a given period of time has elapsed. To assure that the water under observation is the weak link, glass-housed acoustic transducers are used both for cavitating and subsequent stressing of the sample to rupture. The tank containing the water is also of glass.

\*Rectified diffusion (the growth of a bubble in a sound field because a larger surface area is present during the negative part of the pressure wave than during the positive part (26)) was not in evidence where one could compare bubble sizes determined by tensile strength measurements versus sizes obtained by rise in accordance with Stokes law. Rectified diffusion should tend to decrease the tensile strength of the water. The implication is that heavy surface films make nuclei behave as solid spheres when rising. Also, the surface of a nucleus when ruptured must have an effective surface tension equal to that of water. This would apparently present a picture of a surface film that is rigid and strong under compression but that can rupture readily under tension, leaving "ice floes" of material held together by the surface tension of water. This is not inconceivable from what is known of monomolecular surface films.



a. TEST SET-UP



b. IDEAL TEST RESULTS

Fig. 4 - Test for the effect of surface films on gas nuclei dissolving (tensile strength)

The nuclei produced are sorted out as to size by rising toward the open top of the test tank in accordance with Stokes' law. The maximum diameter of nuclei retained in the water is approximately 0.01 cm. Dissolving is produced essentially by the surface tension of the water creating pressure within the nuclei. The water is maintained at saturated equilibrium with dissolved air. The transducers are of such size and frequency (18 kc/s) that the acoustic intensity decreases rapidly with increasing distance from the active face of the transducer. Thus nuclei and water are stressed at maximum intensity very close to the face of the transducer at a given height in the water. This height is held constant.

If no surface films are present, the curve for "ideal dissolving" is followed, making for a rapid acquisition of tensile strength, as shown in Fig. 4b. If no dissolving takes place, the subsequent increase in tensile strength due to rising of nuclei to the surface is slow, following the "no dissolving" curve shown in Fig. 4b. Waters possessing different film effects fall between these two extremes.

#### Tests on Nuclei Dissolving in Different Waters

The first step in looking for surface film effects in water would be to establish test results for nuclei when no films are present, and thus be able to



conclude that surface films must be acting for all slower dissolving waters. Concentrated nitric acid was therefore tested to provide a standard for ideal dissolving, since it is inconceivable that the conditions required for surface film formation can exist. Nitric acid is highly oxidizing and should break down organic materials to simple molecules and ions, whereas relatively long, high-molecular-weight surfactant molecules are necessary for film formation. Corrections were made for differences in diffusion coefficients, viscosity, and surface tension where necessary.

Some results are given (Fig. 5) to show the range of nuclei dissolving encountered. Three general categories of water were identified, covering approximately a 100:1 variation in the time required to reach appreciable tensile strength. They were:

*Rapid tensile strength gain*, typified by the artificial case of no surface films obtained with nitric acid (5); approximating ideal dissolving by diffusion through the water surrounding a nucleus, as solved by Epstein and Plesset (8).<sup>\*</sup> Distilled water dissolved almost as fast when care was taken not to contaminate it. Waters of this type reached high tensile strengths (> 50 psi) in 6 to 7 minutes.

*A slower strength gain* for natural, "prototype" waters, typified by pure lake water (2). Distilled water contaminated by moderate exposure to air fell on this curve (1a). The sea water sample tested was somewhat similar. These waters continued to dissolve for 5 to 6 hours after cavitation, slowing down

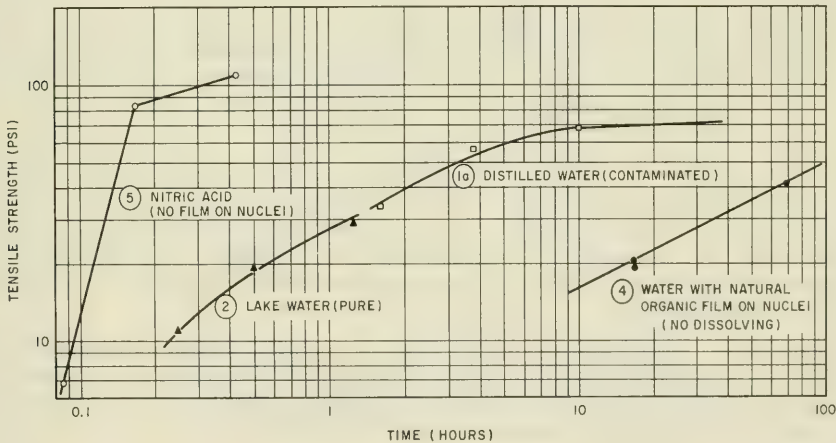


Fig. 5 - Summary of the trends of tensile strength after cavitation

\*However, at high tensile strengths close to 100 psi, dissolving did slow down somewhat. The cause of this was not investigated, since it occurred at high strengths beyond the range of major interest.



considerably after approximately a 1000:1 contraction in the surface area of the nuclei, as measured by the change in tensile strength, ultimately reaching a maximum tensile strength of about 45 to 60 psi, at which dissolving ceased. This gives one an idea of the maximum tensile strength likely to be achieved by these waters in practice, since the attendant tensile strength gain due to the rise of nuclei to the surface is little.

*No dissolving*, typified by an extremely slow tensile strength gain. This occurred as a consequence of natural films that had been produced by algae allowed to grow in tap water (4).

These tests were done in still water; turbulent flow should considerably increase the rate of dissolving.

### Nuclei Dissolving in Water from a Water Tunnel (David Taylor Model Basin\*)

Now, having an appreciation of the effects of surface films, it was desirable to determine the behavior of water taken from a cavitation test facility. This was important to evaluate the first part of the following argument: if fast dissolving waters exist, and if in addition small nuclei are produced during cavitation, then tensile strength might be likely to occur in the test section of a water tunnel, provided sufficient time occurred during transit of the water in the tunnel to appreciably dissolve nuclei.

Water Tunnels - The 36-in.-tunnel† water, the 24-in.-tunnel water, and drinking water from the David Taylor Model Basin were tested, and were found to be identical in surface film behavior (Fig. 6). There was a rapid, high gain of tensile strength after cavitation, much higher in fact than any natural, "prototype" water that had been tested. This is directly opposite to what one would prefer to see: little or no tensile strength gain. The rate of dissolving was similar to distilled water (Fig. 7).

The water supply for the David Taylor Model Basin is principally Potomac River water that has been treated. Chlorination is one of the treatments used. The implication is that chlorination has destroyed the organic contaminants that would normally be responsible for surface film formation. The tunnels are closed systems using well water for makeup, so that there is little likelihood of subsequent contamination.

A peculiarity of the test results is that the tensile strength versus time data randomly fell on two curves. It was as if two types of surface films were capable of being formed, but the start of one — a random event — excluded the other.‡ Both of the curves obtained represent rapid dissolving.

\*After March 31, 1967, renamed Naval Ship Research and Development Center.

†Throat size of nozzle in test section. See Ref. 10 for a description of the David Taylor Model Basin tunnels and the towing basins.

‡This effect would not be noticeable at a low tensile strength since many nuclei are present. Water possessing appreciable tensile strength generally has only a few nuclei of a given strength in the volume of water (approximately 1 cc) stressed to rupture.

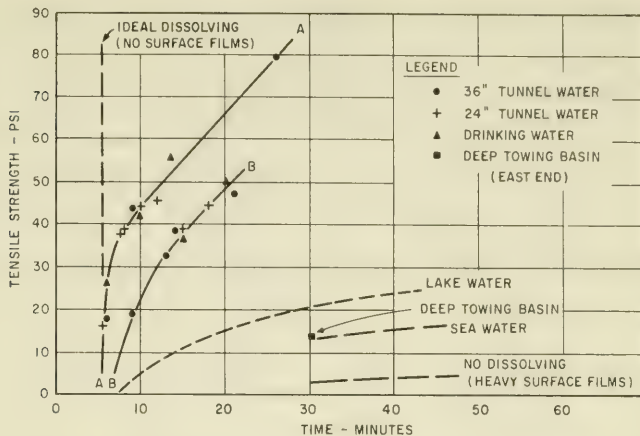


Fig. 6 - Tensile strength of David Taylor Model Basin tunnel water after cavitation

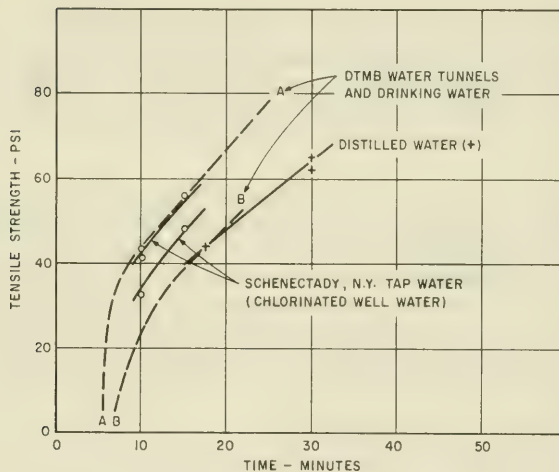


Fig. 7 - Comparison of the tensile strength of David Taylor Model Basin water and other purified waters

**Towing Basin** — The Deep Water Towing Basin (east end) of the David Taylor Model Basin had considerable film formation (Figs. 8 and 6). This water in this sense was similar to natural waters, i.e., water obtained from the sea or lake. Roughly speaking, the water is an order of magnitude slower in acquiring tensile strength by dissolving than the tunnel or drinking water. One must conclude that exposure of a large surface of water in the basin to contamination from the air,

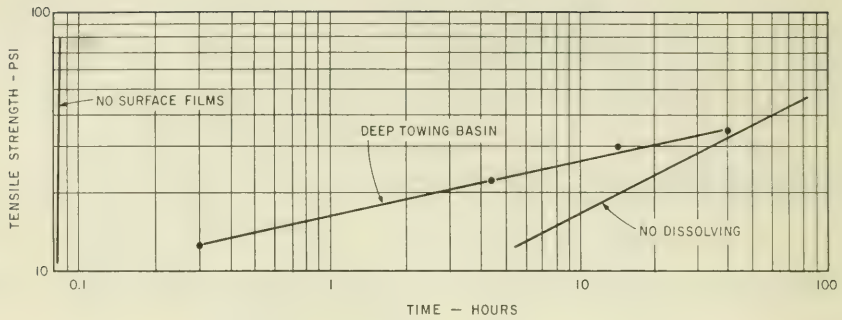


Fig. 8 - Tensile strength of the Deep Towing Basin water of the David Taylor Model Basin

along with the existence of plant and animal life in the water, have resulted in surface films that delay dissolving.

**Extensive Cavitation Removes Film-Forming Materials** — It is of considerable interest to note that extended cavitation of the Deep Water Towing Basin water made it similar to the tunnel water and drinking water (Fig. 9). A similar effect was obtained when using a protein surface film (egg albumin) to retard dissolving (Fig. 10). That is, cavitation removed the materials causing surface films.

This substantiates the expectation that protein films are largely responsible for surface film formation. A protetin film becomes insoluble when compressed, or agitated mechanically (11-13). The action is not reversible. Hence, a protein by becoming insoluble cannot form a film from solution, and so is gradually removed from action. *Thus, unless the dissolved materials making surface films are replenished, water that is normally slow dissolving should eventually change its characteristics to become a fast dissolver.*

#### PRODUCTION OF SURFACE FILMS ABOUT GAS NUCLEI

It is rather difficult to analyze trace materials if structural details of the molecules are desired. Also nuclei are too small to be observed readily in their habitat. Hence an understanding of surface film action can most readily be achieved if known materials are used to form surface films and the gross results are observed. Furthermore, the type of materials necessary for film formation can be fairly well delineated in advance by considering the essential properties required to obtain the characteristics nuclei films must possess. In addition, this approach leads to acquiring a means of controlling surface film effects by using additives in water to produce the surface film desired.

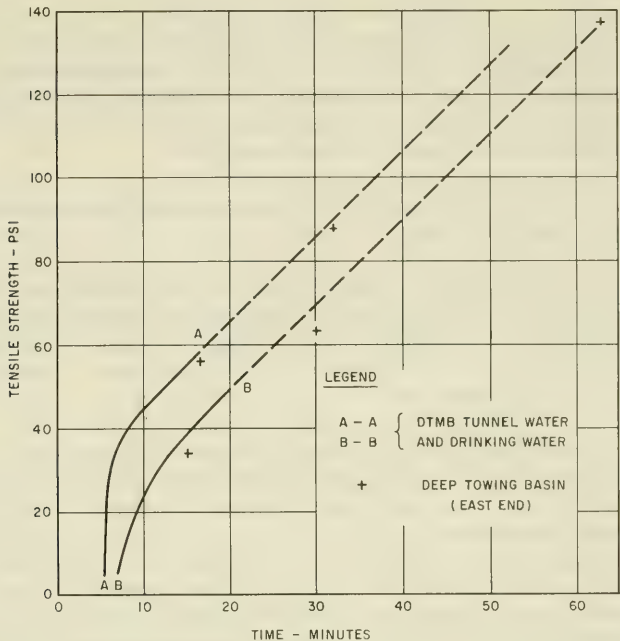


Fig. 9 - Tensile strength of the Deep Towing Basin water after extended cavitation

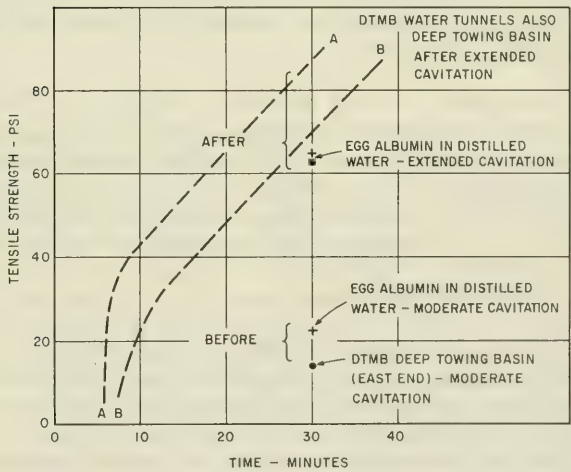


Fig. 10 - Effect of cavitation on removing film-forming materials



## Characteristics of Surface Films

One can conclude that certain types of hydrocarbons and proteins should be suitable for forming effective surface films about gas nuclei, were it not for two important uncertainties: Can a hydrocarbon dissolve and form an adequate film from solution? Can a protein film prevent dissolving? These gaps in the picture were subsequently investigated experimentally.

It seems reasonable that a surface film about a nucleus can oppose dissolving in various ways, i.e., by:

1. *Reducing surface tension.* Surface tension is reduced as soon as certain materials reach the nuclei surface because of the lower inherent "surface energy" of the material. Water has a high surface tension of 73 dynes/cm. An oil, hydrocarbon, or protein film — when completely formed — reduces surface tension to about 20 to 30 dynes/cm.
2. *Acting as a barrier to gas diffusion.* Certain hydrocarbon, monomolecular films are known to be relatively impermeable and act as a barrier to diffusion. Thus, cetyl alcohol has been used with success to retard evaporation from water storage reservoirs.
3. *Structurally opposing surface tension with a solidlike film.* Certain surface films when compressed acquire characteristics similar to solids. The film possesses considerable strength in compression, is elastic, and progressively opposes surface tension as the surface is contracted. Thus Fox and Herzfeld put forth the premise that fatty acid films could form a rigid shell about a nucleus, and that the life of the nucleus would end when the film was crushed in compression (9).

In addition to these primary actions, certain secondary characteristics should also be important. They are:

1. *Solubility.* The material must dissolve in water in at least trace quantities so that it is available to form the surface film. Materials normally considered relatively insoluble can form surface films. For instance, egg albumin (a protein) forms excellent surface films when coming out of solution; its solubility is somewhat less than 1 part in 1000. For a given type of chain molecule, solubility decreases with increasing molecular weight. One therefore expects to find a maximum molecular weight beyond which a material cannot dissolve in adequate quantities to supply a surface film.
2. *Being Surfactant.* The material must be "surfactant." That is, it must be capable of migrating to the surface of the water and forming a film. By this process, an extremely low concentration of material in solution is brought to a high concentration at the surface; i.e., a little has a large effect. For a chain hydrocarbon, this sets the requirement that the molecules comprising the film be at least about 12 atoms long, and that each end of the molecule be different. One end must be highly attractive to water, i.e., polar, such as an OH or carboxyl group (COOH). The other end must be less attractive, such as a hydrocarbon group. Being surfactant thus sets the general type of molecule and

specifies a minimum molecular weight. Here again, there should also be a limit to the maximum molecular weight suitable. The ability of a molecule to diffuse and be transported to the surface decreases as the molecule weight increases. In addition, the molecule may become too cumbersome to align readily at the surface. Thus, for a given type of molecule, there should be a maximum molecular weight beyond which film formation cannot take place readily.

3. *Mechanical Properties.* It would seem desirable that a surface film be strong and highly elastic, so that the film in its early stages can follow the motions of a cavitation void or bubble surface without being destroyed. Also, strength would be desirable to prevent the crushing postulated by Fox and Herzfeld as the end of the life of a nucleus. Thus, different mechanical properties should affect nuclei performance. One would expect a thicker hydrocarbon film (higher molecular weight) to be stronger in compression and more rigid.

4. *Barrier to Gas Diffusion.* A thicker film offers a greater resistance to gas diffusion (16). Thus, for a given type of a hydrocarbon film, increasing the molecular weight decreases the permeability of the film.

In a hydrocarbon, most of the above properties are related to molecular weight. Thus finding an effective film should simplify to determining the effects of molecular weight on nuclei dissolving.

Hydrocarbons — Hydrocarbon films, if they could be formed about nuclei, appear suitable in many ways. That is, certain types of hydrocarbon materials are used to form monomolecular films on the surface of water which act as barriers to the diffusion of water vapor. They also reduce tension and produce solidlike films upon compression. These hydrocarbons are long chain linear alcohols, fatty acids, and amines (14,15). These materials are readily obtainable. Hence they would be of interest if found to be usable in a cavitation test. They would also provide a standard of known molecular composition to compare against a protein.

However, hydrocarbons are generally insoluble. As a consequence, hydrocarbon films are not spread from a water solution; other techniques are used. Nevertheless, nuclei can undergo a large surface contraction when dissolving. This should markedly increase the concentration of hydrocarbons at the surface even if ordinarily only present in nominal quantities.

Proteins — The available evidence does point to proteins as being important in the creation of surface films. That is, protein films meet the requirements of reducing surface tension and forming a solidlike film upon compression (12,13). Although the majority of proteins are insoluble in water, some proteins (egg albumin for instance (18,19)) are slightly soluble and do form monomolecular surface films by coming out of solution. These films form fairly rapidly, and are strong, elastic, and durable to a greater degree than a hydrocarbon film. Film possessing characteristics typical of proteins (as well as hydrocarbons) have been noted in streams. That is, films are formed that, upon compression, whiten, fold, and become insoluble (20). And, as previously mentioned, water capable of producing surface films ceased to do so after

extensive cavitation, thus implying that an insoluble protein film was being produced that subsequently could not redissolve to continue the film forming action. A hydrocarbon film upon compression behaves similarly, but should ultimately redissolve.

Opposed to this is that little information is available about the resistance of protein films to gas diffusion. It is in fact occasionally stated in the literature that proteins do not offer a barrier to diffusion. This belief must be overcome if one wishes to justify that protein films play a role in retarding dissolving.

A barrier to gas diffusion should be selected on the basis that it have few holes in it, and/or be thick. The molecules comprising the film should be tightly packed. If the surface film of a protein could be highly compressed, as may occur when a nucleus dissolves, examination of the molecular structure leads one to conclude that some proteins should have a resistance to diffusion of about the same order of magnitude as a hydrocarbon. A few tests in the literature on *compressed protein films* (on gases other than air) do show appreciable resistance to diffusion (21), tending to support this supposition.

Unfortunately, intimate details of molecular structure and composition are not well known for many proteins. Furthermore, the change in protein structure as it becomes insoluble reorients the molecule (12,13). This lack of knowledge poses a difficulty in understanding protein action about a nucleus, and selecting suitable proteins for forming a film.

### Structure of Surface Films

The general form of various linear, chain hydrocarbons capable of film formation, and known to offer a barrier to gas diffusion, are given in Fig. 11a. The molecules have no side branches, and so pack well together. The alcohol, fatty acid, and amine shown each have different polar ends, i.e., OH, COOH,  $\text{CH}_2\text{NH}_2$ . Otherwise the composition and form are similar. The length of the molecule is varied by changing the number of repeating hydrocarbon units. To give an idea

of the length we are concerned with, cetyl alcohol is 16 carbon units long. At a surface, these molecules align side by side to form a film as shown in Fig. 11b. In studies of the diffusion of water

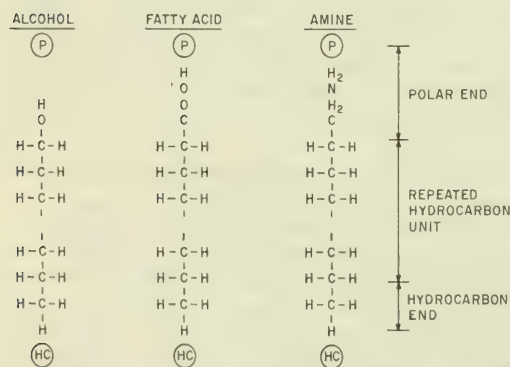


Fig. 11a - Structure of long-chain, surfactant hydrocarbons

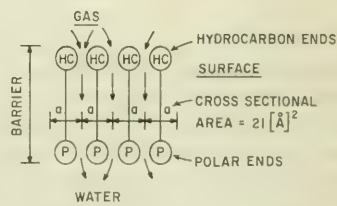


Fig. 11b - Monomolecular film at the surface of water



vapor through hydrocarbon surface films, it has been found that a compressed film is somewhat more resistant to the diffusion of water vapor than an uncompressed film (14). When the film is tightly packed by compression, the cross-sectional area occupied by each molecule is about  $21 \text{ \AA}^2$ .\*

The composition of a protein film is summarized by Adam (12). It is diagrammatically pictured in Fig. 12 as a plan view, looking down upon the surface of the water. The linkages forming the backbone of the structure are called peptide links. They bind in sequence the repeating CH, NH, CO, ... groups. Attached to this backbone are hydrocarbon chains from one to six repeating units long, or chains with a COOH or  $\text{NH}_2$  group at the end. Each chain is represented by the term R. The endless variety of proteins encountered in nature is due to the various different chains (R), and the different sequences and patterns in which these chains are attached to the polypeptide backbone (23). In an uncompressed form, the side chains R are stretched out on the surface of the water. The R chains are identified by examining the breakdown products of proteins. Several dozen of these amino acids are known (24).

If considerable compression of the protein were to occur, then the peptide links would tend to pack closely, as shown, with the R chains forced perpendicularly out of the surface. The significance of this is that the reoccurring groups in the backbone to which the R chains are attached would then be on as little as  $3.5 \times 4.5 \text{ \AA}$  spacings. This gives an area allotted to each R chain of about  $17 \text{ \AA}^2$ , somewhat less than the spacing of the chains in a compressed hydrocarbon film. However, the length of the R chains is less than the chain length for a hydrocarbon.

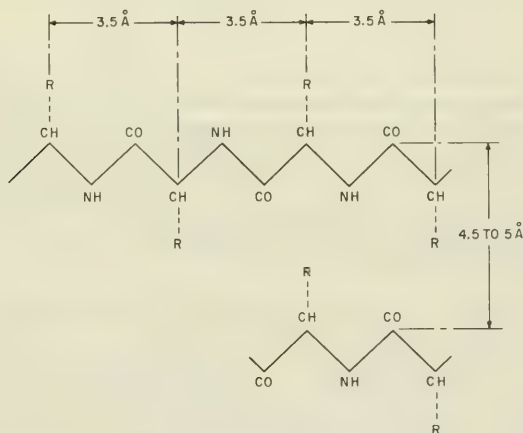


Fig. 12 - Plan view of the structure of a protein film

\*One angstrom unit ( $\text{\AA}$ ) equals  $10^{-8}$  centimeters.



Thus, a protein film is thinner than a hydrocarbon such as cetyl alcohol, but under favorable circumstances could have smaller spacings between adjacent molecular groups. Hence, because the basic dimensions of the two types of films can be "roughly" the same given the proper R chains in the protein, the resistance to diffusion under such conditions should be of approximately the same order of magnitude. It is conceivable that various mechanical properties might also be somewhat similar. The problem of course is to find proteins of the proper structure.

### Film Formation from Solution

The first experimental step was to see whether hydrocarbon films would form unaided from solution, as would have to take place in nature. Some information was available for proteins (19).

Saturated solutions were used to promote film formation. They were made by stirring an excess of the material at length in water, after using a wetting agent. Desired was the maximum molecular weight capable of forming a film within a few seconds rather than, as can be the case, minutes or hours. This would then ensure that a surface film is present (in the cavitation test) from the inception of a nucleus.

Testing for the existence of a surface film was done by placing the solutions in a Langmuir trough and highly compressing the surface to see if a film had been formed. The film was then swept off the surface of the water and the creation of a new film checked immediately thereafter. The Langmuir trough is a device for restraining a film between barriers, compressing the film to occupy a smaller surface area, and measuring the reaction of the film to compression.

Hydrocarbons — For the hydrocarbons, the molecular weights tested ranged from about the minimum expected to be surfactant to the maximum readily available through commercial channels for the linear alcohols, fatty acids, and amines. The test results are given in Table 7. The maximum molecular weights found capable of rapid film formation were: cetyl alcohol ( $\text{CH}_3(\text{CH}_2)_{15}\text{OH}$ ), lauric acid ( $\text{CH}_3(\text{CH}_2)_{13}\text{CH}_2\text{NH}_2$ ), and pentadecylamine ( $\text{CH}_3(\text{CH}_2)_{13}\text{CH}_2\text{NH}_2$ ). Each of these materials plus a higher molecular weight in the same series was subsequently used in the cavitation test.

Proteins — A similar procedure was followed for selecting proteins, four of which were used in the cavitation test. Egg albumin was chosen because more is known about its spreading than any other protein (18,19,22). Langmuir has studied the film formation of the enzymes pepsin and trypsin, which are protein-like in structure (19). Zein, a corn derivative, also formed films from solution.

Various gelatins and gelling agents were also tried with some success, as listed in Table 8. Gelatins are obtained from animal proteins and are composed mainly of amines and carboxyls, i.e., proteins or protein fragments. Polyhall is a synthetic polyacrylamide ( $\text{CH}_2\text{CHCOHN}_2$ ). Jaguar, a plant derivative, was included even though it is primarily a carbohydrate containing only 5% protein.

Table 7  
Surface Film Formation\* of Linear Hydrocarbons  
from Saturated Water Solutions

| No. of Carbon Atoms | Alcohol<br>$\text{CH}_3(\text{CH}_2)_n\text{OH}$ | Fatty Acid<br>$\text{CH}_3(\text{CH}_2)_n\text{COOH}$ | Amine<br>$\text{CH}_3(\text{CH}_2)_n\text{CH}_2\text{NH}_2$ |
|---------------------|--------------------------------------------------|-------------------------------------------------------|-------------------------------------------------------------|
| 12                  | —                                                | Lauric (G)                                            | —                                                           |
| 13                  | —                                                | —                                                     | —                                                           |
| 14                  | —                                                | Myristic (P)                                          | Tetradecylamine (G)                                         |
| 15                  | —                                                | —                                                     | Pentadecylamine (G)                                         |
| 16                  | Cetyl (G)                                        | Palmitic (P)                                          | —                                                           |
| 17                  | —                                                | —                                                     | Heptadecylamine (N)                                         |
| 18                  | Octadecanol (N)                                  | Stearic (P)                                           | —                                                           |
| 20                  | —                                                | Arachidic (P)                                         | —                                                           |

\*Code for film formation: good - G; fair - F; poor - P; none - N.

Table 8  
Surface Film Formation\* of Gelatins and Gelling Agents

| Gelatin                   | Source       | Concentration | Film |
|---------------------------|--------------|---------------|------|
| "U.S.P."                  | Fisher G-7   | 1%            | F    |
|                           |              | 0.1%          | F    |
| Jaguar 508                | Stein Hall   | 1%            | G    |
|                           |              | 0.5%          | P    |
|                           |              | 0.1%          | F    |
| Purified Calfskin         | Eastman 1099 | 0.1%          | N    |
| Purified Pigskin          | Eastman 5247 | 0.1%          | N    |
| "For Bacteriological Use" | Fisher G-8   | 0.1%          | N    |
| Polyhall 27               | Stein Hall   | 0.5%          | N    |

\*Code for film formation: good - G; fair - F; poor - P; none - N.

### Cavitation Tests: Nuclei Dissolving

The film-forming materials selected were used to create monomolecular surface films about nuclei. The nuclei were produced by cavitating saturated water solutions, and the retardation of nuclei dissolving obtained by measuring the tensile strength that ensued 30 minutes later. The data spread between the extremes of no dissolving and the rapid dissolving of the distilled water used to

make the solutions. The results are given in Table 9. Increasing tensile strength means less retardation of dissolving.

Materials 1 through 7 were "good" film-formers. The data for hydrocarbons is plotted in Fig. 13 against the number of carbon atoms in the molecule in order to show the effects of the length of the molecule. From the shape of the curve, one can conclude that solubility and/or transport to a nucleus controls and limits the behavior obtainable from a surface film. An optimum molecular weight for the least dissolving exists.

Table 9  
Effect of Monomolecular Films on Nuclei Dissolving  
(Tensile Strength 30 Minutes After Cavitation)

| Film-Forming Material                     | Tensile Strength (psi) |                                               |
|-------------------------------------------|------------------------|-----------------------------------------------|
|                                           | First Test             | After Additional<br>or Extended<br>Cavitation |
| DTMB Deep Towing Basin<br>(East End)      | 13.8                   | 63.0                                          |
| 1. Lauric acid                            | 8.82, 10.6             | 39.3                                          |
| 2. Gelatin Fisher G-7 - 1%                | 10.4                   | —                                             |
| 2%                                        | 12.0                   | —                                             |
| 3. Stearic acid                           | 17.0                   | 21.4                                          |
| 4. Pentadecylamine                        | 17.8                   | —                                             |
| 5. Egg albumin                            | 20.6                   | —                                             |
| 6. Cetyl alcohol                          | 21.9                   | —                                             |
| 7. Heptadecylamine                        | 24.3                   | —                                             |
| 8. Octadecanol                            | 29.4                   | —                                             |
| 9. Pepsin - 1/2%                          | 33.2                   | 54.2                                          |
| 10. Zein                                  | 34.8                   | —                                             |
| 11. Gelatin Fisher G-8 - 1%               | 37.6                   | —                                             |
| 12. Trypsin                               | 40.6                   | —                                             |
| 13. Gelatin Fisher G-7 - 0.1%             | 55.7                   | —                                             |
| Distilled Water                           | 62                     | —                                             |
| 14. Gelatin - purified<br>calfskin - 0.1% | 151                    | —                                             |

The most effective films in retarding dissolving were lauric acid and a gelatin (Fisher G-7). Egg albumin fell within the range blanketed by the hydrocarbons, showing that its essential properties relating to nuclei are similar to hydrocarbons.



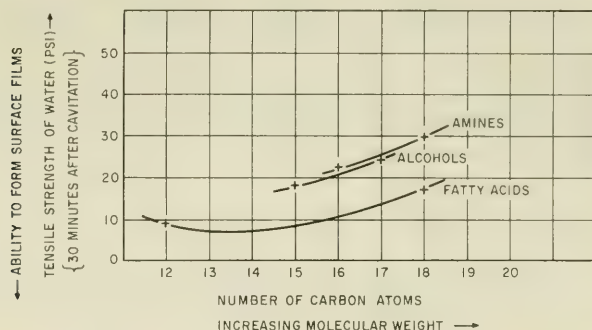


Fig. 13 - Effects of the length of linear hydrocarbons on film formation

### Double Surface Films

In order to prevent appreciable dissolving of nuclei, an effective diffusion coefficient must be obtained for a surface film that is much lower than any realized for a monomolecular film. That such a coefficient must exist is indicated by the behavior of tap water in which algae were growing. This water had negligible tensile strength gain due to dissolving when cavitated.

How can a lower diffusion coefficient be obtained? The monomolecular films applied to nuclei were the thickest and most compact monomolecular films possible. Therefore, to obtain a better film, one is forced to conclude that additional material must be made to collect about a monomolecular film to increase its thickness or perhaps "plug" leaks that may exist. This would improve the effectiveness of the film as a barrier to gas diffusion, and possibly reinforce the film so that it is stronger and more capable of structurally opposing the surface tension acting to cause dissolving. A likely way to do this would be surround the nucleus and its monomolecular surface film with polar materials capable of bonding to the polar groups that face the water in the film.

To test this supposition, various nonsurfactant polar materials were added to water containing either lauric acid or egg albumin. The polar materials used were gelatins or gelling agents that had previously shown no surface film activity (Table 8). They therefore should be unable to interfere with the formation of a monomolecular film by attempting to form a competing film. However, once the monomolecular film is in place, the polar groups in the film should provide a stationary surface to which the nonsurfactant polar material can adhere and start to build a structure.

The combination of a saturated egg albumin solution plus a 0.01% Jaguar solution worked well, resulting in zero tensile 30 minutes after cavitation. The combination of a saturated cetyl alcohol solution plus a 0.1% gelatin (Fisher G-8) solution was not as good, but improved a cetyl alcohol film from 21.9 psi tensile strength to 6.5 psi.



The mixtures, although effective in preventing dissolving, had short life. This is shown in Fig. 14a, where successive tests using the same samples show increasingly higher tensile strengths. One might possibly attribute this increase to attrition of the film forming materials, or to a protein film becoming insoluble. However, just allowing the sample to stand had a similar effect (Fig. 14b).

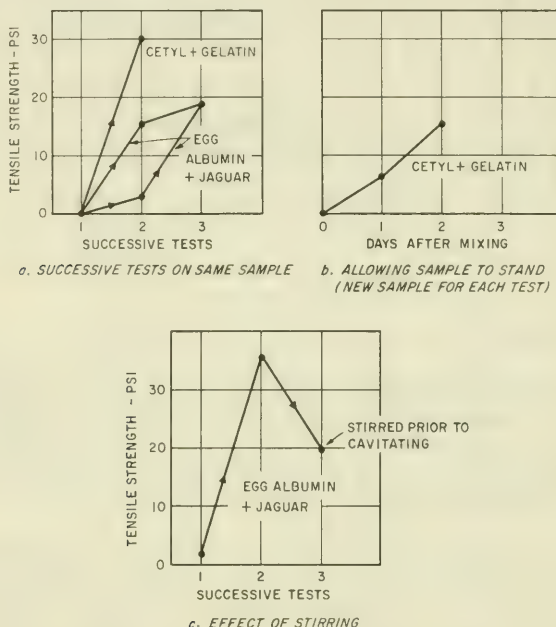


Fig. 14 - Double film formation

Therefore, suppose one assumed that this decrease in film formation is due to bonding taking place between gelatin molecules, or between gelatin and surfactant to alter the character of the surfactant, thereby preventing films from being formed. Bonds so formed probably would be weak. In that case shearing of the fluid as it is stirred might be able to break the bonds. Stirring did in fact improve a sample, and did partially restore its film-forming activity (Fig. 14c). This substantiated the chain of reasoning. However, weaker bonds and/or a continuous shearing motion are apparently needed to maintain the water in its original condition of preventing dissolving.

### Summation

In investigating the surface films of gas nuclei, the following steps were achieved:

1. Monomolecular hydrocarbon and protein films were placed about nuclei produced by cavitation. Most of the film-forming materials used were moderate in price.

2. The thickest possible hydrocarbon films were used. Thus the resistance of the film to gas diffusion and the strength of the film were the maximum possible, so as to obtain the greatest retardation in dissolving possible.

3. A few proteins — from those chosen more or less at random as far as knowledge of their structure was concerned — were found to be equivalent to the hydrocarbons in preventing nuclei dissolving. By comparison, this implied that the structure of these proteins had to be of a certain general type. Hence it is unlikely that thicker or "better" protein monomolecular films exist.

4. The "best" hydrocarbon and protein films reduced the rate of nuclei dissolving from the ideal by almost an order of magnitude. This should be a useful difference.

5. The lowest "effective" permeability coefficient for gas diffusion (least dissolving) obtained for a monomolecular film placed about a nucleus was  $1.8 \times 10^{-6}$  cm/sec. This contrasts with the lowest values in the literature of  $1 \times 10^{-3}$  obtained under "ordinary" conditions, showing that a film in situ is exceedingly effective in retarding dissolving.

6. Nevertheless, dissolving of the nuclei was not fully retarded. Tap water in which algae were growing had negligible tensile strength gain due to dissolving. Thus nature had apparently done much better than we had. Since the monomolecular films used were the best possible, one had to conclude that the only means by which dissolving could be further retarded would be that additional materials must surround the monomolecular film. Therefore a "double" film was devised that prevented virtually all dissolving.

In doing all this, one obtained a picture of what must take place in nature when a heavy surface film prevents dissolving of gas nuclei. That is:

1. A monomolecular film must form from organic surfactant material.

2. The film must have a structure built upon it by gel-like materials which are in themselves nonsurfactant.

3. The molecular bonds formed must be weak and shear readily, so that each type of material retains its original characteristics as the water moves and flows, so as to be able to act when a nucleus is encountered.

## COMMENTARY

It has been shown that surface films that retard the dissolving of nuclei come from trace organic materials normally considered insoluble. The term "trace materials" implies considerable variability in the ability of water to form surface films, as has been found to be the case. It is reasonable to expect

wide variations in the amounts of trace materials carried in water, and differences in the characteristics of these materials.

Yet, the conclusion that a double film structure exists in nature implies that almost any surfactant material is capable of forming monomolecular films that are effective in retarding dissolving, provided that a second film forms about the monomolecular film. The second film should be producible from a wide variety of ionic/polar structural building materials such as gelatins, starches, and sugars. Hence, the ability to form effective surface films about nuclei could be widespread — as also appears to be the case. By comparison, the surfactant hydrocarbon materials used in this investigation for monomolecular film formation were specially chosen to be capable of close packing — a category that may not necessarily be widely found in nature.

The testing of water from various water tunnel test facilities showed a relative absence of film formation. As a consequence, when small nuclei are produced by cavitation action, tensile strength probably can result in the water approaching a test section. The characteristics of the body being cavitation tested — as well as the attendant flow conditions — should markedly affect the size of the nuclei produced. Testing for incipient cavitation should produce small cavitation voids with small nuclei as the end product. Heavy cavitation and long dwell times in a low pressure region should produce large nuclei. Thus one is left with the conclusion that nuclei action in a tunnel must be correlated to the type of test at hand, as well as the film-forming tendencies of the water.

The preceding discussion points out the gross variability of surface film effects, whereas one would like to be in a position to control or limit them. Several approaches might be possible in a water tunnel, such as: (a) marking off an undesirable region of operation to be avoided, (b) having uniform effects that can be corrected for, (c) using film-forming additives to retard dissolving, and (d) deliberately injecting suitable nuclei. Using film-forming materials would require periodic replenishment of the materials. Depending upon the materials chosen, one can probably expect some spoilage due to bacteria, some breakdown of the molecules due to cavitation forces, and, for a protein, insolubility causing withdrawal from action. Also, fatty acids, acting as a soap, can combine with minerals in the water.

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\* \* \*

## DISCUSSION

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In connection with this extremely interesting paper I think it might be relevant to summarize briefly the results of some comparative tests which have been performed as part of the work of the Cavitation Committee of the 11th International Towing Tank Conference (ITTC). These tests studied primarily the cavitation inception on a head of a body of revolution. Results were obtained from 14 establishments including results from 13 different bodies tested in 17 different cavitation tunnels.

The results for gas content ratio  $\alpha/\alpha_s = 0.6$ , corrected for wall influence on water speed and cavitation number, are shown in Fig. D1. You can see that there is a considerable dispersion around the value of the pressure minimum for the body, which corresponds to  $\sigma = 0.6$ . Most of the tunnels have inception values below 0.6, indicating negative cavitation pressures, but there are two tunnels having high positive values. The pressure minimum being known in this case, an approximate calculation of the cavitation pressures can be made. The resulting pressures are shown in Fig. D2 together with results of measurements of cavitation pressures in nozzles, reported elsewhere.

Some of the differences in cavitation pressure, shown here, are probably due to differences in technique, body size, height of the tunnels and so on,

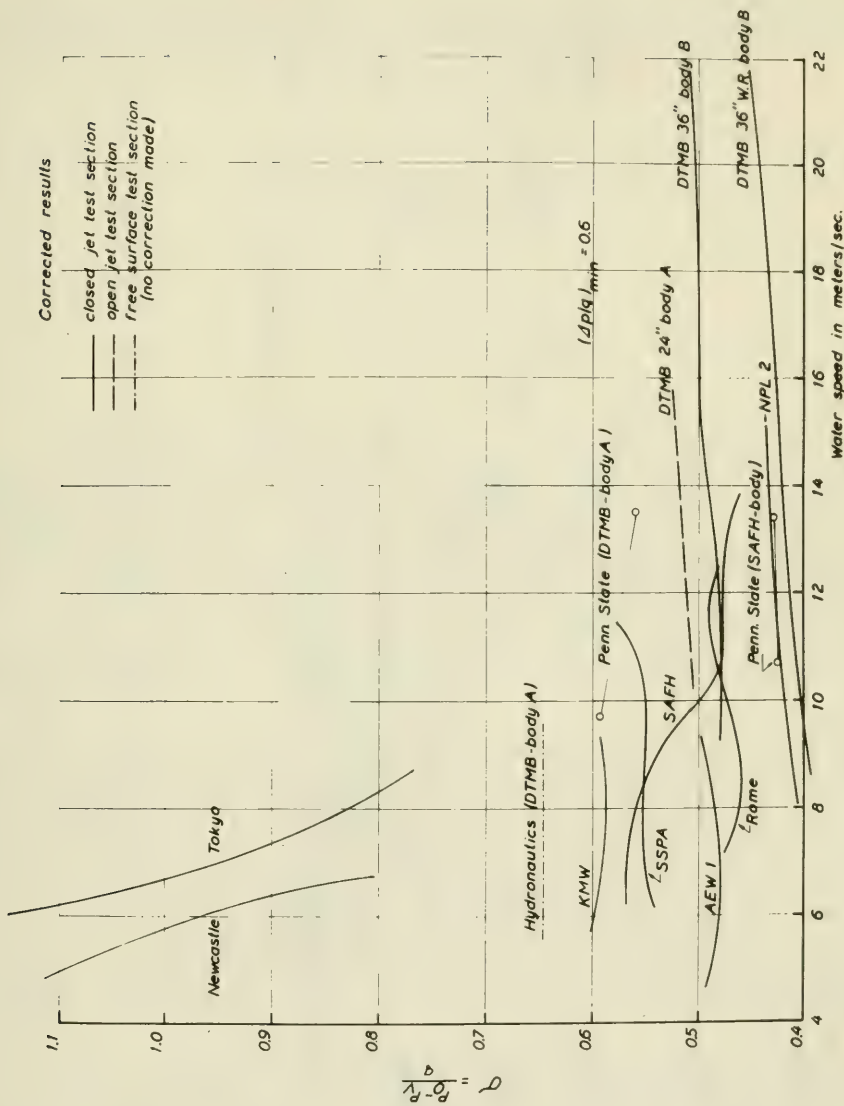


Fig. D1 - Inception cavitation number for the head of body of revolution for different tunnel-body combinations



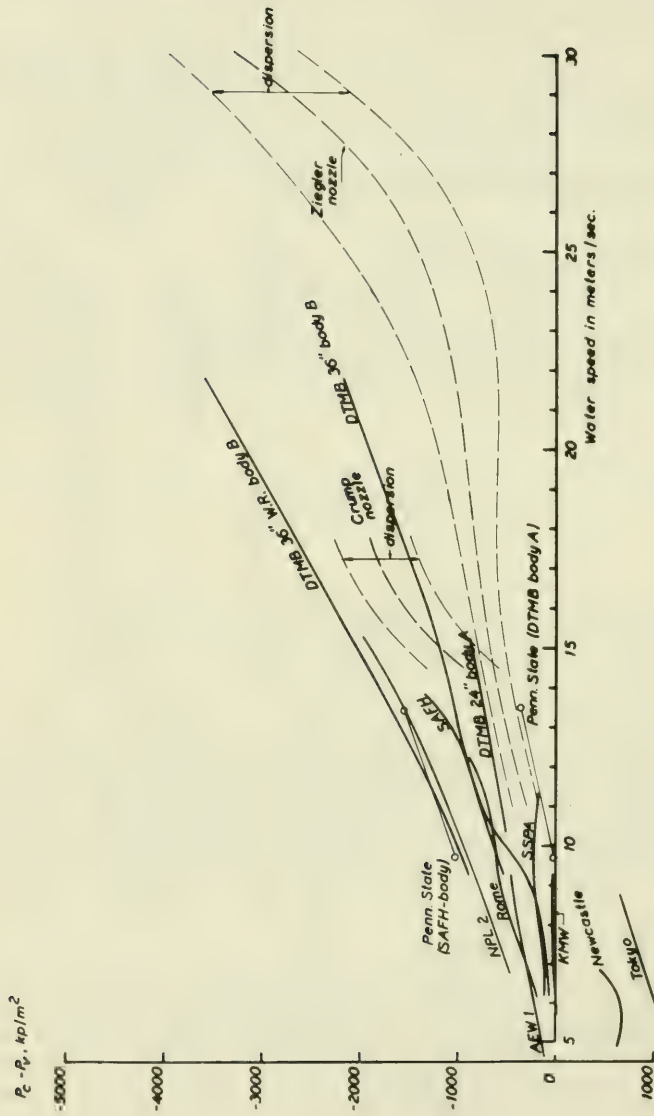


Fig. D2 - Cavitation pressures for the different tunnel-body combinations of Fig. D1 along with results for nozzles

things which have been discussed in the report to ITTC, but I suppose that part of the dispersion can be attributed to differences in water quality according to the mechanism we have seen outlined here. However, the range of pressure values is small in comparison with what we have seen reported here.

Finally one question. What is the reason for the increase in negative pressure with increasing water speed? Is it increased diffusion speed due to destruction of the film because of mechanical agitation?

\* \* \*

## DISCUSSION

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I am interested in the results obtained using Teflon to induce cavitation upon an ultrasonic transducer which contrasts with our work using spots of Teflon to induce similar bubbles for the boiling process (a procedure for which I hold the U.S. patent). Excess energy is similarly required to form a bubble, which in boiling comes from excess temperature, while with cavitation it comes from an excessive pressure reduction. For boiling, the Teflon spots are used to reduce this temperature excess for bubble formation to negligible proportions, resulting in easily a tenfold increase in heat transfer coefficient. An important difference between cavitation and boiling is that for boiling the bubbles do not collapse.

Our boiling work and I believe that of Dr. Richard Gaertner also of the General Electric Company has been under conditions which should not only effectively remove gas from the site, but also degas the liquid; but, with sufficiently fresh Teflon enamel as starting material and proper processing, the activity of the sites do not change with time, whether or not (a) boiling is continuous or (b) is periodically stopped with or without subcooling of the liquid. However, at a certain age which is generally considerably short of the six-month expiration date given by duPont, the Teflon enamel (which appears to go on properly, and which gives initial activity before degassing is complete) will become inactive for producing and maintaining bubbles for boiling within a period of an hour or so.

I wonder if the present deactivation of the Teflon surface might be due to the use of Teflon after it has become relatively ineffective as occurs in boiling, whether inactivation is due to the pounding of cavitation in changing the porosity or surface of the Teflon then reducing the effectiveness for vapour generation or retention, or whether this inactivation is in fact due to the difference between cavitation and boiling.

With regard to the hypothesis that stabilized gas nuclei could be formed with some protein-hydrocarbon shell, such nuclei can aid cavitation only to the extent they provide an unrestrained gas bubble. If the shell were impervious, then its efficacy in aiding bubble formation on the exterior would not differ whether gas were held internally or not. On the other hand, if gas is readily released by rupture or rapid penetration of the shell, it will be rapidly depleted.

\* \* \*

## REPLY TO DISCUSSION

Leslie H. Bernd

### CAVITATION INCEPTION LEVEL OF WATER TUNNELS

Mr. Johnson's data on the cavitation inception levels obtained in different water tunnels are most interesting. It shows a difference in the cavitation levels. It implies that up to a few psi of tensile strength may exist in the water in these tunnels. If so, the trend is to increasing tensile strength with increasing water speed. The implication of the data is that comparative tests are desirable and necessary. But, one expects that considerable research will be necessary before the means of producing uniform test results become obvious and before the role of gas nuclei is adequately delineated.

Assuming the tensile strength in the tunnels reported upon is controlled by gas nuclei carried in the water, investigations we have made on the dissolving of gas nuclei in turbulently flowing water would lead one (as a possibility) to attribute an increase in tensile strength with increasing water speed to a general decrease in the size of the nuclei produced by the cavitation occurring during the tests. One would be interested, as pertinent factors, in the sequence in which water speed was varied, as well as the duration of the cavitation tests. Also, for a small test body, one might wish to invoke as a cause of apparent tensile strength the time required to expand a nucleus against the inertia of the surrounding water, as a nucleus passes into the low pressure region at the body to form a cavitation void.

As relevant information, our investigations on the dissolving of gas nuclei in turbulently flowing water showed the following:

1. After the formation of gas nuclei by cavitation, rapid dissolving of these nuclei takes place.
2. The resultant change in the diameter of a nucleus due to dissolving, for a given distance traveled downstream, is generally a constant approximately independent of water speed. The time rate of dissolving increases proportionally as the water speed is increased.



3. After a brief period of time, this rapid dissolving may be abruptly halted (or greatly slowed down) by surface films. The size at which this change takes place is highly dependent upon the initial size of nucleus. A small nucleus may dissolve rapidly to obtain high tensile strength, whereas a large nucleus is arrested from its rapid dissolving before high tensile strength can be developed. The tensile strength obtained in the comparative tests are approximately in accordance with the nuclei sizes obtained in our tests when surface films acted to produce the above mentioned slow dissolving.

4. The subsequent slow dissolving of a nucleus surrounded by a surface film remains as a task requiring investigation. If the surface film retards dissolving primarily by acting as a barrier to gas diffusion, then for a given set of conditions the time rate of gas transport should be a value sensibly independent of water speed. If the surface film prevents dissolving primarily because it is stressed in compression as the surface area is reduced, and this compression opposes the contraction of the gas nucleus, then the time rate of transport should be proportional to the water speed.

Thus one expects tensile strength, as produced by the reduction in size of a nucleus caused by dissolving, to be approximately independent of water speed, or possibly to increase somewhat with decreasing water speed.

#### TEFLON SURFACE TO CAUSE BOILING OR CAVITATION

Dr. Hummel notes that the Teflon coatings he used to promote boiling worked well for an indefinite period of time when fresh but stopped after an hour when old. It was our experience that certain rubberlike, castable polymers cavitated profusely when freshly cast, but did not do so after considerable drying. The presence of volatile solvents was blamed for this action.

The rapid exhaustion of a Teflon coating in causing cavitation, as shown in Fig. 2b, was obtained with an old Teflon coating (by Dr. Hummel's definition). The ability to cavitate was lost after 2 hours of cavitation. I cannot comment on the role of age as it may affect a Teflon surface, but I would suggest that the gas content of the Teflon be measured and the surface nucleation sites be counted. These had been found to be the primary factors causing cavitation originating from polymers. As an example, Teflon sheet stock (virtually no nucleation sites) did not cavitate, whereas a Teflon coating (many nucleation sites) cavitated as mentioned above. Both Teflon samples had the same gas content.

#### THE MECHANISM OF A STABILIZED GAS NUCLEUS

Dr. Hummel questions the hypothesis of a stabilized, long-life gas nucleus consisting of gas enclosed by a protective, relatively impervious organic "shell." Reasonably he asks how a gas nucleus can be surrounded by a shell and yet subsequently act to produce cavitation when a gas/water interface is



needed to initiate cavitation. I am sure that many, including myself, have asked the same question. However, tests conducted by myself, as well as others, force one to conclude that stabilized nuclei do exist. The problem is to obtain a believable physical model.

It is reasonable to expect a nucleus to acquire a monomolecular surface film. The characteristics of surface films observed away from a nucleus imply that the necessary properties for a stabilized nucleus capable of initiating cavitation are obtainable by a surface film at a nucleus. For example, investigations on the behavior of surface films at a gas/water interface have shown that some monomolecular films can be highly compressed (in the order of 50:1) as the interface is contracted, and then subsequently re-expand to their original condition as the interface is expanded. Some surface films may however change their physical structure upon compression, possibly acquire a permanent set, slowly redissolve in the water, or yield slowly in a viscous fashion. Thus hysteresis may be exhibited for various reasons. In some cases, a slight expansion may reduce the compressive stress to zero and cause rupture in tension. Langmuir was the original worker in this field.

Furthermore, because of the spherical shape of the nucleus, the surface film should be stressed efficiently (good use of material) before failure in compression can occur. Also, the spherical form is a stable shape; it prevents failure due to buckling of the film.

It is therefore reasonable to suppose that a film can be sufficiently compressed to (a) greatly decrease the transport of gas because of the reduced spacing between molecules, as well as (b) exert a compressive stress that prevents the internal gas pressure necessary for dissolving from being developed.

If the stabilized nucleus enters a low-pressure area, the gas will attempt to expand. The surface film may also elastically attempt to expand. The surface film will then be expanded until essentially the surface tension of the water alone acts to contract the nucleus. The protective "shell" no longer acts. It has either been ruptured upon expansion, or it has expanded until it is ineffectual. Thus the gas nucleus should be capable of expanding to form a cavitation void or acquire additional gas in the form of water vapor or dissolved gas.

\* \* \*

# LINEAR AND NONLINEAR DISPERSION OF PRESSURE PULSES IN LIQUID- BUBBLE MIXTURES

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## 1. INTRODUCTION

Thirty years ago Minnaert (1) derived a formula for the oscillation frequency  $\omega_B$  of an air bubble in water:

$$\omega_B = \frac{2}{R_0} \left( \frac{p_0}{\rho} \right)^{1/2} \quad (1.1)$$

In (1.1)  $R_0$  is the bubble radius,  $\rho$  the density of water and  $p_0$  the ambient pressure.

When many small bubbles are present in a cubic meter of water, in say, it is equally well known that the mixture can be considered as a homogeneous fluid with mass density nearly equal to the water density and with compressibility arising from the gas content. For a small gas volume in a unit of volume of the mixture, the velocity with which infinitesimal disturbances of pressure propagate through the mixture (the sound velocity) is given by

$$c = \left( \frac{p_0}{\pi n \rho R_0^3} \right)^{1/2} \quad (1.2)$$

Equation (1.2) follows from considerations in which the bubble character of the air content is not taken into account — in other words for wavelengths which are large with respect to the bubble radius. How do waves with shorter wavelengths propagate? This question is considered here.

Furthermore, it is well known that for finite pressure variations, growth and collapse of a bubble are highly nonlinear. High pressures occurring during

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the collapse of cavitation bubbles, filled with vapor and a small amount of air, are in fact the cause of cavitation damage. For a bubble with initial pressure  $p_0$  and collapsing under an outside pressure  $p_\infty$  the rate of change of the radius  $R$  with time  $t$ , neglecting vapor content, will be seen from the next section to be given by

$$\left(\frac{dR}{dt}\right)^2 = \frac{2p_0 R_0^3}{\rho R^3} \int_{R/R_0}^1 \left(\frac{p_\infty}{p_0} - x^{-4}\right) x^2 dx. \quad (1.3)$$

The minimum radius appears to be about  $3p_0 R_0 / p_\infty$  and the associated pressure in the bubble can raise considerably above  $p_\infty$ . The question arises how in a mixture of collapsing bubbles the average pressure in the fluid behaves. This question was considered in Van Wijngaarden (2) and is discussed here again in connection with the results of linear theory.

The investigation presented here attempts to link single bubble theory with homogeneous fluid theory both for linear and nonlinear oscillations of the individual bubbles in the mixture.

## 2. LINEAR THEORY I

To fix ideas we consider the two-dimensional problem of a bubble-water mixture bounded at  $y' = 0$  by a solid wall and at  $y' = h$  by pure water. At times  $t' \leq 0$  the pressure both in the pure water and in the bubble-water mixture is  $p_0$ .

At  $t' = 0^+$  the pressure in the region  $y' \geq h$  is increased by an amount  $\Delta p$ . This gives in the mixture rise to variation of average pressure  $p$ , and average velocity  $v$ . The averaging is over a region small with respect to  $h$  but still containing many bubbles, so that it is required that

$$n^{-1/3} \ll h. \quad (2.1)$$

The time history of the radius of an individual bubble, neglecting the effects of viscosity and surface tension, is given by (see e.g., Lamb (3))

$$\rho \left\{ R \frac{\partial^2 R}{\partial t'^2} + \frac{3}{2} \left( \frac{\partial R}{\partial t'} \right)^2 \right\} = p_g - p, \quad (2.2)$$

in which  $p_g$  is the pressure of the gas in the bubble. Equation (2.2) has been used by many authors in order to deal with the collapse of a single bubble under influence of pressure  $p_\infty$  far from the bubble.

Writing  $p = p_\infty$  and assuming a polytropic change of the bubble pressure with bubble volume given by

$$p_g R^4 = p_{g_0} R_0^4, \quad (2.3)$$

the relation (1.3) can be easily derived from (2.2).



The essential point in the present theory (see also Van Wijngaarden (2)) is that in (2.2) we insert for  $p$  the *average* pressure in the mixture at the location of the bubble, so that for each bubble the "pressure at infinity" is the average pressure in the volume element in which the bubble is one of the large number of constituents.

Insertion of (2.3) in (2.2) and linearization about the equilibrium radius  $R_0$  results in

$$\frac{\rho R_0^2}{4p_0} \frac{\partial^2 p_g}{\partial t'^2} + p_g = p. \quad (2.4)$$

Recognizing from (1.1) that the factor in the first term of this equation is  $1/\omega_B^2$  we may also write

$$\frac{1}{\omega_B^2} \frac{\partial^2 p_g}{\partial t'^2} + p_g = p. \quad (2.5)$$

For the average quantities  $p$  and  $v$  we now write down the continuity and momentum equations:

$$\frac{\partial v}{\partial y'} = 4\pi n R^2 \frac{\partial R}{\partial t} \quad (2.6)$$

and

$$-\frac{1}{\rho} \frac{\partial p}{\partial y'} = \frac{\partial v}{\partial t'}. \quad (2.7)$$

In formulating the momentum equation, (2.7), convective acceleration terms are neglected because for a small amount of gas (as is assumed to be the case) the average velocity  $v$  will be small.

When small changes in the bubble radii are involved (resulting from a small value of  $\Delta p$ ) we may linearize (2.6). Then upon crossdifferentiation of (2.6) and (2.7) we obtain

$$\frac{\partial^2 p}{\partial y'^2} = \frac{1}{c^2} \frac{\partial^2 p_g}{\partial t'^2}, \quad (2.8)$$

where  $c^2$  is given by (1.2).

For  $R_0 \rightarrow 0$  the bubble frequency  $\omega_B$  tends to infinity. In that case (2.5) reduces to  $p = p_g$  and (2.8) to the well-known acoustical wave equation. Hence the mixture, as described in the present model, behaves in the limit of infinite bubble frequency as a homogeneous fluid with sound velocity  $c$ .

The boundary and initial conditions of the problem posed at the beginning of this section are

$$t' = 0: p = p_g = p_0; \quad \frac{\partial p}{\partial t'} = \frac{\partial p_g}{\partial t'} = 0 \quad (2.9)$$

and

$$y' = 0: \frac{\partial p}{\partial y'} = 0; \quad y' = h: p = p_0 + \Delta p H(t') . \quad (2.10)$$

In (2.10)  $H(t')$  is the unit step function. In case of small values of  $\Delta p/p_0$  the linearized equations (2.5) and (2.8) may be used. The boundary at  $y' = 0$  can be removed, provided we extend the mixture till  $y' = -h$  and apply at  $y' = -h$  the same boundary conditions as hold for  $y' = h$ . If  $\omega_B \rightarrow \infty$ , we know from acoustic theory that at  $y' = 0$  the fluid is quiescent until  $t' = h/c$ . At that moment the pulses emitted from  $y' = h$  and  $y' = -h$  arrive, and the pressure remains  $p_0 + 2\Delta p$  until at  $t' = 3h/c$  a rarefaction wave of strength  $2\Delta p$  arrives, etc. Therefore at  $y' = 0$  and  $\omega_B \rightarrow \infty$

$$p = p_g = p_0 + 2\Delta p [H(t' - h/c) - H(t' - 3h/c) + \dots] . \quad (2.11)$$

As long as  $t' < 3h/c$  the first two terms on the right side of (2.11) are nonzero, which we can write as

$$p_g - p_0 = \frac{4\Delta p}{\pi} \int_0^\infty \frac{\sin kh (1 - \cos kct')}{k} dk \quad (2.12)$$

or

$$p_g - p_0 = 2\Delta p \left\{ 1 - \left[ \frac{1}{\pi} \int_0^\infty \frac{\sin k (h + ct')}{k} dk + \frac{1}{\pi} \int_0^\infty \frac{\sin k (h - ct')}{k} dk \right] \right\} . \quad (2.13)$$

The expression (2.13) allows the interpretation for  $p_g - p_0$  as the integrated result of a continuous progressing wave spectrum with wave number distribution  $1/k$ , each wave traveling with the same velocity  $c$ .

In dealing with finite values of  $\omega_B$  it is convenient to introduce dimensionless variables by

$$t = \omega_B t' \quad (2.14)$$

and

$$y = \frac{\omega_B y'}{c} . \quad (2.15)$$

With help of these relations elimination of either  $p_g$  or  $p$  from (2.5) and (2.8) yields

$$\left( \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial t^2 \partial y^2} - \frac{\partial^2}{\partial t^2} \right) \begin{Bmatrix} p \\ p_g \end{Bmatrix} = 0 . \quad (2.16)$$

Seeking solutions in the form of progressive waves

$$\exp i(ky - \sigma t)$$

we obtain the dispersion equation

$$\sigma = \frac{k}{(1 + k^2)^{1/2}}. \quad (2.17)$$

Waves are dispersed. The phase velocity is given by

$$\frac{\sigma}{k} = \frac{1}{(1 + k^2)^{1/2}}, \quad (2.18)$$

whereas the group velocity is

$$\frac{d\sigma}{dk} = \frac{1}{(1 + k^2)^{3/2}}. \quad (2.19)$$

Only for very long waves ( $k \rightarrow 0$ ) the propagation is nondispersive, but otherwise the group velocity is smaller than the phase velocity, indicating a normal dispersion pattern.

For the present investigation it is convenient to apply a Laplace transform to (2.16). Defining  $\mathcal{L}[f(t)]$  by

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-St} f(t) dt \quad (2.20)$$

we obtain from (2.16), using (2.9), (2.10), (2.14) and (2.15),

$$\mathcal{L}(p - p_0) = \frac{\Delta p}{S} \frac{\cosh \left[ \frac{Sy}{(S^2 + 1)^{1/2}} \right]}{\cosh \left[ \frac{S\lambda}{(S^2 + 1)^{1/2}} \right]}, \quad (2.21)$$

and

$$\mathcal{L}(p_g - p_0) = \frac{\Delta p}{S(S^2 + 1)} \frac{\cosh \left[ \frac{Sy}{(S^2 + 1)^{1/2}} \right]}{\cosh \left[ \frac{S\lambda}{(S^2 + 1)^{1/2}} \right]}. \quad (2.22)$$

The parameter  $\lambda$  is the (dimensionless) thickness of the bubble-water mixture

$$\lambda = \frac{\omega_B h}{c} \quad (2.23)$$



The parameter  $\lambda$  expresses the ratio between the period of free oscillations of a bubble and the time for a disturbance to travel over a distance  $h$  with velocity  $c$ . With help of (1.1) and (1.2)  $\lambda$  may be written as

$$\lambda = \frac{h}{R_0} (4\pi n R_0^3)^{1/2}. \quad (2.24)$$

For a gas volume fraction  $10^{-3}$  and  $h = 10^{-2}$  m we obtain  $\lambda \approx 10^{3/2}$  for  $R_0 = 10^{-5}$  m and  $\lambda \approx 10^{1/2}$  for  $R_0 = 10^{-4}$  m. These figures pertain for example to a cloud of cavitation bubbles, as considered in Van Wijngaarden (2). Here also we shall consider values of  $\lambda$  of about  $10^1$  or  $10^2$ .

The transforms (2.21) and (2.22) cannot be inverted as they stand, but sufficient information for the present purpose can be derived from these expressions. For small times, we obtain allowing  $S$  to tend to infinity

$$p_g - p_0 \approx \frac{\Delta p t^2}{2} \frac{\cosh y}{\cosh \lambda} \quad (2.25)$$

and

$$p - p_0 \approx \Delta p \frac{\cosh y}{\cosh \lambda} H(t). \quad (2.26)$$

Pressure changes occur instantaneously, because in the present approximation we have considered the fluid as incompressible. From (2.17) we infer that whereas for real  $k$ ,  $\sigma$  is real also,  $k$  is only real for  $\sigma < 1$ . Frequencies higher than  $\sigma = 1$  are strongly absorbed, since from (2.17) it follows that

$$k = \frac{\sigma}{(1 - \sigma^2)^{1/2}}.$$

For  $\sigma \gg 1$ ,  $k = \pm i$ , which explains the nature of the signal for small values of  $t$ .

For later reference we note that at  $y = 0$  (2.25) may be written as

$$p_g - p_0 \approx \Delta p t^2 e^{-\lambda} (1 - e^{-2\lambda} + \dots). \quad (2.27)$$

In order to investigate the development of  $p_g - p_0$  at  $y = 0$  further we write the transform of this quantity (as given by (2.22)) as

$$\frac{2\Delta p}{S(1+S^2)} \sum_{m=0}^{\infty} \exp \left[ - \frac{(2m+1)\lambda S}{(1+S^2)^{1/2}} \right]. \quad (2.28)$$

The first term ( $m = 0$ ), as is shown in the Appendix, can be inverted exactly, which results in

$$p_{g1} - p_0 = \frac{4\Delta p}{\pi} \int_0^\infty \frac{\sin \lambda k}{k} \left\{ 1 - \cos \left[ \frac{kt}{(1+k^2)^{1/2}} \right] \right\} dk \quad (2.29)$$

or

$$p_{g1} - p_0 = 2\Delta p \left( 1 - \frac{1}{\pi} \int_0^\infty \left\{ \frac{1}{k} \sin \left[ \lambda k + \frac{kt}{(1+k^2)^{1/2}} \right] + \frac{1}{k} \sin \left[ \lambda k - \frac{kt}{(1+k^2)^{1/2}} \right] \right\} dk \right). \quad (2.30)$$

The integrands in (2.29) and (2.30) reduce for small  $k$  (long waves) to the corresponding expressions in (2.12) and (2.13).

By expansion of the terms in braces in (2.29) a series convergent for all  $t$  can be obtained. The first term is

$$\frac{2\Delta p}{\pi} \int_0^\infty \frac{kt^2 \sin \lambda k}{1+k^2} dk = \Delta p t^2 e^{-\lambda}, \quad (2.31)$$

in agreement with the first term in (2.27). Higher terms are of the type  $t^{2k} \lambda^{k-1} e^{-\lambda}$ . Although these higher terms can be obtained relatively easily, the result is not easy to derive conclusions from. For larger values of  $t$  we proceed therefore otherwise and use (2.30).

At  $t \approx \lambda$  the main contribution to the first integral in (2.30) stems from the neighborhood of  $k=0$  for the values of  $\lambda$  considered here. The first integral in (2.30) can therefore at  $t \approx \lambda$  be replaced by

$$\frac{1}{\pi} \int_0^\infty \frac{1}{k} \sin (\lambda k + kt) dk = \frac{1}{2}.$$

The second integral has a similar contribution from  $k=0$  but also an important one from the stationary phase point where  $\lambda/t$  equals the group velocity or with help of (2.19)

$$\frac{\lambda}{t} = \frac{1}{(1+k^2)^{3/2}}. \quad (2.32)$$

A stationary phase exists only for  $t > \lambda$ . At  $t = \lambda$  the stationary phase coincides with  $k=0$ . Therefore at  $t = \lambda$  the second integral in (2.30) is approximately given by

$$\frac{1}{\pi} \int_0^\infty \frac{1}{k} \sin \frac{1}{2} k^3 t dk = \frac{1}{6}.$$

The total signal at  $t = \lambda$  is

$$p_{g1} - p_0 = \frac{2}{3} \Delta p$$

which signal can be considered as the "forerunner" (see e.g., Brillouin (4)).

For  $t > \lambda$  the contributions by  $k = 0$  to the integrals in (2.30) cancel just as the integrals in (2.13) cancel for  $t' > h/c$ . The total signal is, however, not  $2\Delta p$  for these values of  $t'$ , as in (2.13), but  $2\Delta p$  minus the contribution of the stationary point in the integrand of the second integral in (2.30). The arrival of the total signal  $2\Delta p$  is therefore not determined by the "sound" velocity  $c$  but by the group velocity as given by (2.32). Writing

$$\int_0^\infty \frac{1}{k} \sin \left[ \lambda k - \frac{kt}{(1+k^2)^{1/2}} \right] dk = \text{Im} \int_0^\infty \frac{\exp i t \left[ \frac{\lambda}{t} k - \frac{k}{(1+k^2)^{1/2}} \right] dk}{k} \quad (2.33)$$

and evaluating the contribution of the point of stationary phase, we obtain for  $t > \lambda$

$$p_{g1} - p_0 = 2\Delta p \left( 1 - \frac{(t/\lambda)^{5/6}}{(6\pi t)^{1/2} \left[ (t/\lambda)^{2/3} - 1 \right]^{3/4}} \cos \left\{ t \left[ 1 - (\lambda/t)^{2/3} \right]^{3/2} + \frac{\pi}{4} \right\} \right). \quad (2.34)$$

The expressions (2.31), (2.33), and (2.34) show how the pressure  $p_g$  in the bubbles increases with time at the plate  $y' = 0$ . This forms an adequate description as long as  $t < 3\lambda$ , since at that moment the "forerunner" associated with the second term in (2.28) arrives at  $y' = 0$ . Schematically  $p_g$  as a function of  $t$  is shown in Fig. 1.

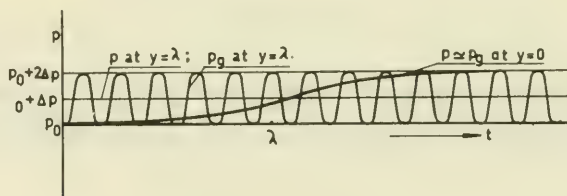


Fig. 1 - Behavior of  $p$  and  $p_g$  as a function of  $t$  and  $y = 0$  and  $y = \lambda$

Next we consider the behavior of the pressure in the bubbles at the outer boundary of the mixture. From the boundary conditions at  $y' = h$  and from (2.5) it follows that at  $y' = h$

$$p_g - p_0 = \Delta p (1 - \cos \omega_B t'). \quad (2.35)$$

Because the ambient pressure is prescribed at  $y' = h$  the bubbles oscillate as if in an infinite fluid with pressure  $p_0 + \Delta p$ . The frequency of the oscillations is  $\omega_B$ . The relevant part of the phase variation in (2.34) is given by

$$\cos \left\{ \omega_B t' \left[ 1 - \left( \frac{h}{ct'} \right)^{2/3} \right]^{3/2} \right\}. \quad (2.36)$$

Comparison of the arguments of the cosines in (2.35) and (2.36) shows that in the time interval between  $h/c$  and  $3h/c$  the pressure  $p_g$  in the gas bubbles varies much slower at  $y' = 0$  than at  $y' = h$ . Then it follows from consideration of (2.5) that, while at  $y' = h$  the difference  $p_g - p$  varies rapidly, this difference is small at  $y' = 0$ , because the term  $(1/\omega_B^2) (\partial^2 p_g / \partial t'^2)$  in (2.5) is small at  $y' = 0$ .

The average pressure  $p$  in the fluid at the plate therefore rises at a slow pace compared with  $\omega_B^{-1}$  to the value  $p_0 + 2\Delta p$ . In the nonlinear case the same occurs qualitatively, but since the gas pressure reaches much higher values (than the value  $p_0 + 2\Delta p$  in the linear case), also the average pressure in the fluid can assume high values, as will be made plausible in section 4.

### 3. LINEAR THEORY II

With a view to the nonlinear case, with which we occupy ourselves in the next section, we deal with the linear case in another way.

The nondimensional form of (2.4) is (cf. (2.14))

$$\frac{\partial^2 p_g}{\partial t^2} + p_g = p. \quad (3.1)$$

Introducing

$$U = \frac{\partial p_g}{\partial t} \quad (3.2)$$

we integrate (3.1) to obtain

$$U^2 = 2 \int_{p_0}^{p_g} (p - \xi) d\xi, \quad (3.3)$$

while combination of (3.1) and the dimensionless form of (2.8) yields

$$\frac{\partial^2 p}{\partial y^2} = p - p_g. \quad (3.4)$$

We now consider  $U$  and  $p$  as dependent variables and  $p_g$  and  $y$  as independent variables. In this case we should write (3.4) as



$$\left( \frac{\partial}{\partial y} + \frac{\partial p_g}{\partial y} \frac{\partial}{\partial p_g} \right)^2 p = p - p_g. \quad (3.5)$$

In the linear case  $\Delta p$   $p_0$  is a small quantity,  $\epsilon$  say. The pressure  $p_g$  is of order  $p_0(1 + \epsilon)$ ,  $\partial p_g / \partial p_g$  is of order 1, and  $\partial p_g / \partial y$  is of order  $\epsilon$ .

We now assume that the operator working on  $p$  in the left side of (3.5) may be approximated by  $\partial^2 / \partial y^2$ . Then we may write instead of (3.5)

$$\frac{\partial^2}{\partial y^2} (p - p_g) - (p - p_g) = 0. \quad (3.6)$$

Solving this equation yields using the boundary conditions (2.10),

$$p - p_g = (\Delta p + p_0 - p_g) \frac{\cosh y}{\cosh \lambda}. \quad (3.7)$$

Substituting (3.7) in the integrand in (3.3), we obtain

$$p_g - p_0 = \Delta p \left\{ 1 - \cos \left[ t \left( \frac{\cosh y}{\cosh \lambda} \right)^{1/2} \right] \right\}. \quad (3.8)$$

Inserting (3.7) and (3.8) in (3.5) shows that reducing the differential operator in (3.4) to  $\partial^2 / \partial y^2$  is a reasonable approximation for small values of  $t$  and independently for small values of  $y$ .

If we take  $t \ll 1$ , it follows that at  $y = 0$  for small  $t$

$$p_g - p_0 \approx \Delta p t^2 e^{-\lambda}, \quad (3.9)$$

where from the hyperbolic function in (3.8) only the first term in the expansion in terms of  $\lambda$  has been taken into account. This expression is identical with (2.31), confirming what is said above about the validity of the used approximation for small  $t$ .

For other values of  $t$ , (3.8) yields at  $y = 0$  ( $\cosh \lambda^{-1}$  being replaced by  $2e^{-\lambda}$ )

$$p_{g1} - p_0 = \Delta p \left\{ 1 - \cos \left[ 2^{1/2} t e^{-\lambda/2} \right] \right\}. \quad (3.10)$$

This expression shows just like the results of the foregoing section a slow variation of the gas pressure at  $y = 0$ . Comparison of (3.10) with (2.34) shows that due to the approximation used in this section the variation of the gas pressure as given by (3.10) is considerably too slow.

Qualitatively, however, the results of this section agree with those of section 2. Equation (3.7) indicates that for  $\lambda \gg 1$ , the difference between  $p$  and  $p_g$  is small at  $y = 0$ , whereas Eq. (3.10) expresses that as time proceeds  $p_g$  increases at  $y = 0$  slowly (compared with  $\omega_B^{-1}$ ) from  $p_0$  at  $t = 0$  to the value  $p_0 + 2\Delta p$ .

#### 4. NONLINEAR THEORY

Starting with the Eqs. (2.2), (2.3), (2.6), and (2.7) for  $p$ ,  $p_g$ ,  $R$ , and  $v$  and rendering  $p$  dimensionless with  $p_0$ ,  $R$  dimensionless with  $R_0$  ( $x = R/R_0$ ), and  $y'$  dimensionless with  $\omega_B/c$  (cf. (2.15)), it may be shown (Van Wijngaarden (2)) that with the approximation used in section 3 and consisting in reducing  $(\partial^2/\partial y^2)_{t=\text{const}}$  to  $(\partial^2/\partial y^2)_{x=\text{const}}$  the following equation holds for the difference between average pressure  $p$  in the fluid and pressure  $p_0 R_0^4/R^4$  or  $x^{-4}$  in dimensionless variables, in the gas bubbles:

$$\frac{\partial^2}{\partial y^2} (p - x^{-4}) - x(p - x^{-4}) = \frac{1}{x^2} \int_1^x \xi^2 (p - \xi^{-4}) d\xi. \quad (4.1)$$

The boundary conditions for (4.1) are (cf. (2.10))

$$y = \lambda: \quad p = \frac{p_0 + \Delta p}{p_0} \quad (4.2)$$

and

$$y = 0: \quad \frac{\partial p}{\partial y} = 0. \quad (4.3)$$

In Van Wijngaarden (2) an approximate solution of (4.1) subjected to (4.2) and (4.3) was given. An exact solution appears, however, to be possible. We define

$$G(x, y) = \int_1^x \xi^2 (p - \xi^{-4}) d\xi. \quad (4.4)$$

Bearing in mind that  $x$  and  $y$  are independent variables, we can combine (4.1) and (4.4) into

$$\frac{\partial^3 G}{\partial y^2 \partial x} - x \frac{\partial G}{\partial x} = 0. \quad (4.5)$$

Introducing the Laplace transform of  $G$  with

$$\bar{G} = \int_0^\infty e^{-Sy} G(x, y) dy \quad (4.6)$$

and denoting the pressure at  $y = 0$  in dimensionless form by  $\Pi(x)$ , transformation of (4.5) yields

$$(S^2 - x) \frac{d\bar{G}}{dx} - \bar{G} = Sx^2 \left[ \Pi(x) - x^{-4} \right].$$

Integration with respect to  $x$  results in

$$\bar{G} = \frac{S}{S^2 - x} \int_1^x \xi^2 \left[ \Pi(\xi) - \xi^{-4} \right] d\xi. \quad (4.7)$$

Inversion then yields, using (4.4),

$$\int_1^x \xi^2 (p - \xi^{-4}) d\xi = \cosh \left[ y(x)^{1/2} \right] \int_1^x \xi^2 \left[ \Pi(\xi) - \xi^{-4} \right] d\xi. \quad (4.8)$$

For the unknown function  $\Pi(x)$  we can now derive a relation, since (4.2) has not been used yet. Doing this now, thereby writing

$$\frac{\Delta p + p_0}{p_0} = p \quad (4.9)$$

we infer from (4.8), by differentiation and substitution,

$$\Pi(x) - x^{-4} = \frac{p - x^{-4}}{\cosh \left[ \lambda(x)^{1/2} \right]} - \frac{\lambda}{x^{5/2}} \frac{\sinh \left[ \lambda(x)^{1/2} \right]}{\cosh^2 \left[ \lambda(x)^{1/2} \right]} \int_1^x (p - \xi^{-4}) \xi^2 d\xi. \quad (4.10)$$

While (4.10) describes how the pressure in the mixture changes at the wall with the pressure in the bubbles, an equation for the rate of change of the bubble radii can be obtained in the following way. Inserting  $p$  instead of  $p_\infty$  in (1.3), we obtain

$$\frac{\rho}{p_0} \left( \frac{\partial R}{\partial t} \right)^2 = - \frac{2}{x^3} \int_x^1 (p - \xi^{-4}) \xi^2 d\xi. \quad (4.11)$$

Using the boundary condition at  $y = \lambda$  we infer from (4.8)

$$\int_1^x \xi^2 \left[ \Pi(\xi) - \xi^{-4} \right] d\xi = \frac{\int_1^x \xi^2 (p - \xi^{-4}) d\xi}{\cosh \left[ \lambda(x)^{1/2} \right]}. \quad (4.12)$$

Using this result, combination of (4.8) and (4.11) yields

$$\frac{\rho}{p_0} \left( \frac{\partial R}{\partial t} \right)^2 = - \frac{2}{x^3} \frac{\cosh \left[ y(x)^{1/2} \right]}{\cosh \left[ \lambda(x)^{1/2} \right]} \int_x^1 \xi^2 (p - \xi^{-4}) d\xi. \quad (4.13)$$

The expressions (4.10) and (4.13) contain the information necessary for our present purpose. First we note that for values of  $x$  near unity (4.10) reduces to (3.7), if in the last expression we take  $y = 0$ . It can easily be shown that also for  $x$  near unity (4.13) can be integrated to the relation (3.8). It is interesting to note that while in the linear case  $\cosh \lambda$  occurs in the expressions for the gas pressure, in the nonlinear case  $\lambda$  is multiplied with  $x^{1/2}$  which expresses the

physical fact that during collapse of the bubbles the volume of the gas content decreases, which fact is ignored in the linear case.

Equation (4.13) shows that at  $y = 0$  the rate of change of  $R$  is much slower than at  $y = \lambda$ . The maximum gas pressure occurs when  $\partial R / \partial t = 0$ . From (4.13) we infer that the value  $x_{\min}$  at which this occurs is the same for all  $y$ . This value is for  $p \gg 1$  approximately given by

$$x_{\min} = \frac{3}{p}. \quad (4.14)$$

According to the relation (4.10) we have therefore at  $y = 0$

$$\Pi_{\max} - x_{\min}^{-4} = \frac{p - x_{\min}^{-4}}{\cosh \left[ \lambda (x_{\min})^{1/2} \right]}. \quad (4.15)$$

On account of (4.14) the maximum gas pressure is given by

$$x_{\min}^{-4} \approx 10^{-2} p^4. \quad (4.16)$$

Note for comparison that in the linear theory the maximum gas pressure is  $1 + (2\Delta p/p_0)$ . If the value of  $\lambda (x_{\min})^{1/2}$  is not small, it follows from (4.15) that the maximum pressure at the wall  $\Pi_{\max}$  can increase to a significant fraction of the maximum pressure in the bubbles. High *average* pressures can therefore be expected to occur.

## 5. CONCLUSION

Although the approximations of the present analysis do not permit accurate quantitative results, the linear and nonlinear theories developed here make it clear that water-bubble mixtures are strongly dispersive and that due to the dispersion high average pressures in the mixture can occur during the nonlinear collapse of individual bubbles.

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# APPENDIX

To invert the first term in (2.28) we consider the integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{St+\lambda S/(1+S^2)^{1/2}}}{S(1+S^2)} dS \quad (A1)$$

along a straight path a distance  $c$  to the right of the imaginary axis in the  $S$  plane (Fig. A1). We deform this path to a closed contour as indicated in the figure. The contribution from the pole in  $S = 0$  equals unity. To evaluate the contribution from the branch points  $S = \pm i$  we have to integrate around the branchcut indicated in the figure. We transform now the  $S$  plane into a  $\mu$  plane\*

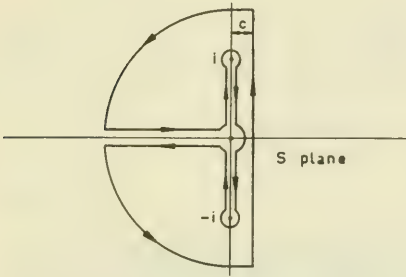


Fig. A1 - Closed contour in the  $S$  plane

$$\frac{S}{(1+S^2)^{1/2}} = \mu. \quad (A2)$$

Because of the branchcut we put the left side of (A2) equal to  $+\mu$  on the left side of the branchcut and equal to  $-\mu$  on the right side. The region between the branch points in the  $S$  plane maps on the imaginary axis in the  $\mu$  plane:

$$S = \mu(1 - \mu^2)^{1/2}.$$

Then (A1) becomes

$$1 - \frac{1}{2\pi i} \left[ \int_{-i\infty}^{i\infty} \frac{e^{[\mu t/(1-\mu^2)^{1/2}] + \lambda\mu}}{\mu} d\mu + \int_{i\infty}^{-i\infty} \frac{e^{[\mu t/(1-\mu^2)^{1/2}] - \lambda\mu}}{\mu} d\mu \right].$$

Introducing  $\mu = ik$  this can be written as

$$1 - \frac{1}{\pi} \left\{ \int_0^\infty \frac{\sin \left[ \lambda k + \frac{kt}{(1+k^2)^{1/2}} \right]}{k} dk + \int_0^\infty \frac{\sin \left[ \lambda k - \frac{kt}{(1+k^2)^{1/2}} \right]}{k} dk \right\}.$$

\*For this transformation the author is indebted to R. Timman.

## DISCUSSION

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Dr. van Wijngaarden's paper introduces several interesting new ideas, in particular that a suspension of gas bubbles in liquid is a *dispersive* medium for the propagation of pressure waves and that nonlinear effects may compete with the effects of frequency dispersion. I noticed a fairly complete analogy between this problem and the problem of nonlinear dispersion of long water waves, about which an extensive body of theory is available, and I shall now outline an alternative treatment which brings out this helpful connection. The analysis applies only to waves whose length is considerably greater than the characteristic length  $h/\lambda = c_0/\omega_B$ , where  $c_0$  is the sound velocity given by (1.2) in the paper and  $\omega_B$  is the bubble resonance frequency given by (1.1). But within this limitation a full account of essential wave behavior can be made quite easily.

### INFINITESIMAL LONG WAVES

Let  $x$  and  $t$  denote dimensional distance and time, and define corresponding dimensionless variables

$$X = x/h; \quad T = c_0 t/h. \quad (1)$$

Here  $h$  is merely understood to be some overall length typical of the physical system, but the specific meaning given to  $h$  in the paper is of course included in this more general definition. In terms of  $X$  and  $T$ , the linearized Eq. (2.16) becomes

$$\frac{\partial^2 p}{\partial X^2} - \frac{\partial^2 p}{\partial T^2} + \frac{1}{\lambda^2} \frac{\partial^4 p}{\partial X^2 \partial T^2} = 0, \quad (2)$$

where

$$\frac{1}{\lambda^2} = \left( \frac{c_0}{\omega_B h} \right)^2 \quad (3)$$

is a parameter representing the relative importance of dispersion. For long waves, as now in question, we have  $\lambda^{-2} \ll 1$ . In Eq. (2),  $p$  is the local mean pressure in the liquid-bubble mixture, but clearly in the linearized approximation every other dependent variable will satisfy the same equation.

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Regarding simple-harmonic solutions in the form

$$\exp [i(\alpha X - \nu T)] ,$$

we obtain from (2) the dispersion relation

$$\nu = \frac{\alpha}{[1 + (\alpha^2/\lambda^2)]^{1/2}} \quad (4)$$

(cf. (2.17) in the paper). This result shows that the phase velocity  $\nu/\alpha$  and group velocity  $d\nu/d\alpha$  are a maximum for  $\alpha \rightarrow 0$ , i.e., for very long waves; and we note that this is a property in common with infinitesimal surface waves on water of uniform undisturbed depth. According to a well-known principle which has often been used in water-wave theory, the front of any wave propagating into the undisturbed medium will eventually be composed only from Fourier components with group velocities near the maximum. Hence a good asymptotic approximation to the properties of the wave front may be obtained by taking a first approximation to (4) for small  $\alpha^2$ , thus

$$\nu = \alpha - \frac{1}{2\lambda^2} \alpha^3 , \quad (5)$$

and using this in a Fourier-integral representation of the wave (see H. Jeffreys and B. S. Jeffreys, "Methods of Mathematical Physics," Cambridge University Press, 1946, § 17.09).

For example, consider the solution  $p(X, T)$  of (2) that is initially a step function, such that

$$p(X, 0) \begin{cases} = 1, & X < 0, \\ = 0, & X > 0. \end{cases}$$

This solution is

$$p(X, T) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i(\nu T - \alpha X)} \frac{d\alpha}{\alpha} , \quad (6)$$

with the path of integration indented under the origin. Substituting (5) for  $\nu$  and then writing

$$\zeta = \left( \frac{3}{2\lambda^2} T \right)^{1/3} \alpha , \quad z = (X - T) / \left( \frac{3}{2\lambda^2} T \right)^{1/3} ,$$

we obtain as the asymptotic approximation

$$p \sim F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left[ -i \left( z\zeta + \frac{1}{3} \zeta^3 \right) \right] \frac{d\alpha}{\alpha} = \int_z^{\infty} \text{Ai}(z) dz , \quad (7)$$

where

$$\text{Ai}(z) = \frac{1}{\pi} \int_0^{\infty} \cos \left( z\zeta + \frac{1}{3} \zeta^3 \right) d\zeta$$

is the Airy function of the first kind. The result (7) represents a waveform advancing as a whole at unit velocity (note  $p$  keeps a constant value at  $X = T$ , i.e., at  $x = c_0 t$ ), but with a length scale increasing with time like  $T^{1/3}$  (cf. T. B. Benjamin and B. J. S. Barnard, "A Study of the Motion of a Cavity in a Rotating Liquid," *J. Fluid Mech.* 19:193 (1964), p. 205).

If  $\lambda^{-4}$  is negligible, the approximation (5) to the dispersion relation is reliable for any waves whose dimensionless length is  $O(1)$ ; and for waves traveling in the positive  $x$  direction, it is equivalent to  $p$  and the other dependent variables satisfying

$$\frac{\partial p}{\partial T} + \frac{\partial p}{\partial X} + \frac{1}{2\lambda^2} \frac{\partial^3 p}{\partial X^3} = 0. \quad (8)$$

We note, for example, that (7) is a solution of (8). Equation (8) may be regarded, therefore, as the most general governing equation in the case of one-way propagation by long waves of very small amplitude. The effects of dispersion are represented by the third term on the left side, which can be regarded as a first-order correction to the "acoustic" wave equation. We go on to find a corresponding correction for the effects of finite wave amplitude, and finally we combine the two.

## LONG WAVES OF FINITE AMPLITUDE

An exact nonlinear theory can be formulated very readily if one assumes  $O(\lambda^{-2})$  can be neglected and hence ignores the mechanism that causes frequency dispersion (i.e., the lag between local mean pressure  $p$  and the bubble pressure  $p_g$ ). Thus one takes  $p = p_g$ , implying that the local net density  $\rho$  of the mixture (which of course is less than the liquid density, say  $\rho_w$ ) is a function of  $p$  only. The mixture can then be treated as a continuum, by the same method as is familiar in elementary gas dynamics.

The equation of motion and the equation of mass conservation are (in terms of dimensional variables)

$$\rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \frac{\partial p}{\partial x} = 0 \quad (9)$$

and

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0. \quad (10)$$

As is well known, these two equations may be transformed to yield two in characteristic forms, thus



$$\left[ \frac{\partial}{\partial t} + (c + v) \frac{\partial}{\partial x} \right] (v + f) = 0 \quad (11)$$

and

$$\left[ \frac{\partial}{\partial t} - (c - v) \frac{\partial}{\partial x} \right] (v - f) = 0, \quad (12)$$

where

$$c = \left( \frac{dp}{d\rho} \right)^{1/2} \quad (13)$$

and

$$f = \int \frac{c}{\rho} d\rho = \int \frac{1}{\rho c} dp. \quad (14)$$

Equations (11) and (12) show that the "Riemann invariants"  $v \pm f$  propagate with constant values along the respective characteristics

$$\frac{dx}{dt} = v \pm c.$$

The net density of the mixture is given very closely by the expression

$$\rho = \rho_w (1 - nV), \quad (15)$$

in which  $nV = (4/3) \pi n R^3$  is the volume fraction occupied by the bubbles. Hence, assuming adiabatic compression or expansion of the gas in the bubbles, we have

$$\rho = \rho_w \left[ 1 - nV_0 \left( \frac{p_0}{p} \right)^{1/\gamma} \right], \quad (16)$$

where the zero suffix refers to the undisturbed state of the mixture. (Here an arbitrary value  $\gamma$  of the ratio of specific heats is represented, rather than the value  $4/3$  assumed in the paper.) Upon substitution of this relationship between  $\rho$  and  $p$ , (13) gives

$$c = c_0 \left( \frac{p}{p_0} \right)^{(\gamma+1)/2\gamma}, \quad (17)$$

where

$$c_0 = \left( \frac{\gamma p_0}{\rho_w n V_0} \right)^{1/2} = \left( \frac{3\gamma p_0}{4\pi \rho_w n R_0^3} \right)^{1/2} \quad (18)$$

(cf. (1.2) in the paper), and (14) gives

$$f = \frac{2\gamma}{(\gamma-1)} \frac{P_0}{\rho_w c_0} \left[ \left( \frac{P}{P_0} \right)^{(\gamma-1)/2\gamma} - 1 \right]. \quad (19)$$

(Note that the propagation velocity  $c$  increases much more rapidly with the pressure ratio  $P/P_0$  than it does when the whole fluid is a gas, in which case  $c = (\gamma P/\rho)^{1/2} \propto P^{(\gamma-1)/2\gamma}$ .)

The preceding exact equations for very long waves may be helpfully simplified if attention is restricted to fairly weak pressure pulses. Writing

$$\frac{P}{P_0} = 1 + P,$$

we assume  $P$  to be a fairly small fraction and hence obtain from (17) and (19) the approximations

$$c = c_0 \left\{ 1 + \left( \frac{\gamma+1}{2\gamma} \right) P \right\} \quad (20)$$

and

$$f = \frac{P_0}{\rho_w c_0} P. \quad (21)$$

Consider now a wave propagating in the  $x$  direction toward an undisturbed region. After a time the frontal parts of the wave will be covered by backward-going characteristics along which the invariant  $v - f$  has the value zero determined at the origin of these characteristics in the undisturbed region ahead of the wave. Thus we have  $v + f = 2f$ , and on substituting (20) and (21) we get from (11)

$$\frac{\partial P}{\partial t} + c_0 \left\{ 1 + \left( \frac{\gamma+1}{2\gamma} + \frac{P_0}{\rho_w c_0} \right) P \right\} \frac{\partial P}{\partial x} = 0. \quad (22)$$

Writing, for the sake of neatness,

$$Q = \left( \frac{\gamma+1}{2\gamma} + \frac{P_0}{\rho_w c_0} \right) P$$

and reintroducing the dimensionless variables  $T$  and  $X$ , we then have

$$\frac{\partial Q}{\partial T} + \frac{\partial Q}{\partial X} + Q \frac{\partial Q}{\partial X} = 0. \quad (23)$$

The third term on the left side of (23) is the required simple representation of the nonlinear effects.

Both Eqs. (8) and (23) have precise counterparts in water-wave theory, and various schemes have been developed in that context for *simultaneously* obtaining approximations to the effects of both frequency dispersion and nonlinearity.

By analogy, the outcome of such an investigation in the present instance is obvious: the "correction" terms in (8), rewritten as an equation for  $Q$ , and in (23) are simply added. Thus we have

$$\frac{\partial Q}{\partial T} + \frac{\partial Q}{\partial X} + Q \frac{\partial Q}{\partial X} + \frac{1}{2\lambda^2} \frac{\partial^3 Q}{\partial X^3} = 0. \quad (24)$$

This third-order nonlinear equation is often called the Korteweg-de Vries equation (see G. B. Whitham, "Nonlinear Dispersive Waves," Proc. Roy. Soc. A283: 238 (1965), p. 252), and a great deal is now known about its solutions. Its occurrence here indicates the possibility of several interesting wave phenomena in a liquid-bubble mixture that would be analogous to water-wave phenomena known to be described approximately by the equation. In particular, there appears the possibility of oscillatory shock waves (analogous to undular bores or hydraulic jumps) and of solitary waves (i.e., nonperiodic waves of finite amplitude but permanent form). Note that the solitary-wave solution of (24) is

$$Q = a \operatorname{sech}^2 \left[ \left( \frac{1}{6} a \right)^{1/2} \lambda (X - CT) \right], \quad (25)$$

with

$$a = 3(C - 1).$$

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## REPLY TO DISCUSSION

L. van Wijngaarden

I wish to thank Dr. Brooke Benjamin very much for his comments. His suggestion, substantiated by his analysis, that there is an analogy between long water waves and waves in a liquid-bubble mixture is most useful, both from a theoretical and a practical point of view. Both types of waves display indeed frequency dispersion and amplitude dispersion. However, upon looking deeper in this matter, it appears that whereas in the theory of water waves there are three characteristic lengths, viz., amplitude, wavelength and depth, there are four lengths here, namely wavelength, amplitude, bubble size, and inter-bubble distance. In the theory of water waves there are consequently *two* dimensionless parameters, one giving the ratio between amplitude and wavelength, the other the ratio between depth and wavelength. In the case of bubble-fluid mixtures there are *three* parameters. If e.g., bubble size plays the role of depth, then the third parameter is the ratio between bubble size and bubble distance. It appears that only when this ratio is of order 1, the analogy holds strictly. The above mentioned ratio is essentially the cubic root of the volume fraction  $nV_0$  of the gas. Now in some of Dr. Benjamin's equations this fraction is assumed to be of order 1 (Eqs. (9) and (10)), whereas in others it is assumed to be

small, for example in the derivation of Eqs. (19) and (21) from (16) and (17), where  $\rho$  is taken equal to  $\rho_w$ .

In fact (a)  $nV_0 = O(1)$  and (b)  $nV_0 \ll 1$  have to be considered separately. Inspired by Dr. Benjamin's comments I will publish the theory for both of these cases in the near future. Also other aspects of the comments, as for example the reduction to the Korteweg-de Vries equation will be dealt with in this forthcoming paper.

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# ON JETS AND SHOCK WAVES FROM CAVITATION

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Historically, the type of cavitation which consists of relatively small, transient cavities in a liquid first received attention by Euler (1) in the eighteenth century. Very likely he and others had it forcefully called to their attention because of its well demonstrated capability of causing extreme damage to solid surfaces such as turbine blades. This brief review will also concern itself primarily with those hydrodynamic aspects of cavitation which are believed to be of importance in the theory of cavitation damage.

The early theoretical treatments by Besant in 1859 (2) and Rayleigh in 1917 (3) quite naturally adopted the simplest possible model, namely that of a spherically symmetric cavity collapsing to very small or zero radius. It was easily found theoretically that the bubble wall velocity approached infinity as the radius approached zero. This result was readily seized upon as a firm explanation of the amazingly high pressures which would be required to account for the observed damage to solids. The pressure was, of course, expected to reach enormous values as the inward motion was arrested either by gas in the cavity or by wall to wall impact. The case of isothermal gas compression was included in Rayleigh's work as well as the empty cavity.

Some investigators at first refused to believe that the damage was not of an exclusively chemical nature, but their number has been steadily declining since the early part of this century. However, quantitative estimates of the pressure involved covered an extremely wide range varying from thousands (4) to millions (5) of pounds per square inch. For a while there was also a strong reticence to admit that both corrosion and erosion might play prominent roles depending upon cavitation intensity and length of exposure. This feeling has lately been largely dispelled due to good experimental work which is aimed at separating these effects (6). Of course no one ever really believed that pressures actually became infinite, but the problem was neatly bypassed by the convenient fact that something involving real fluid effects or thermodynamic behavior was always neglected.

An undoubtedly better approximation for spherical collapse than the Rayleigh theory was made by Gilmore (7) who included compressibility in his equations and put in surface tension and viscosity through boundary conditions. The main

result was merely that collapse velocity became infinite at a lower rate (proportional to the inverse one-half rather than the inverse first or three-halves power for the acoustic or incompressible cases, respectively). The Kirkwood-Bethe hypothesis (8) was used as a basis for Gilmore's theory, and this also has not been proved or universally accepted (9) for the conditions involved in cavitation. It should also be pointed out that the radiation condition (no incoming waves) which is assumed in this approximation is obviously not applicable when one considers the effects of nearby boundaries. If one is interested in the study of cavitation damage, then boundaries must be considered.

There has, in the past, been an unjustified confidence in assuming that cavitation damage was entirely caused by large-amplitude pressure waves in the liquid due to an isolated, collapsing spherical bubble. It is true that the possibility of instability of the spherical shape of a bubble has received a great deal of attention in the last fifteen years or so (10,11) but true instability of an initially spherical bubble isolated in an infinite medium was always the problem considered. This theoretical work was a very important and basic step in the progress of our understanding, but isolated bubbles by definition can hardly be expected to damage anything. Of course, this isolation is relative and is assumed only to make the theory more tractable under the assumption that the radiated shock wave could travel the distance from the "isolated" bubble and still be strong enough to do damage. A quantitative answer to this question is therefore required before the validity of the assumption can be properly assessed. Experimental answers are unfortunately still inadequate or dependent on other assumptions, although some progress has been made (12). The theoretical situation is somewhat better due to the efforts of many investigators (13,14,15) as long as a spherical collapse is assumed.

One of the most recent and best treatments is that of Hickling and Plesset (16). They make the assumption that the bubble contains a small amount of gas ( $10^{-3}$  or  $10^{-4}$  atmosphere pressure at maximum radius). They include compressibility and find numerical solutions including flow during rebound from a Lagrangian formulation. They find the acoustic approximation to be valid at high enough pressures that the peak intensity of the outgoing wave can be calculated. The results of these calculations are most informative and show that peak pressures are very sensitive to gas content. Typical results are that at a distance of the bubble maximum diameter from the collapse point the peak pressure is 200 atmospheres for  $10^{-3}$  atmosphere initial gas content and 1000 atmospheres for  $10^{-4}$  atmospheres initial gas content when the collapsing pressure at infinity is 1 atmosphere. In the author's opinion, these values are quite reasonable and should be expected to be confirmed by experiment if it proves possible to obtain a spherical collapse. They are sufficient to cause the typical peening appearance or depressions of relatively large curvature observed in soft materials and which cannot be reasonably explained by a jet impact mechanism because of this large curvature (17). They are also capable of causing "cold work" and ultimate fatigue damage in harder materials.

However, in view of the demonstrated violence of cavitation attack on some very resistant materials it does not seem likely that the shock wave damage mechanism is the only one. It might be argued that if the peak pressure has a  $1/r$  dependence, then the bubble need only collapse closer to the surface to



develop higher pressures. The trouble with this is that the assumption of spherical collapse ceases to be valid due to proximity of the solid surface. That this is true, at least for bubble collapse which is not flow generated, has been demonstrated by Naudé and Ellis (12). Very little work has been done on single bubble collapse in a flowing liquid with the notable exception of Maurice Rattray (18). His perturbation analysis breaks down during the later stages of collapse but does indicate that spherical symmetry is not maintained if the bubble is closer than about one maximum diameter from a wall or if the bubble has a relative velocity to the free stream such as one would expect in the presence of pressure gradients, which usually exist in cases of practical interest.

It appears, then, that the case of hemispherical collapse on a solid, which for ideal fluids is theoretically equivalent to spherical collapse in an infinite fluid, is the only case where spherical symmetry might be preserved and yet keep the collapse point near or on the solid. Some experimental work has been done by Jones and Ellis (19) using a quartz force gage and simultaneous high-speed photography. The big objection to their work was that the bubbles were spark generated and thus contained relatively large and unknown amounts of permanent gas from electrolysis. Despite this, a lower bound of about 10,000 atmospheres pressure was obtained. This was in agreement with work done by Jones and Edwards (20) without benefit of high-speed photography.

Although the hemispherical mode of collapse is undoubtedly very damaging, such experimental observations as are available indicate it is not a very likely occurrence either in flow (21) or acoustically generated (22) cavitation. It appears much more common, at least for pressure distributions on ogival bodies such as were used by Knapp and Hollander (23) and Plesset (24) and Ellis (21), that when the cavity is in actual contact with the solid it may be of roughly spherical shape but it is likely to be either less or more than a hemisphere (i.e., the center of curvature is within or without the solid surface). Such examples are shown in Fig. 1 from the work of Ellis (21). A theoretical and experimental study of the latter type but in the absence of flow was made by Naudé and Ellis (12). The method of bubble generation used was a spark, but as was indicated theoretically, high-speed photography showed that a jet developed and struck the solid before there was enough cavity volume reduction to increase the internal pressure significantly. It was thus felt that spark generation with resultant permanent gas was not objectionable for this mode of collapse, whereas it would be for the spherically symmetric mode. Figure 2 shows the collapsing jet. Although the photographs in this figure had been taken at the time of the original paper, they were not published until later (25). If one takes a conservative estimate of the jet velocity by assuming that it has just reached the solid in the next frame, a value of about 600 feet per second is obtained. The impact pressure would then be about 3000 atmospheres. This is about a third of the experimental estimate for the hemispherical bubble of Ref. 19, but it is quite adequate to pit aluminum, and indeed this was done, as shown in Fig. 3.

It should be mentioned that in similar subsequent work by others (26) no jet damage was observed but instead cavitation damage due to vortex generated cavitation was evident. In the author's opinion this did nothing to disprove the jet mechanism because of a simple difference in the spark generator. The discharge was from a 0.02-microfarad capacitor in the case of Naudé and Ellis and



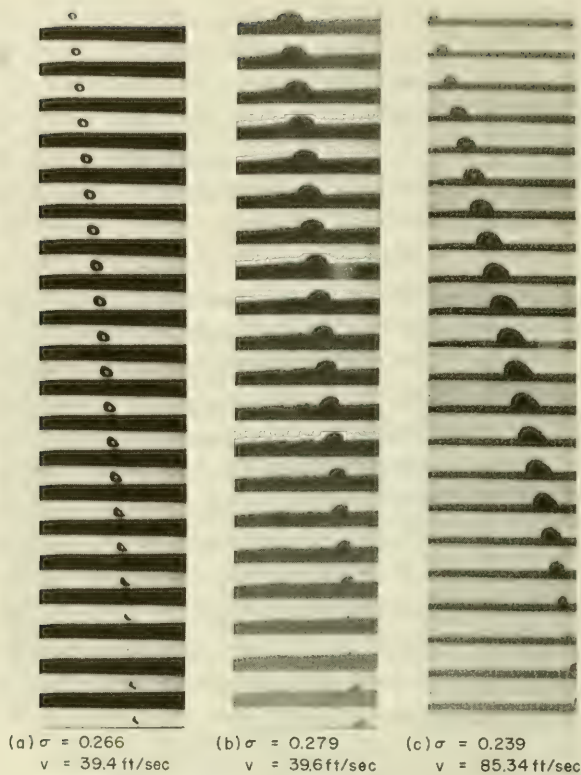


Fig. 1 - Cavitation on 1.5-caliber ogive  
(10,000 frames/sec; 0.75X)

was 1.0 microfarad in the case of Shutler and Mesler. Other things assumed similar, this means that the period of the electrical discharge was fifty times longer and therefore the hot arc was maintained during most of the bubble history. This conclusion is born out by the photographs which show the arc. It does not seem reasonable to expect the jet to pass through the hot gas with any vigor, and in fact it might try to move out to some radial distance and do peripheral damage. However, a maintained arc should also be an excellent vortex generator, and it is probably quite true that much damage producing cavitation, especially in turbomachinery, has been vortex generated.

One should be very careful not to assume that all damage is done by jets and none by shock waves, however. It is a natural tendency to associate shock waves with spherically symmetric collapse and then to discard the idea of damage from shock waves when one finds that such collapse near a boundary is an extremely unlikely event. The point to be emphasized is that shock waves can arise from nonsymmetric impact of portions of the walls of a cavity with other portions or with solid boundaries. In the former case the shock wave may still

Fig. 2 - Spark bubble with a collapsing jet within it. The thin line above the bubble is the path of two tungsten wires which carried the spark current (100,000 frames/sec; 2.5X)

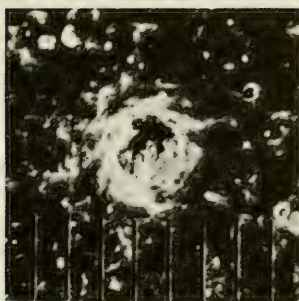


Fig. 3 - Pit left by single bubble collapse (1 division is 0.004 inch)

travel to the boundary and do damage. In the latter the damage would be done by the impacting wall which may or may not be in the shape of a jet. Figure 4 shows a spark-generated bubble from unpublished work of Naudé and Ellis which collapses unsymmetrically and yet generates a shock wave. The origin of the shock wave is obviously not at the solid surface and hence must be at a colliding liquid-liquid interface. The lack of sharpness of the shock is due to the relatively long exposure time of about 0.1 microsecond in this case. It should be emphasized that the behavior of spark-generated bubbles in the absence of a flow was not necessarily expected to duplicate conditions of flow-generated cavitation. They were studied mainly because theoretical treatment was more tractable than the general case of collapse in a flow field, but it was felt that

such general features as jets might well exist in the latter case as well.

The brief outline of cavitation damage research given here is merely intended to show motivation for work presently under way. Very complete and excellent summaries are available elsewhere, notably those of Flynn (27) and Eisenberg and Tulin (28).

It was felt by the author that a logical attack on the complete problem should first consider those situations where comparison of theory and experiment might be possible though difficult. Toward this end, experimental methods of generating and observing collapsing cavities both in flow and in stationary liquid were studied. The first step was to duplicate the Naudé-Ellis work but in a flowing system. Accordingly a glass-lined blow-down type of water tunnel was constructed which has been described elsewhere (29). This tunnel was designed to provide velocities of over 100 feet per second through a 2-3/4-inch-square working section. Very-low-speed flows can also be generated covering a Reynolds number range from zero to over two million. Initial cavitation experiments included an initially hemispherical spark-generated bubble at the low pressure point of a circular-arc hydrofoil. The radius of curvature and chord length was chosen such that the ideal fluid pressure distribution would correspond to that of the 1.5-caliber ogive of Fig. 1, since this distribution had yielded large well-defined bubbles in the axially symmetric case.

Figure 5 shows the resultant behavior. It can be seen that even though the initial shape is hemispherical the shape near collapse is not. The center figure shows a shape very much like a theoretical shape calculated by Naudé and Ellis for an initially nonhemispherical bubble without flow. The situation 10 microseconds later (bottom figure) is seen to consist of a smaller and more asymmetrical bubble remnant and a radiated shock wave. Although rather speculative, it seems likely that the presence of flow caused a perturbation from the hemispherical shape to develop in later stages and that a jet occurred, hitting the solid at an



Fig. 4 - Bubble showing asymmetrical collapse and shock wave centered off the surface (200,000 frames/sec)



angle pointing downstream. At least in the no-flow case, theory indicates that if a bubble is shaped like in the center figure, then a jet will develop.

The theory for the flow situation is not yet complete, but it was felt that observations on an initially greater than hemispherical bubble in a flow would also be helpful. Tungsten wires were extended into the stream from the minimum pressure point of the circular foil, and the resultant sequence is shown in Fig. 6, where the center and lower frames occur just before and just after collapse. In this case, the shape in the center frame differs mainly from the no flow theory in the development of an upturned tail on the downstream side. No explanation has yet been advanced for this effect, although it seems to occur rather



Fig. 5 - Initially hemispherical spark bubble generated at the minimum pressure point of a circular-arc hydrofoil ( $\sigma = 0.3$ ;  $v = 30$  ft/sec; 2.5X)



Fig. 6 - A spark bubble initially greater than hemispherical (approximately the same conditions as in Fig. 5)



consistently for this type of bubble generation. Figure 6 also indicates that shock waves from liquid-solid impact occur, since the collapse could hardly be called spherically symmetric enough to cause appreciable gas compression.

The preliminary work just described on single bubble collapse in flow was done by the author and Dr. B. J. S. Barnard just before both left for a year at the University of Cambridge. Subsequent work done there in collaboration with Dr. T. Brooke-Benjamin was on the collapse of single bubbles in the absence of flow but with a new technique for bubble generation (30). In the author's opinion this new method is much superior to spark generation for most experiments. The basic idea was quite simple, but much time was spent in mastering the technique required. The method was to generate a single hydrogen bubble by electrolysis to act as a nucleus for the desired relatively large spherical cavity. The hydrogen bubble, which is usually chosen to be about  $10^{-3}$  cm in radius, at  $1/20$  atmosphere pressure, is subjected to a tension wave in the liquid which causes it to grow to typically 2 cm in radius, if desired. Thus, the volume dilution of the gas is of the order of 1 part in  $10^{10}$ . Since the growth occurs in a few milliseconds and the water must have been deaerated previously to remove extraneous air nuclei, there is little chance for additional gas to diffuse into the cavity. Under these conditions the bubble should contain less gas than the usual flow-generated cavitation bubble in nondeaerated water. The gas content can be increased, of course, and the amount present is known from the observed size of the nucleus bubble at known pressure and temperature. Figure 7 shows a typical bubble of about 1.5 inches diameter. A scale in the background with large divisions of one inch may be seen.

With such large controlled cavities available it became possible to conduct experiments that shed new light on the modes of bubble collapse. Time was not available for study of any but the no-flow behavior. However, the effects of pressure gradient and solid wall proximity were observed. It was found that in a gravity hydrostatic pressure gradient the collapse was nearly symmetric except for a flat on the bottom (high-pressure side) forming very nearly at minimum volume. Evidently this was the beginning of a vertical jet, since pictures taken after collapse and during rebound exhibited the rather surprising behavior evident in Fig. 8, which also appears in Ref. 30. A very thin jet has apparently



Fig. 7 - An example of a spherical bubble generated by the tension technique

Fig. 8 - Rebound of a bubble in a pressure gradient showing the resultant jet (4X)



formed and impacted the opposite side of the growing rebounding cavity. This behavior leads to speculation that rebounding cavities previously thought harmless might indeed be capable of damage if conditions were just right. Some evidence for damage during rebound has been cited by Shutler and Mesler (26), but they ascribed it to other causes than a jet.

To confirm that the jet was due to a pressure gradient pictures were taken by Dr. T. Brooke-Benjamin and the author when the entire liquid and container was in free fall. No jet was observed in this case.

The effect of a plane solid boundary at various distances was also investigated and yielded results as shown in Figs. 9, 10, and 11. In Fig. 9 the cavity has developed an elongated shape as predicted theoretically by Rattray (18), but in addition a jet is seen to develop at the end farthest away from the rigid wall and to travel toward the wall. Obviously the wall was too far away to be damaged. Figure 10 shows a bubble closer to the boundary. The jet is seen to develop at an angle both to the wall and to the vertical as should be expected since both effects were present. (The container was not in free fall in this case.) Figure 11 shows a bubble still nearer the boundary, which is of interest because of the relatively thin and rapid jet which has formed and impacted the wall between the two consecutive photographs at the left in the bottom sequence. A lower bound on the impact pressure in this case is about 8000 psi, but if the collapsing pressure had been as high as one atmosphere, which is conservative in practice, then pressures of at least 40,000 psi should be expected. This is of the same order as the pressures estimated from the previously mentioned experimental piezoelectric data of Jones and Ellis for hemispherical collapse of a spark generated bubble.

The experimental work of Benjamin and Ellis at Cambridge thus revealed some new modes of bubble collapse, notably that jets could occur in rebounding

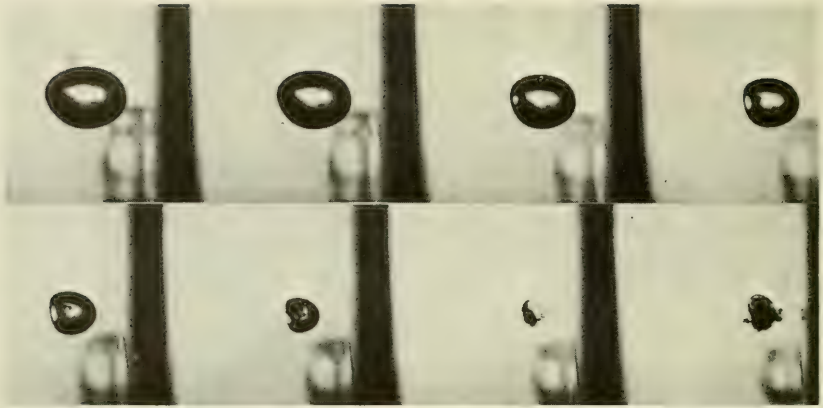


Fig. 9 - Bubble collapsing at approximately one maximum diameter from the wall showing initial elongation normal to wall and subsequent jet formation (5000 frames/sec; actual size)



Fig. 10 - Collapse of a bubble touching a wall (to the left) and in a vertical hydrostatic pressure gradient. The jet axis angle is about midway between vertical and the normal to the wall (5000 frames/sec; 2X).

bubbles in a pressure gradient and that jets occurring during collapse due to solid wall proximity were sometimes much thinner than could be treated by the perturbation theory of Naudé and Ellis. In addition, theoretical considerations showed that for many deformable shapes such as may be assumed by bubbles, the kinetic energy of the liquid may advantageously be written in terms of bubble deformation rates and speed of the centroid. In this way the importance of momentum conservation and the Kelvin impulse in considering cavitation damage mechanisms become more apparent. It appears that there are many more factors to be considered in studying the hydrodynamics of actual cavitation damage than have been considered in the spherically symmetric case. One oversimplified but intuitively satisfying point of view is to regard a bubble moving toward



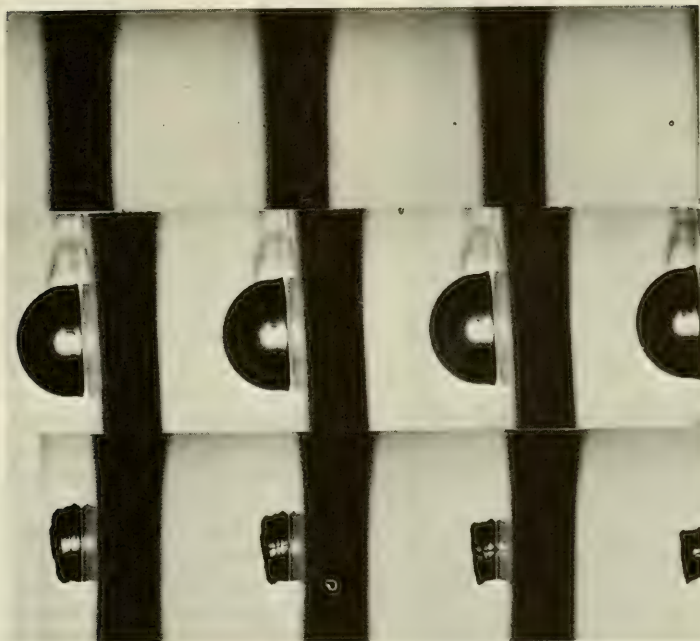


Fig. 11 - Sequences taken at beginning, middle, and end of a film at 5000 frames per second showing a thin, high-velocity jet which has formed and hit between the first two frames of the last sequence (actual size)

a wall much as a massive but deformable missile ready to deliver a definite impulse by virtue of its translational velocity. Concentration of this impulse over a small area of the wall through the jet mechanism will depend critically on a large number of factors, especially the location of stagnation points in the flow and the flow velocity. Such dependence might well explain the observations by Knapp (17) that cavitation damage in the vicinity of a re-entrant flow stagnation point was extremely sensitive to flow velocity (he stated it varied as the sixth power of the velocity). He and other investigators have also found that an extremely small fraction of the bubbles present actually did damage. This would fit in well with the ideas presented here. Of course it should be noted that the possibility of damage by shock waves has not been ruled out, and the mechanism considered here is offered in addition to and not in place of the shock-wave hypothesis. However, shock wave generation is probably an infrequent event itself; and if only spherical collapses were effective, it would be more infrequent still.

In view of the many new qualitative observations and ideas it seemed most desirable to construct new equipment at the California Institute of Technology in which single bubbles could be generated and subjected to a known and repeatable





Fig. 12 - Cavitation on the end of a 15-kilocycle magnetostriction transducer. Note that shock waves radiate from centers on the interface (300,000 frames per second; 15-nanosecond exposure time; taken with repetitively pulsed ruby laser using back illumination; 3.6X).

pressure field. In addition, it was desired to use several atmospheres pressure to collapse the cavity and thus be able to actually damage materials. At Cambridge the objective was to use low pressures to slow down the collapse and permit easier observation. This, of course, also limited the damage producing capabilities of the collapsing bubble. At Caltech high-speed photographic equipment has been especially developed (31) to facilitate observation of more rapid events. Figure 12 shows one of the first high-speed movie sequences taken with ruby laser illumination. The subject is the specimen end of a magnetostriction type cavitation apparatus operating at about 15 kilocycles. The quality is poor mainly due to excessive scattering of the beam by scratches, etc., on the Lucite window. However, the relative sharpness of the cavitation-generated shock waves are of interest. This is due to the shortness of the exposure time, which in this case is about 15 nanoseconds. The interval between pictures is about 3 microseconds, and resulting measurements of the shock velocities indicate they are not much greater than sonic. Since this was merely a test sequence, quantitative data was not taken, but it is of some interest because all of the shock waves appear to originate on the surface and what can be seen of the surface cavitation appears very unsymmetrical.

However, other pulsed laser photo-

graphs such as Fig. 13 have been taken of cavitation generated by a barium titanate ring apparatus (22), and some of these cavities do appear to collapse to small volume and radiate shock waves. It may be of significance that damage by the barium titanate ring apparatus is less severe than by the magnetostriction apparatus (32,33).

Fig. 13 - Acoustic cavitation at 18 kilocycles generated by a ceramic ring method. Note the shock waves from small collapsed bubbles away from any interface. (Photographic details are the same as in Fig. 12.)



The equipment recently set up at Caltech to continue single-bubble-collapse studies at higher and more completely controlled pressures is shown in Fig. 14. It consists of a 4-inch-diameter Lucite cylinder about 5 feet long mounted on an electromagnetic shaker of 5-kilowatt power rating. The relatively high rating permits easy degassing of the water by vertical sinusoidal motion of the entire column without the necessity of operating at discrete resonant frequencies of the apparatus. To grow larger cavities from small gas nuclei a large downward acceleration of the tube (about 30 g) is obtained by discharging a capacitor through the energizing coil of the shaker by firing an ignitron rectifier. Since the ignitron passes current in one direction only, a half sine wave of current results,

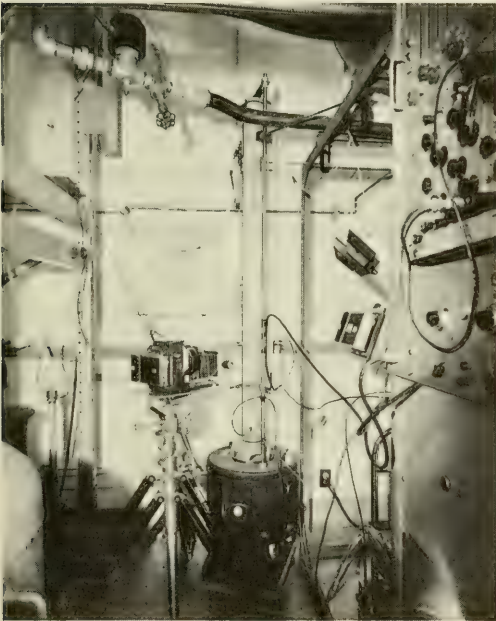


Fig. 14 - Electromagnetic shaker and vertical liquid column. The bubble nucleus generator is close to the bottom of the tube.

and the accelerometer record is much the same except that over a relatively long period the time integral of the acceleration must be zero. A typical record is shown in Fig. 15. The sweep rate is 2 milliseconds per large division, and the spike on the upper oscilloscope trace indicates when a photograph was taken of the cavity under observation.

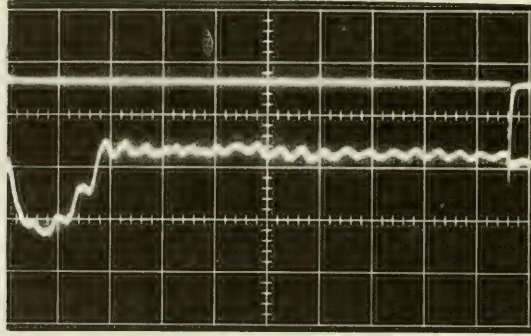


Fig. 15 - Oscilloscope record of vertical acceleration of the tube in Fig. 14. The grid calibration is 25 g/cm and 2 msec/cm horizontally.

Typical cavities are shown in Fig. 16. The magnification is about four times, but since flat windows had not yet been installed in the tube, there was an apparent flattening in the horizontal direction. The cavity at the top is actually quite spherical. The second and third photographs show flattening of the bottom, and after collapse the vertical jet and later the vortex ring may be seen. This is in accord with the observations made at Cambridge.

In conclusion it may only be said that the quantitative aspects of the work are still being performed and that it is hoped that they will be presented at some later time. The generalities which have been discussed should then appear in sharper perspective.

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Fig. 16 - Typical bubble photographs. Note the jet and vortex ring cavitation in the last two frames. Distortion is present due to photography through the cylindrical tube.



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\* \* \*

## DISCUSSION

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The author is to be congratulated on his very perceptive review of the present situation regarding cavitation damage mechanisms as well as upon his excellent photographs of bubbles in various stages and modes of collapse. Dr. Ellis's extensive contribution to this area of research over the years is too well known to require elaboration here.

However, I do question his implication that large curvature, i.e., shallow craters as observed by Knapp (Ellis, Ref. 17) in soft aluminum cannot be caused by jets rather than shock waves. Figure D1 of this discussion shows profiles of shallow craters produced in austenitic stainless steel by a water jet with a velocity of about 4000 ft/sec (reproduced from Robinson and Hammitt\* for convenience) compared with profiles of cavitation craters produced on the same type of material in a cavitating venturi using in one case water and in the other mercury. The impact profiles are due to DeCorso.† For all these pits the diameter-to-depth ratio is of the order of 100. Thus the pits are indeed relatively very shallow, although this fact is obscured by the magnified depth scale used in the figure. According to DeCorso the jet diameter is about 1/4 that of the resulting pit, an approximate value also confirmed by Brunton‡ using lead as a target and a somewhat higher water impact velocity (~5000 ft/sec). We estimate from DeCorso's data as well as that of Engel§ that the jet velocity necessary to create the cavitation pits shown is about 4000 ft/sec for water and 600 ft/sec for mercury. This is not inconsistent with our experiment,\* since the venturi throat velocity used for the water tests was typically about 6 times that for the mercury tests due to equipment limitations.

As shown by Dr. Ellis's Fig. 11 especially, jets of a diameter very small compared to that of the bubble from which they form are observed and may well be an important damaging mechanism. This formation of a "microjet" from an accelerating concave liquid surface is very reminiscent of the Monroe jets observed by Bowden• in his jet impact experiments, and also recently observed in

\*M.J. Robinson and F.G. Hammitt, "Detailed Damage Characteristics in a Cavitating Venturi," ASME Paper 66-WA/FE-39, to be published Trans. ASME, J. Basic Engr.

†S.M. DeCorso, "Erosion Tests of Steam Turbine Blade Materials," ASTM Proc., ASTM, Vol. 64, pp. 782-796.

‡J.H. Brunton, "The Physics of Impact and Deformation: Single Impact," Phil. Trans. Roy. Soc. A260(No. 1110):79-85 (July 28, 1966).

§O.G. Engel, "Pits in Metals Caused by Collision with Liquid Drops and Soft Metal Spheres," Research Paper 2985, J. Res. Natl. Bur. Stds. 62(No. 6):229-246 (June 1959).

•F.P. Bowden, "The Formation of Microjets in Liquids Under the Influence of Impact or Shock," Phil. Trans. Roy. Soc. A260(No. 1110):94-95 (July 28, 1966).

## Jets and Shock Waves from Cavitation

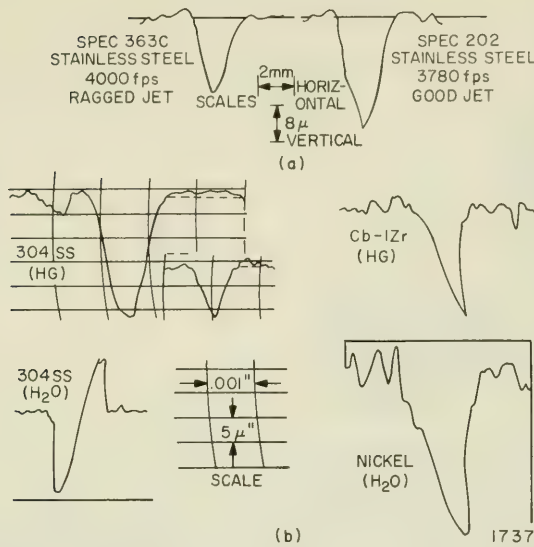


Fig. D1 - Comparison of water jet impact damage with cavitation pit profiles  
 (a) water impact profiles from DeCorso,  
 (b) cavitation pit profiles from Robinson

our own laboratory from a somewhat similar experiment (Fig. D2).<sup>\*</sup> Since the cavitation microjets also emanate from an accelerating concave liquid surface, is not the similarity perhaps more than coincidental?

Finally, as shown by Dr. Ellis's Fig. 9 and also in various pictures of non-symmetrical collapses which we have obtained<sup>†</sup> there is an indication of a protuberance both on the side of the presumed jet penetration and on the opposite side. We have interpreted these to be local cavitation generated by the passage of a high-velocity jet through a relatively quiescent liquid. Thus there is an indication of local high liquid velocities both on the side toward which the jet is traveling as well as that from which it has come. From simple considerations of continuity this does not appear to be necessarily unreasonable.

<sup>\*</sup>F.G. Hammitt, J.F. Lafferty, M. Pitek, P. Drucker, H.G. Olson, and M.J. Robinson, "A Facility for the Study of High Velocity Jet Impact," ORA Technical Report 08153-2-T, Nuclear Engineering Department, University of Michigan.

<sup>†</sup>H.G. Olson, "High-Speed Photographic Studies of Ultrasonically-Induced Cavitation and Detailed Examination of Damage to Selected Materials," Ph.D. Thesis, Nuclear Engineering Department, University of Michigan, Aug. 1966. (Also available as ORA Technical Report 07738-2-T, University of Michigan, Aug. 1966.)



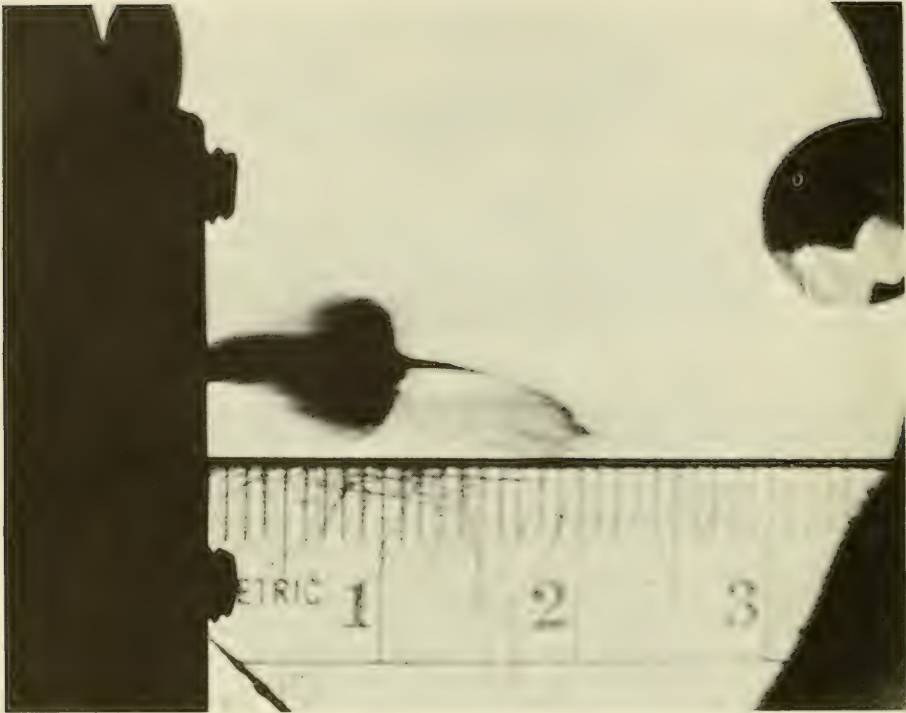


Fig. D2 - Monroe-type jet from an air-gun  
momentum exchange device

\* \* \*

## DISCUSSION

R. Hickling

*General Motors Research Laboratories  
Warren, Michigan*

Professor Ellis's work over the past few years has clearly indicated the essentially asymmetric nature of cavity collapse and the undoubted role that liquid impact plays in the cavitation damage process. The lack of symmetry in the collapse of a cavitation bubble, however, does not necessarily imply that centers of high compression cannot occur within the collapse. One can easily imagine an irregular nonspherical collapse that may involve several centers of high compression as well as the liquid impact effects discussed in Professor

Ellis's paper. It is necessary, of course, to have such centers of high compression in order to generate strong shock waves.

Centers of high compression are also required to explain another well-known phenomenon associated with cavitation, namely, the faint luminescence that is emitted from cavitation, otherwise known as sonoluminescence. Currently it is believed that this emission is thermal in origin and is caused by the rapid compression of small traces of permanent gas in the collapsing bubbles. Temperatures as high as  $10,000^{\circ}\text{K}$  are considered likely. If the effects of heat conduction and vapor condensation are considered, very high compressions indeed would be required to reach such temperatures. Observations have been made of the luminescence from the cavitation cloud at the end of a magnetostrictive transducer.

I am indebted once more to Professor Flynn for Fig. D3 that he took of such a cloud through a light intensifier. The intensification factor is of the order of  $10^4$  and the exposure time is  $1/5$  sec. The transducer surface is



Fig. D3 - Intensified view of luminescence emitted from cavitation

oscillating in water at 25 kHz. This is, of course, a rather familiar photograph to some of you but it gives, I believe, a possible indication of the density of centers of high compression that may exist within a cavitation cloud.

If we accept that such centers of high compression exist, do they occur close enough to damage an adjacent solid surface with shock waves emanating from the rebound? Could a highly compressive collapse occur on a surface? It would be interesting, I believe, to look at the emission from a collapse on or near a solid surface, of single bubbles that contain a sufficient proportion of monatomic or diatomic gas.

In the calculation of the intensities of shock waves emanating from a spherical collapse center, by Professor Plesset and myself, a fairly elaborate procedure had to be followed for each example. We were satisfied, therefore, to establish order-of-magnitude results for an adiabatic gas inside the bubble without considering more realistic models that would allow for heat conduction and vapor condensation. It was pointed out in our paper, however, that these effects would induce a much more violent collapse than was shown in the calculated examples, and that the resulting shock waves under these more realistic conditions should certainly be strong enough to account for any known type of cavitation damage.

These results coupled with the apparently high density of centers of high compression in a cavitation cloud would make it appear reasonable to believe that the shock-wave mechanism of cavitation damage is not a negligible possibility. I believe that the centers of high compression and the liquid impact effects discussed by Professor Ellis both have a role to play in the cavitation damage process. Other mechanisms have also been discussed. It seems clear that cavity collapse is a rather complicated process. I believe that it would be an error at this time to single out any one hydrodynamic effect as the principal cause of cavitation damage.

\*     \*     \*

## DISCUSSION

D. C. Gibson

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*(presented at the symposium by T. B. Benjamin)*

During recent experiments with spark-generated cavities in water, a curious phenomenon was observed which apparently has not before been put on record. The phenomenon can be seen in the accompanying sequence of frames from a high-speed ciné film, showing a cavity at four instants during the rebound that followed its first collapse (Fig. D4). To generate this cavity, a 0.725- $\mu$ f capacitor at 4.5 kV was discharged through a spark between the aligned thin electrodes that appear in the photographs. The cavity was formed about

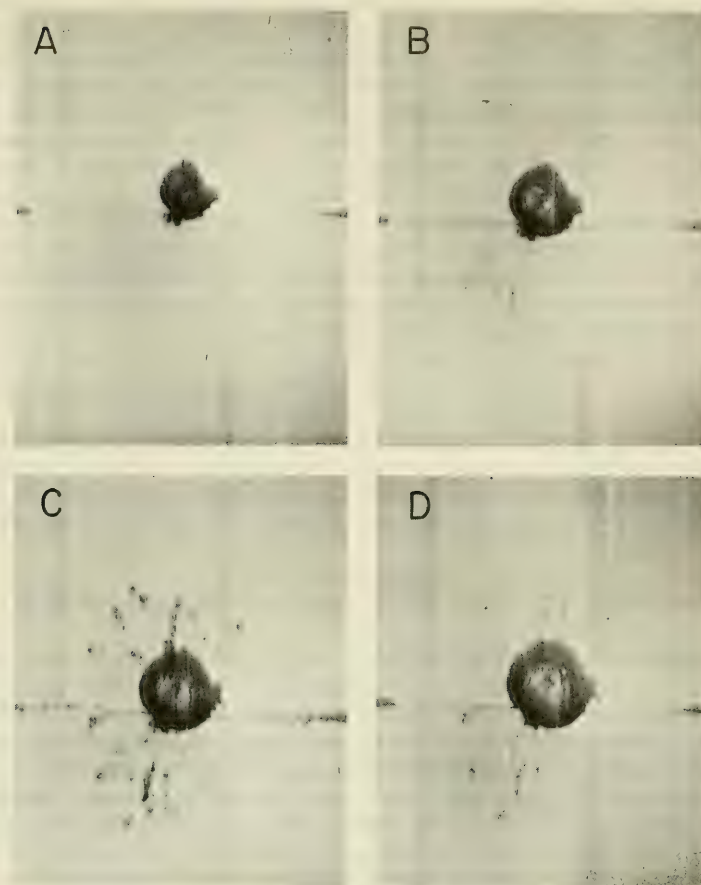


Fig. D4 - Cavity during rebound

20 cm below the free surface of the water, and the walls and bottom of the container were at comparable distances away. The ambient pressure and the vapour pressure of the water were respectively 82.5 and 14.5 mm Hg absolute. Earlier frames from the film show that the cavity initially expanded to a radius of 1.76 cm and then collapsed in the familiar manner, with enormous rapidity in the final stages of contraction. Frame A was taken about 0.15 msec after the instant when the collapse was arrested, and the others were taken later at successive intervals of 0.125 msec (i.e., the framing rate was 8000 per second). The mean radius of the cavity when recorded in frame D is estimated to have been 0.55 cm.

The feature to which attention is drawn is that whereas in frame A the water surrounding the cavity appears transparent, in frame B it appears speckled by a multitude of minute bubbles. These have grown to much greater



prominence in frame C, but frame D shows them finally waning. It seems most likely that this phenomenon was an effect of the shock wave arising from the first collapse of the main cavity. Presumably, after arriving at the free surface as a compression wave, this was reflected as a wave of tension, which subsequently exploded an array of microscopic nuclei somehow accumulated in the vicinity of the main cavity. The main evidence in favour of this explanation is that the time from the end of the collapse to the first appearance of the effervescence was approximately the same as the time of transit at the speed of sound from the cavity to the free surface and back. The evident abundance of nuclei near the rebounding cavity is not yet understood; but possible sources may have been, first, electrolysis accompanying the spark discharge and, second, the remains of needlelike protuberances of the cavity surface formed during its initial, explosive expansion around the spark track.

\*       \*       \*

## REPLY TO DISCUSSION

A. T. Ellis

### REPLY TO PROFESSOR HAMMITT

Professor Hammitt's view that the craters shown by Knapp (17) could be caused by jets rather than shock waves is well taken. Reexamination of the pit-to-bubble diameter ratios quoted in that paper confirm this. However, from personal observations made at the time I would not like to exclude the possibility of depressions caused by shock waves. This point is now being investigated.

The author has believed for a long time that Monroe jets are very closely akin to the jets which occur during bubble collapse. The details are different from the usual shaped-charge theory because inertial forces more complicated than those of a thin liner are involved and because one does not rely on a passing detonation shock front to determine initial speed of the interface. Current work does show that the concave liquid surface and the sign of the acceleration are important. Dr. Hammitt is certainly correct in suggesting that the similarity is more than coincidental.

It should be pointed out that there is a thin jet in Fig. 11 but not, for example, in Fig. 2. The jet is thicker in Fig. 2 and moves toward the solid as the Naudé-Ellis theory predicts, but there is no present theory which would describe the thin jet in Fig. 11.

During cavity collapse there are likely to be tensions generated locally by high flow velocities and accelerations. Professor Hammitt's suggestion that additional cavitation is generated by local flows appears quite reasonable.

REPLY TO DR. HICKLING

It was not meant to imply that centers of high gas compression could not occur in an asymmetric collapse. However, it seems reasonable that a given cavity would not be as effective in this regard after being broken up into a number of smaller ones.

Figure 4 was introduced to illustrate the fact that there can be shock waves generated from more than one center during a bubble collapse. This is evident from the two shock fronts in Fig. 4. These shocks were assumed to be formed from liquid-liquid impact because of the evident failure of the bubble to collapse completely. It should also be pointed out that light flashes are produced by a liquid jet striking a solid surface.\* The source of this light is still in question, but even if it does come from gas compression, it is compression caused by an impacting jet.

I agree with Dr. Hickling that cavity collapse is a complicated process and that there are several hydrodynamic effects involved in cavitation damage. I do believe that sufficient evidence has been accumulated to show that jet impact is a major cause of damage even if it is not the only one.

REPLY TO D. C. GIBSON

I particularly want to thank Mr. Gibson and Dr. Benjamin for their illustration of the effectiveness of shock waves from bubble collapse in generating new cavities. I have long felt that this might be an important mechanism in cavitation hysteresis. My experience, especially in acoustically generated cavitation, is that the presence of one cavitation cloud is very effective in starting another, even at some distance away. I have also noticed secondary bubble formation both from spark generated bubbles and bubbles originating from gas nuclei in a varying pressure field. I agree completely with the explanation given, especially since the effect also seems to exist with other than spark generated bubbles.

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\*S.M. DeCorso and R.E. Kothmann, "Erosion by Liquid Impact," Symposium on Erosion and Cavitation, ASTM STP 307, Am. Soc. Testing Materials, 1962; S.M. DeCorso and R.E. Kothmann, "Characteristics of the Light Flash Produced Upon Impact of a Liquid with a Surface," Materials Research and Standards, Oct. 1965, pp. 525-528; J.H. Brunton, "High Speed Liquid Impact," Phil. Trans. of the Royal Soc. of London, Ser. A, Vol. 260, pp. 79-85, July 1966.

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# THE INFLUENCE OF THE TRAJECTORIES OF GAS NUCLEI ON CAVITATION INCEPTION

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## INTRODUCTION

As discussed in Ref. 1, it is understood that cavitation is the result of the unstable growth of gas volumes caused by the reduction of the pressure surrounding these "nuclei." The nuclei may be of three types: (a) quasi-spherical entrained gas bubbles, (b) gas volumes stabilized in the crevices of entrained solid particles, or (c) gas volumes stabilized in the crevices of the boundary material. This report is directed to the first of these nuclei — the entrained gas bubble. As pointed out in Ref. 2, this type of nucleus is probably the important one in most practical cases.

As pointed out in Ref. 1 spherical gas nuclei may be characterized by definite conditions of instability. For a given ambient radius a pressure exists below vapor pressure for which a bubble will grow in an unstable manner. In most practical cases, the times to which such nuclei are subjected to the critical pressure are adequately long (greater than a few microseconds) so that conditions for instability are adequately defined by the results of simple static equilibrium theory. Consequently, for this type of nucleus, the critical cavitation inception pressure may be readily determined. If the minimum pressure in the flow field is also prescribed, the conditions for cavitation inception should also be prescribed.

In flow where the boundary layer is not separated the minimum pressure can be determined, and thus if the nuclei sizes present in the flow are known, the condition for inception should be determinable. However, experience has indicated that scale effects exist even in water tunnels where nuclei are of such a size that critical pressure variations should not be expected. It is speculated that in such cases the forces acting on bubbles entrained in the free stream may tend to move the larger bubbles away from the body and thus leave only smaller bubbles with more negative critical pressures to be active in the inception process. The forces acting in this screening process are such that the screen will be graded according to the relative size of the body, the speed of the flow, and the oncoming nuclei sizes present in the flow.



To achieve at least an approximate quantitative understanding of the screening process in which the body and its pressure gradients influence the nuclei distribution in the flow field, we have set up and solved the following problem. We assume a size distribution of spherical gas bubbles in the oncoming free stream. It is assumed that the bubbles remain spherical throughout their trajectory. It is assumed that the flow field is potential; that is, no boundary layer exists on the body, and the velocities and pressures surrounding the body are given by potential theory and are not influenced by the presence of the entrained, small gas bubbles. It is not, however, assumed that the liquid is inviscid insofar as the bubbles are concerned; that is, it is assumed that one of the forces acting on the bubble is the drag created by the relative velocity between the bubble and the liquid and the drag coefficient which determines this force is assumed to be that which would exist for a solid sphere at the Reynolds number based on the bubble diameter, the relative speed, and the viscosity of the liquid. The other forces which act on the spherical bubble are, of course, the pressure forces established by the potential field of the body and the inertial forces associated with the acceleration and virtual mass of the spherical bubble. Using this model, trajectories of various bubble sizes are determined and the stability of each bubble is examined during its passage, for various body sizes and flow conditions. Computations have been made only for a two-dimensional half body; but the results reveal important scale effects on the inception of cavitation which are caused by the screening process that the pressure gradient field imposes on the oncoming entrained gas nuclei.

#### STATIC STABILITY OF SPHERICAL GAS BUBBLES

The conditions for the static stability of a spherical gas volume surrounded by a liquid has been analyzed in Ref. 1. The equation relating the bubble size and the surrounding pressure is given by

$$\frac{p - p_v}{p_0 - p_v} = \left( \frac{R'_0}{R'} \right)^3 \left[ 1 + \frac{8}{\sigma_v W} \left( 1 - \frac{R'^2}{R_0'^2} \right) \right], \quad (1)$$

where

$p$  = surrounding pressure of the bubble

$p_v$  = vapor pressure of the liquid

$p_0$  = initial equilibrium pressure for  $R_0$

$R'_0$  = initial bubble size

$R'$  = bubble size at pressure  $p$

$W = 2\rho U^2 R'_0 / \gamma$  = Weber number

$U$  = free stream velocity

$\gamma$  = surface tension of liquid

$$\sigma_v = \frac{P_0 - P_v}{\rho U^2 / 2} = \text{vapor cavitation number.}$$

Figure 1 shows the relation between the bubble size and the pressure given by Eq. (1).

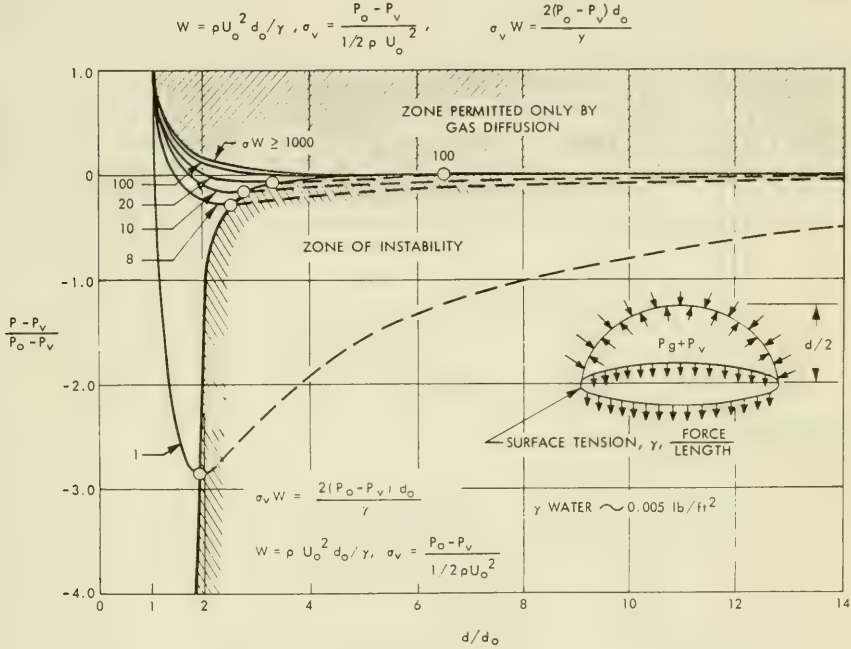


Fig. 1 - The quasi-steady growth of spherical gas bubbles in a liquid

Consider a bubble which is moving in a flow field; its size will vary due to the pressure variation from place to place along its path. In order to find the bubble size at a specific point  $P(x, y)$  in the flow field, Eq. (1) can be rewritten as

$$r^3 - \frac{8/W}{\sigma_v + \frac{8}{W}} r - \frac{\sigma_v + C_p}{\sigma_v + \frac{8}{W}} = 0, \quad (2)$$

where

$$r = \frac{R'}{R'}, \quad C_p = \frac{P - P_0}{\rho U^2 / 2}.$$

It should be noted that  $p$  is now the pressure at point  $P(x, y)$  in the flow field, and  $R'_0$  is the initial bubble size at  $p_0$ , which is the free stream pressure. Using Eq. (2), the size of bubble can be calculated for the following physically possible cases:

I.  $C_p + \sigma_v > 0$ .

(a) When  $Q^2 > \frac{4}{27} P^3$ ,

$$R' = R'_0 / \left[ \left( \frac{1}{2} \sqrt{Q^2 - \frac{4}{27} P^3} \right)^{1/3} + \left( \frac{1}{2} Q - \frac{1}{2} \sqrt{Q^2 - \frac{4}{27} P^3} \right)^{1/3} \right], \quad (3a)$$

where

$$P = \frac{8/W}{\sigma_v + \frac{8}{W}},$$

$$Q = \frac{\sigma_v + C_p}{\sigma_v + \frac{8}{W}}.$$

(b) When  $Q^2 < \frac{4}{27} P^3$ ,

$$R' = R'_0 \cdot 2 \sqrt{\frac{P}{3}} \cos \theta, \quad (3b)$$

where

$$\theta = \frac{1}{3} \cos^{-1} \left( \frac{Q}{\sqrt{\frac{4}{27} P^3}} \right).$$

II.  $C_p + \sigma_v < 0$ . When  $Q^2 < \frac{4}{27} P^3$ ,

$$R' = R'_0 \cdot 2 \sqrt{\frac{P}{3}} \cos \left( \frac{\pi - \theta'}{3} \right), \quad (3c)$$

where

$$\theta' = \cos^{-1} \left( \frac{Q}{\sqrt{\frac{4}{27} P^3}} \right).$$

It is interesting to note that  $Q^2 = 4P^3/27$  in Case II corresponds to the critical point for the instability of the gas nuclei; that is, cavitation inception occurs. This critical point is given by

$$C_p^* + \sigma_v = - \frac{2\sigma_v \left( \frac{8}{\sigma_v W} \right)^{3/2}}{3\sqrt{3} \left( 1 + \frac{8}{\sigma_v W} \right)^{1/2}}, \quad (4)$$

where  $C_p^*$  is the critical pressure coefficient.

## BUBBLE TRAJECTORY

The calculation of water droplet trajectories in air as they approach a body has been carried out in Ref. 3. However, the trajectories of bubbles in a liquid are entirely different from those of water droplets in air because (a) the bubble is lighter than water, so that the pressure gradient produced by any obstacle in the flow field may significantly affect its trajectory, and (b) the bubble size is changing along its path as its surrounding pressure varies. By considering all the forces acting upon a spherical bubble in a two-dimensional flow, the equation of motion of the bubble is

$$\begin{aligned} \frac{1}{2} \times \frac{4}{3} \pi R'^3 \rho \frac{dw_b}{dt} &= \frac{1}{2} \rho (w - w_b) |w - w_b| C_D \pi R'^2 \\ &- \frac{3}{2} \times \frac{4}{3} \pi R'^3 \nabla p + 2\pi \rho R'^2 (w - w_b) \frac{dR'}{dt}, \end{aligned} \quad (5)$$

where

$w_b = iu'_b + jv'_b$  = bubble velocity vector

$w = iu' + jv'$  = fluid particle velocity vector

$C_D$  = drag coefficient of bubble in liquid

$\rho$  = liquid density

$R'$  = radius of bubble

$p$  = pressure surrounding the bubble

$t$  = time.

In Eq. (5) the mass of the gas inside the bubble has been neglected, since it is small compared to that of the added mass of the fluid. Since the spherical bubble under consideration is in an accelerating field, Taylor (5) showed that the pressure force acting on the bubble is  $3/2$  of its volume times the pressure gradient. The last term in the right side of Eq. (5) was estimated to be negligible throughout the entire path of the bubble except immediately prior to the occurrence of instability and therefore was not included in subsequent trajectory computations.

The drag of small, spherical gas bubbles in a liquid was found by Haberman (4) to coincide with those of corresponding rigid spheres. Thus, the experimental drag coefficient curve for a rigid sphere can be used, or



$$\frac{C_D R_b}{24} = 1 + 0.197 R_b^{0.63} + 2.6 \times 10^{-4} R_b^{1.38} \quad (6)$$

Equation (6) is an empirical equation which fits the experimental data very well (3).

By nondimensionalizing the velocity components and linear dimensions by the free stream velocity  $U$  and the body size  $h$  respectively, the equations governing the bubble trajectory become

$$\frac{du_b}{d\tau} = \frac{18}{R^2 R_f} \frac{R_b C_D}{24} (u - u_b) - \frac{3}{2} \frac{dC_p}{dx} \quad (7a)$$

$$\frac{dv_b}{d\tau} = \frac{18}{R^2 R_f} \frac{R_b C_D}{24} (v - v_b) - \frac{3}{2} \frac{dC_p}{dy}, \quad (7b)$$

where

$$\tau = \frac{Ut}{h}, \quad u = \frac{u'}{U}, \quad u_b = \frac{u'_b}{U}, \quad v = \frac{v'}{U}, \quad v_b = \frac{v'_b}{U}, \quad x = \frac{x'}{h}, \quad y = \frac{y'}{h}, \quad R = \frac{R'}{h}$$

$$R_b = \frac{2R' |w - w_b|}{\nu} = R R_f \left[ (u - u_b)^2 + (v - v_b)^2 \right]^{1/2}$$

equals Reynolds number of the bubble, and

$$R_f = \frac{2hU}{\nu}$$

equals Reynolds number of the body. Equations (7) determine the bubble trajectory.

Therefore, the bubble trajectories are seen to depend on the body size  $h$ , the free stream velocity  $U$ , the vapor cavitation number  $\sigma_v$ , the bubble radius  $R_0$ , and the density and viscosity of the fluid.

## APPLICATION TO A TWO-DIMENSIONAL HALF BODY IN AN INFINITE FLUID

### Equations for Bubble Trajectory

Let  $h$  be the semithickness of the two-dimensional half body in an infinite fluid; then the body can be expressed by

$$x = -y \cot \pi y, \quad (8)$$

where  $x$  and  $y$  are normalized by  $h$ . The nose of the half body is located at  $x = -1/\pi$ . The nondimensional velocity components and pressure gradients in

the  $x$  and  $y$  direction at any point in the flow field can be expressed by

$$u = 1 + \frac{1}{\pi} \frac{x}{x^2 + y^2} \quad (9a)$$

$$v = \frac{1}{\pi} \frac{y}{x^2 + y^2} \quad (9b)$$

$$\frac{\partial C_p}{\partial x} = \frac{3}{\pi} \frac{x^2 - y^2 + \frac{x}{\pi}}{(x^2 + y^2)^2} \quad (9c)$$

$$\frac{\partial C_p}{\partial y} = \frac{3}{\pi} \frac{y \left( 2x + \frac{1}{\pi} \right)}{(x^2 + y^2)^2} \quad (9d)$$

From Eq. (7), the bubble motion in the  $x$  and  $y$  direction are expressed by

$$\frac{d\dot{x}}{d\tau} = \frac{18}{R^2 R_f} \frac{R_b C_D}{24} \left[ \left( 1 + \frac{1}{\pi} \frac{x}{x^2 + y^2} \right) - \dot{x} \right] - \frac{3}{\pi} \left[ \frac{x^2 - y^2 + \frac{x}{\pi}}{(x^2 + y^2)^2} \right] \quad (10)$$

$$\frac{d\dot{y}}{d\tau} = \frac{18}{R^2 R_f} \times \frac{R_b C_D}{24} \left[ \frac{1}{\pi} \left( \frac{y}{x^2 + y^2} \right) - \dot{y} \right] - \frac{3}{\pi} \left[ \frac{y \left( 2x + \frac{1}{\pi} \right)}{(x^2 + y^2)^2} \right], \quad (11)$$

where

$$\dot{x} = \frac{dx}{d\tau} = u_b \quad (12)$$

$$\dot{y} = \frac{dy}{d\tau} = v_b \quad (13)$$

$$R_b = R R_f \left\{ \left[ \left( 1 + \frac{1}{\pi} \times \frac{x}{x^2 + y^2} \right) - \dot{x} \right]^2 + \left[ \frac{1}{\pi} \left( \frac{y}{x^2 + y^2} \right) - \dot{y} \right]^2 \right\}^{1/2} \quad (14)$$

For static equilibrium of gas bubbles, the trajectories can be determined by solving Eqs. (10), (11), (12), (13), and (14) together with Eqs. (3a), (3b), and (3c) for a given initial condition of the bubble and the flow field; that is,  $R_0$ ,  $x_0$ ,  $y_0$ ,  $\sigma_v$ ,  $h$ ,  $U$ ,  $\rho$ ,  $\nu$ ,  $\gamma$ . Numerical methods of solving these equations were used.

### Example of Bubble Trajectories

Theoretically, the trajectory of a bubble should be calculated from infinity ahead of the body. Since far upstream of the body, the pressure gradient produced by the body has negligible effect on the trajectory of the bubble, the bubble will just follow the streamline, and the streamline itself is parallel to the flow

direction. To determine a starting point for the calculation of the bubble trajectories, an initial bubble size of  $R_0 = 0.02$  in a flow field with  $U = 50$  fps,  $\sigma_v = 0.4$ ,  $h = 0.6$  in.,  $\rho = 1.94$  slug/ft<sup>3</sup>,  $\nu = 1.05 \times 10^{-5}$  ft<sup>2</sup>/sec, and  $\gamma = 0.00497$  lb/ft was examined. Figure 2 shows the bubble trajectories when  $x_0 = -20, -10$ , and  $-6$  at a fixed  $y_0 = 0.01$ . It is seen that the difference in trajectories is negligibly small. Thus,  $x_0 = -10$  is considered to be far enough upstream to avoid any significant error and is thus used in later calculations. The star in Fig. 2 indicates the location at which a given bubble becomes statically unstable.

The trajectories of the same bubble size were also calculated for different  $y_0$  values, while the remaining parameters were unchanged. The results are shown in Fig. 3 for  $y_0 = 0.01, 0.02, 0.05, 0.10$ , and  $0.20$ . As expected, the trajectories are quite different for different values of  $y_0$ .

In a practical situation, various bubble sizes can be located in any position in the free stream. However, in the problem of determining the conditions for incipient cavitation, we are interested only in the initial bubble position which will lead the bubble in its trajectory, to a pressure point lower than any other point which the bubble could reach from another initial position. From the examples of bubble trajectories given in Fig. 3, it is noted that the closer the initial position of the bubble to the dividing streamline, the sooner the bubble will

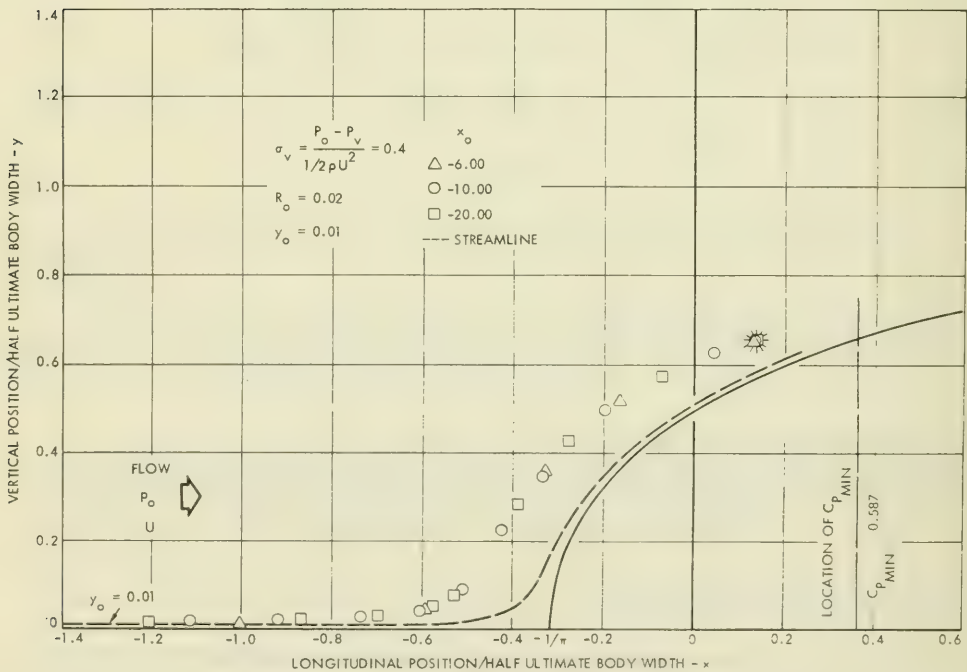


Fig. 2 - The influence of the starting point in the  $x$  direction on the gas nuclei trajectories ( $h = 0.6$  inch;  $U = 50$  fps)

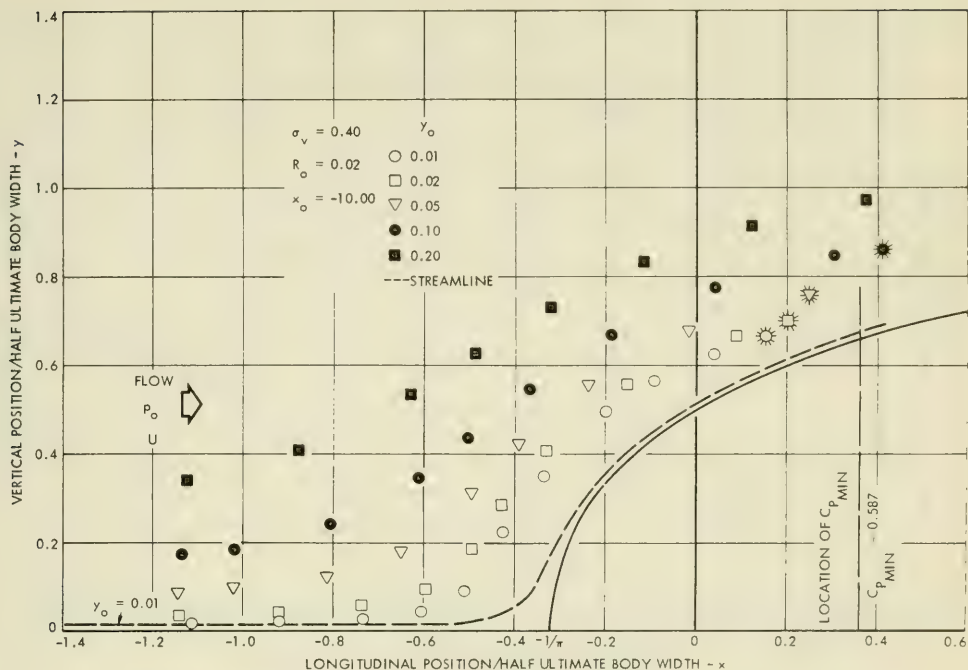


Fig. 3 - The influence of the starting point in the  $y$  direction on the gas nuclei trajectories ( $h = 0.6$  inch;  $U = 50$  fps)

become statically unstable. Since the radius of bubbles being pushed away from its streamline are of the order of several hundredth of  $h$  in the present problem, it is therefore decided to use  $y_0 = 0.01$  for all the calculations.

Figures 4 and 5 show the influence of initial bubble size on its trajectory and thus its stability at  $U = 50$  fps and  $\sigma_v = 0.4$  and  $0.58$ . Figures 4 and 5 illustrate that the larger bubbles may indeed remain stable while smaller ones become unstable. In Fig. 4, two bubbles of initial radius  $R_0 = 0.04$  and  $0.24$  were traced along the dividing streamline. It is interesting to note that these bubbles will temporarily stand still at a point where the drag force and the pressure gradient acting on the bubble are in equilibrium. This occurs at  $x = -0.478$  for  $R_0 = 0.04$  and  $x = -0.833$  for  $R_0 = 0.24$ . Afterward, the bubbles will be moved away from the dividing streamline by the perturbations in the vertical direction. Figure 5 shows that certain bubbles collide with the body at some distance from the nose. It is assumed that after collision the bubble surface will remain tangent to the surface of the body as it moves past the body. Thus, if the collision occurs before the location of minimum pressure of the body, the bubble will definitely encounter the minimum pressure of the body in its trajectory; on the other hand, if the collision occurs after the location of minimum pressure of the body, the bubble will not encounter the minimum pressure in the flow field.



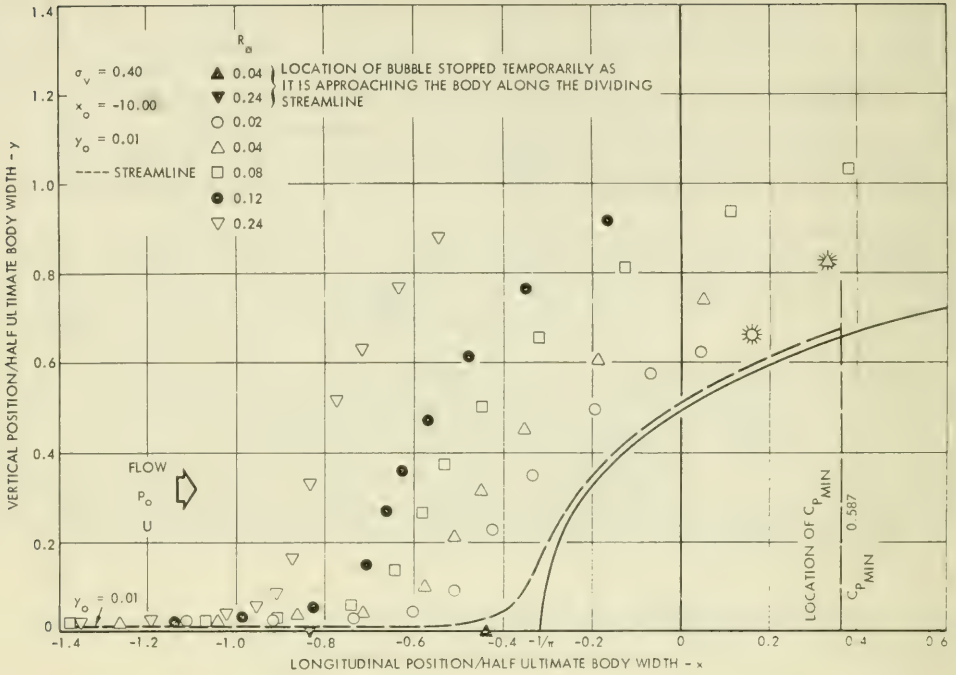


Fig. 4 - Gas nuclei trajectories about a two-dimensional half body ( $h = 0.6$  inch;  $U = 50$  fps)

The results presented in Figs. 4 and 5 also show the influence of vapor cavitation number on the stability and the trajectories of bubbles. For instance, the bubble with initial radius of 0.02 remains stable throughout its entire path at  $\sigma_v = 0.58$  and becomes unstable at  $\sigma_v = 0.4$ . The reason for this effect may be seen in Eq. (4); that is, for the same initial bubble size, the smaller the vapor cavitation number, the higher will be the critical pressure for bubble stability. The influence of  $\sigma_v$  on the bubble path is expected, because at different vapor cavitation numbers, the same initial bubble size will contain different numbers of gas molecules; thus as the pressure surrounding the bubble varies, the bubble size will change a different amount for different vapor cavitation numbers. Since for the same initial bubble size, fewer gas molecules are contained in the bubbles for a lower vapor cavitation number; the same size bubble which initially maintained equilibrium at a lower vapor cavitation number, will decrease more in size as it approaches the stagnation region than it will at the higher cavitation number. Because the smaller bubbles are associated with lower values of the Reynolds number and thus higher drag coefficients; the net effect of lower vapor cavitation number is that the bubble remains closer to its initial streamline. Figure 6 shows the variation of bubble radius along its trajectory for an initial bubble nondimensional radius of  $R_0 = 0.02$  and for  $\sigma_v = 0.2, 0.4$ , and  $0.58$ . It may be noted in Fig. 6 that at  $\sigma_v = 0.2$  the bubble grows at a higher

## Gas Nuclei Trajectories and Cavitation Inception

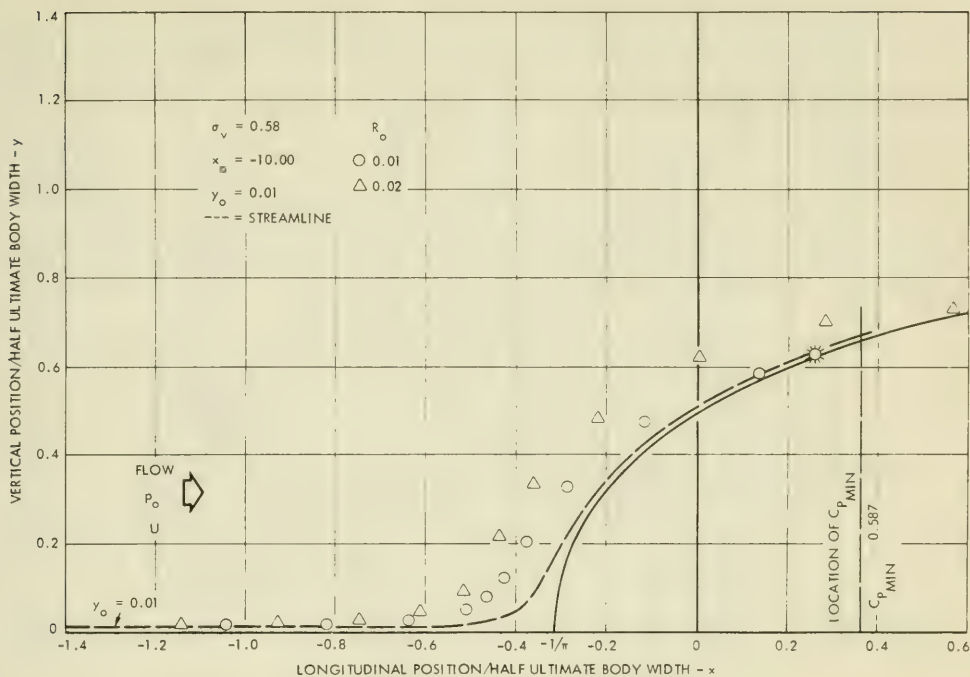


Fig. 5 - Gas nuclei trajectories about a two-dimensional half body ( $h = 0.6$  inch;  $U = 50$  fps)

rate than it does for  $\sigma_v = 0.4$  and  $0.58$  as the bubble passes into the negative pressure zone in the flow field.

### Lower Critical Bubble Radii

In Eq. (4), if  $C_p^*$  is replaced by the value of  $C_{p \text{ min}}$  of the flow field and the definition,  $W = 2R_0' \rho U^2 / \gamma$  is used, then for each value of  $\sigma_v$ , there is a corresponding  $R_0'$  value. Since  $C_{p \text{ min}}$  is the lowest pressure in the flow field, the  $R_0'$  value thus found is the lowest possible critical bubble radii,  $R_{0\ell}^*$ . For those bubble sizes smaller than  $R_{0\ell}^*$ , all bubbles will remain stable throughout the entire flow field for a specific  $\sigma_v$  value. For a two-dimensional half body  $C_{p \text{ min}} = -0.587$ . The value of  $R_{0\ell}^*$  for this body is shown by the solid line in Fig. 7 as a function of  $\sigma_v$ . Since the value of  $C_{p \text{ min}}$  for a half body is independent of body size, the  $R_{0\ell}^*$  curve will also be independent of body size. Although  $C_{p \text{ min}}$  always occurs on the body surface, it is conceivable that the smaller bubbles will "roll over" the body surface, so they do essentially encounter the minimum pressure in the flow field.

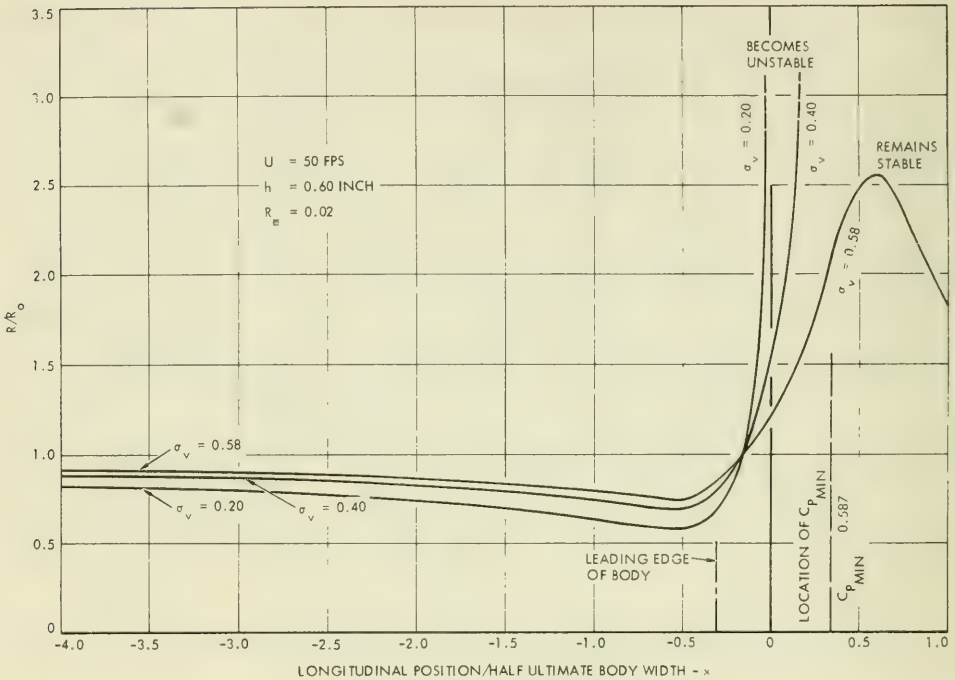


Fig. 6 - Variation of the size of a gas nucleus along its trajectory

### Upper Critical Bubble Radii

As described previously, a bubble coming from upstream is pushed away when it approaches the body. The larger the bubble is, the greater the distance it is pushed out by the pressure gradient. Thus, although the large bubbles have a less critical pressure, they may still not encounter this critical pressure throughout their entire path. Consequently, it is possible to find for a given vapor cavitation number and body size a corresponding value of bubble radius  $R_{0u}^*$  such that all bubbles with radius *greater* than  $R_{0u}^*$  will remain stable in the flow field.  $R_{0u}^*$  is defined as the upper critical bubble radius. That is, for a given value of  $\sigma_v$ , bubbles with radius less than  $R_{0l}^*$  cannot possibly become unstable regardless of their trajectory; for this same value of  $\sigma_v$ , bubbles with radius greater than  $R_{0u}^*$  cannot become unstable because their trajectory removes them from the low-pressure zones required for instability.

The determination of this upper critical bubble size,  $R_{0u}^*$ , for various body sizes and vapor cavitation numbers is tedious work. It was accomplished by the trial and error method, using different sizes of bubbles in the trajectory equations until the critical radius was obtained. The Runge-Kutta method and an IBM 1130 digital computer were used. Results for the two-dimensional half body with  $h = 3, 1.8, 1.2, 0.6, 0.4, 0.2, 0.1$ , and  $0.05$  inch are shown by the dashed lines in Fig. 7.

## Gas Nuclei Trajectories and Cavitation Inception

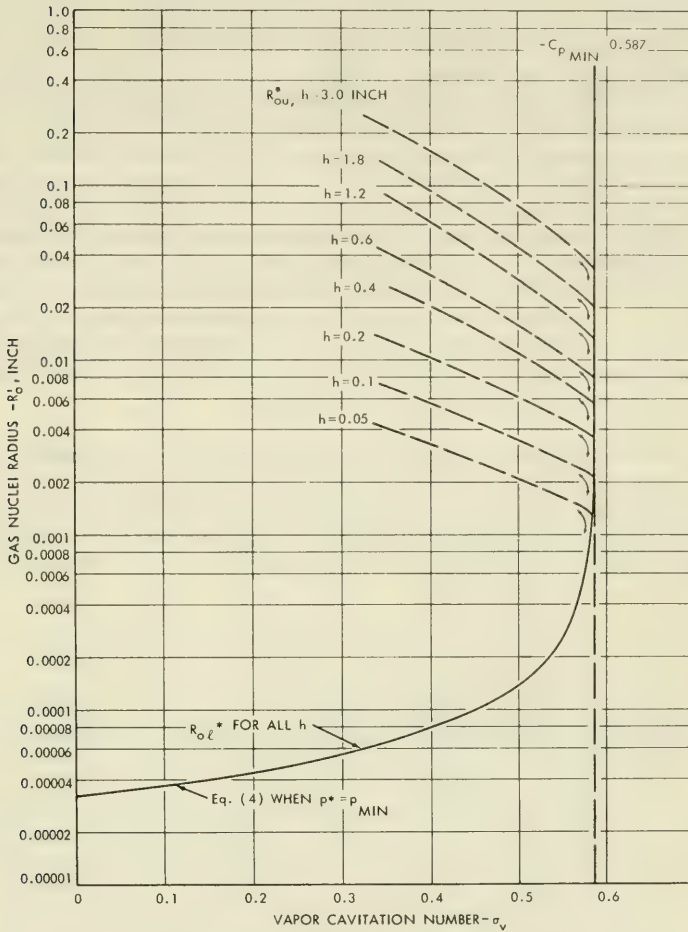


Fig. 7 - Upper and lower critical sizes of gas nuclei as a function of vapor cavitation number and the body size ( $U = 50$  fps)

### Body Size Effect on the Incipient Cavitation Number

In Fig. 7 the initial bubble radii are plotted versus vapor cavitation number for a constant speed of  $U = 50$  fps. There is one curve for the lower critical bubble radii  $R_{0\ell}^*$  and this curve is the same for all body sizes as described previously. In the same figure are eight curves for the upper critical bubble radius; each curve refers to a specific body size. Therefore, for a given value of  $\sigma_v$  and for a free stream bubble population that includes all sizes,  $0 < R_0' < \infty$ , Fig. 7 shows that only the range of bubble sizes  $R_{0\ell}^* \leq R_0' \leq R_{0u}^*$  can actually participate in the inception of cavitation, that is, become unstable. The



intersection of the  $R_{0u}^*$  and  $R_{0l}^*$  curves, therefore, gives the value of vapor cavitation number, for which there is only one size of bubble which can possibly become unstable in the flow field for the specific body size, speed, and entrained bubble population which includes all sizes,  $0 < R_0' < \infty$ . Let this bubble size be defined as  $R_{0i}$ . That is, for a slight increase of the vapor cavitation number (an increase of the free stream pressure) all bubble sizes will remain stable. On the other hand, for a slight decrease of the vapor cavitation number, additional sizes of bubbles will become unstable. Thus, the value of the vapor cavitation number at the intersection of the  $R_{0l}^*$  and  $R_{0u}^*$  curves is the incipient cavitation number  $\sigma_i$ . Consequently, a curve of incipient cavitation number versus body size can be deduced from the results presented in Fig. 7 and this curve (labeled  $0 \leq R_0' \leq \infty$ ) is shown in Fig. 8. In Fig. 8 the dashed curves present the computed values of  $\sigma_i$  obtained when the bubble population in the oncoming stream is void of bubbles below a specified size. It is interesting to note that the body size effect becomes more and more significant as the value of the minimum bubble radii in the bubble population is increased.

Another interesting point to note is that in Fig. 8 for each specific body size, the  $R_{0u}^*$  and  $R_{0l}^*$  form an envelope of an unstable bubble zone. It is seen that to have minimum scale effect, we need the bubble with radii  $R_{0i}$  to be present in the entrained bubble population. Furthermore, for a range of entrained bubble population of radius greater than  $R_{0i}$ , it is always the smallest

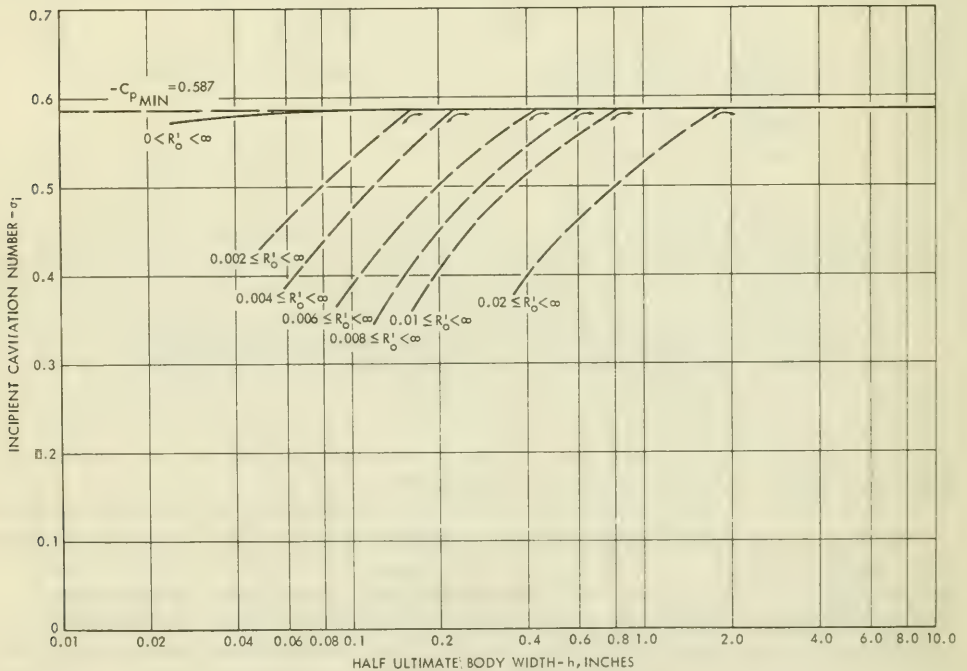


Fig. 8 - The effect of body size on the inception of cavitation as a function of minimum radius of gas nuclei ( $U = 50$  fps)

size which controls cavitation inception; and for a range of entrained bubble population of radius smaller than  $R_{0i}$ , it is always the largest bubble size which fixes the point of cavitation inception.

Calculations, similar to those shown in Figs. 7 and 8, were also carried out for  $U = 25$  fps. These results are presented in Figs. 9 and 10.

### CONCLUDING REMARKS

The results of the analysis of the trajectories of gas nuclei entrained in a fluid flowing past a two-dimensional half-body may be summarized as follows:

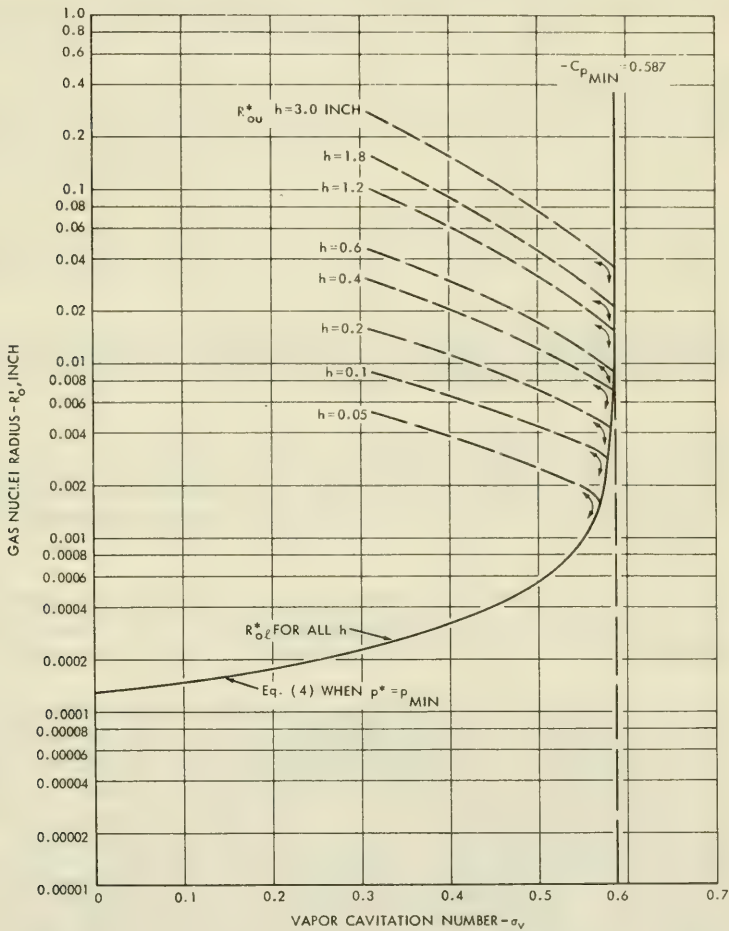


Fig. 9 - Upper and lower critical sizes of gas nuclei as a function of vapor cavitation number and the body size ( $U = 25$  fps)

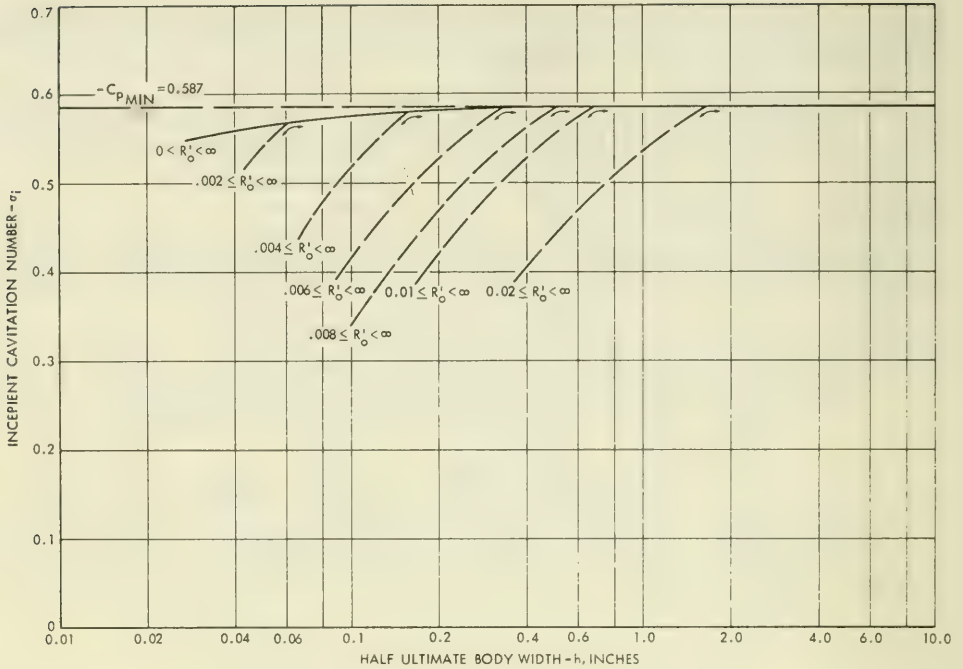


Fig. 10 - Effect of body size on the inception of cavitation as a function of the minimum radius of the gas nuclei ( $U = 25$  fps)

1. For a given size and speed, a unique entrained nuclei size exists whose instability defines the maximum possible incipient cavitation number. This maximum possible incipient cavitation number is always less than  $-C_{p \min}$ .
2. The difference between the maximum possible incipient cavitation number and  $-C_{p \min}$  increases with decrease in body size. This difference also increases with decrease in speed if the free stream contains all sizes of bubbles.
3. If the free stream does not contain the unique nuclei size associated with the maximum possible incipient cavitation number, the difference between the measured incipient cavitation number and  $-C_{p \min}$  is, of course, further increased. For a specific case of half body total width = 0.2 inch, speed = 50 fps, the critical or unique nuclei size is  $R_{0i}^* = 0.0021$  inch, and the maximum possible value of the incipient cavitation number is 0.585, compared with  $-C_{p \min} = 0.587$ . If there is an absence of all entrained nuclei with  $R'_0 \leq 0.004$  inch, the measured incipient cavitation number will be 0.47, or 80 percent of  $-C_{p \min}$ . However, if the same body is operated at a speed of 25 fps, the maximum possible value of the incipient cavitation number is 0.578; and if there is an absence of all entrained nuclei with  $R'_0 \leq 0.004$  inch the measured incipient cavitation number will be 0.517, or 88.3 percent of  $-C_{p \min}$ .

The results presented are, of course, limited because of the simplifying assumptions which were made. The bubble trajectories will be modified by the influence of the boundary layer. Furthermore, the trajectories will be influenced by the distortions from a spherical shape caused by the forces acting on the bubbles. Also, for very small bodies and high speeds, the dynamic effects which were neglected in the analysis presented may become important. Nevertheless, the screening mechanism of the body pressure gradients will persist and this phenomenon must eventually contribute to a final understanding of cavitation inception.

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\* \* \*

## DISCUSSION

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We have seen outlined here a very elegant mechanism for explaining the scale effect at cavitation tests. In this mechanism the minimum diameter of the bubbles present in the tunnel is an important parameter. I would like to mention that another measure of this diameter can be obtained by calculating the resorption capacity of the tunnel circuit. This will give you the largest diameter for complete resorption and thereby the diameter of the smallest bubble present. Equations for such calculations have been given by, for instance, Silverleaf.



The results from such calculations together with curves like those in Figs. 8 and 10 would give the inception cavitation number of the body for a certain body-tunnel combination (with reservation for the assumptions made in the calculations).

In connection with the analysis for the ITTC comparative head form tests which I mentioned in connection with Mr. Bernd's paper, we calculated the maximum bubble diameter for complete resorption for some of the participating tunnels using Mr. Silverleaf's equations. From these results, which are shown in Fig. D1, it appears that for most of the tunnels the value of the largest diameter for complete resorption falls below the lowest  $R_0$  value (0.002 inch) given by the authors, indicating that no serious scale effect can be expected in these tunnels if reasonable body diameters are used. For the resorber tunnels the values of this diameter is appreciably larger. Thus large models have to be used in these tunnels if scale effects should be avoided.

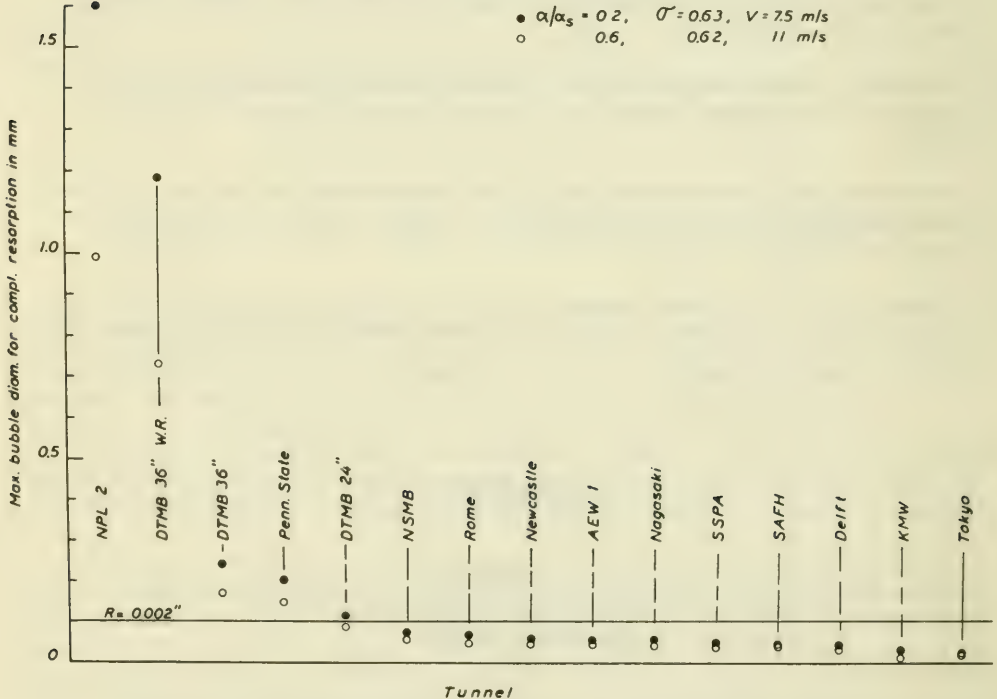


Fig. D1 - Largest bubble diameter for complete resorption for different cavitation tunnels

\* \* \*

## DISCUSSION

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The paper shows very clearly the effects of body size and bubble population on cavitation inception.

At the Netherlands Ship Model Basin, experiments have been conducted with cylindrical bodies with a hemispherical head. The value of  $C_{p\min}$  is equal to -0.74 for these bodies.

The results of the cavitation inception measurements show a bubble radius at incipient cavitation of about  $10^{-5}$  m. Very roughly speaking this is an order of magnitude smaller than those occurring in the report (Figs. 7 and 9) under discussion for comparable sizes.

Can the authors tell us in what direction  $R_{0u}$  varies if, other conditions being held constant,  $|C_{p\min}|$  is increased.

Secondly, can the authors give on the basis of the present work any suggestion as to how cavitation inception results from different water tunnels may be compared?

\* \* \*

## REPLY TO DISCUSSION

Virgil E. Johnson, Jr., and Tsuying Hsieh

With regard to Mr. Johnsson's remarks concerning the necessity for larger models to minimize scale effects when only very small nuclei are present (as in tunnels with resorbers), our study certainly indicates this to be true. A similar conclusion is that higher speeds are also necessary to minimize scale effects when only very small nuclei are present.

Concerning Professor van Wijngaarden's comments relative to the effect of the magnitude of  $C_{p\min}$  on the phenomena discussed in our paper, we submit the following information.

The nuclei screening process described is most pronounced for bodies with strong-stagnation regions followed by a short axial extent of low-pressure region. The effect is probably negligible for bodies with small-stagnation regions followed by low-pressure regions of long-axial extent, such as thin

elliptic cylinders. For bodies having similar stagnation regions, we would expect the screening effect to be less pronounced as  $C_{p \min}$  is made more negative (for similar axial extent of the low-pressure region), because the larger bubbles which are initially pushed out by the stagnation region are acted on by the stronger negative pressure regions to return them. Clearly,  $C_{p \min}$  alone is not adequate to define the magnitude of the screening effect; the entire pressure distribution is necessary.

Relative to suggestions as to how cavitation inception results from different water tunnels may be compared, it is suggested that emphasis be given to the development of instrumentation to determine the nuclei size spectrum in the tunnel free-stream flow. Such information will contribute in a major way to the definition of the true critical pressure for cavitation inception in a particular facility and thus to corrections of the observed values of  $\sigma_v$  incipient to a value based on some standardized critical pressure.

\*       \*       \*

# PREDICTION OF CAVITATION INCEPTION SPEEDS ON ROUGH HYDRODYNAMIC BODIES

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## ABSTRACT

Experimental studies of cavitation inception over two- and three-dimensional roughness elements in a boundary layer (made at the Ordnance Research Laboratory (ORL) of Pennsylvania State University and at the David Taylor Model Basin, respectively), have been used to derive scaling laws for predicting cavitation on these elements under arbitrary boundary layer conditions. It has already been shown that the cavitation number of three-dimensional elements, computed in terms of the velocity of the oncoming flow in the boundary layer at the height of a roughness element, is a function of the local Reynolds number, based on roughness height and local velocity. In the present study, the ORL data have been analyzed in the same way, and analytical curves have been fitted to both sets of data. Methods are derived for computing cavitation inception speeds on rough hydrodynamic bodies. Sample calculations show the degrading effect of the various types of roughnesses on the cavitation inception speed of a typical sonar dome.

## INTRODUCTION

The prediction of cavitation inception on hydrodynamic bodies, such as ship hulls, appendages, sonar domes, and propellers is one of the important unsolved problems in naval architecture. The cavitation inception speeds predicted by model tests and theoretical analyses are never attained under full-scale operating conditions. This disparity is partly due to the physical size of the body and its surface finish. Recent experiments have shown that cavitation inception on bluff bodies and those having sharp pressure minima is primarily a function of Reynolds number. Reference 1 describes experiments on families of ogive and disk models at several water temperatures in which Reynolds number trends were definitely established. Reynolds number effects have also been observed in the scaling of tip vortices from propellers (2).

In a series of experiments recently completed at the David Taylor Model Basin (3), what appeared to be a Reynolds number scaling was found for



cavitation inception over three-dimensional roughness elements embedded in the boundary layer of a flat plate. This relationship became apparent when the cavitation number and Reynolds number were computed in terms of the local velocity in the oncoming flow at the point of cavitation and the roughness height. The Reynolds number dependency was further confirmed when similar two-dimensional data were analyzed in the same way. These data were obtained in two water tunnels, in four different boundary layers, and at temperatures ranging from 76 to 86°F. Although the temperature range was not large enough to establish a true viscous effect in the two sets of boundary layer experiments, it is reasonable to assume that viscosity would play the same role in a shear flow as in a uniform flow. Specifically, when cavitation originates in a vortex resulting from flow separation on a bluff roughness, viscosity of the fluid will affect the cavitation inception speed. There will be no viscous effects on gently curving protuberances on which there is no flow separation. The results discussed here tend to support these hypotheses.

The cavitation inception data obtained at the Model Basin and at ORL have been reduced to local velocity conditions. Analytical curves have been fitted to the data for each roughness geometry. Using Holl's formulas for computing cavitation inception on a rough parent body, sets of curves and tables have been compiled for use in computing cavitation speeds for a cavitating rough body of known pressure distribution and boundary layer characteristics operating at different depths in the ocean.

#### SCALING OF CAVITATION INCEPTION ON ROUGHNESS ELEMENTS

Two sets of experiments have been performed to measure cavitation inception over two- and three-dimensional roughness elements embedded in the boundary layer on a flat plate (3,4). The cavitation number measured in terms of the velocity at the outer edge of the boundary layer has been defined as

$$\sigma_0 = \frac{p - p_v}{\rho U^2/2}, \quad (1)$$

where  $p$  is the static pressure in the water tunnel at the location of the roughness element,  $p_v$  is the vapor pressure of the liquid,  $\rho$  is the density of the liquid, and  $U$  is the velocity at the outer edge of the boundary layer. For a particular roughness element,  $\sigma_0$  is a function of  $h$ , the height of the roughness;  $\delta$ , the boundary layer thickness;  $H$ , the shape parameter of the boundary layer profile; and the velocity of flow. In order to develop scaling laws for predicting cavitation inception, it is necessary to determine the flow in the immediate vicinity of the roughness.

In turbulent flow, the velocity profile of a boundary layer is well approximated by a power law of the form

$$\frac{u_y}{U} = \left( \frac{y}{\delta} \right)^{1/m}, \quad (2)$$

where  $y$  is the distance normal to the surface, and  $u_y$  is the velocity at  $y$ . The shape parameter  $H$  is the following function of the exponent  $m$ :

$$H = \frac{m+2}{m} \quad (3)$$

If  $\delta$  and  $m$  are known, it is easy to find the velocity in the oncoming flow at the tip of the roughness where cavitation inception occurs. Similarly, average velocities over the roughness can be obtained by simple integrations of Eq. (2).

Benson used the velocity at the tip of the roughness element  $u_h$  in defining a local cavitation number and local Reynolds number. Thus

$$\text{local cavitation number } \sigma_h = \left( \frac{h}{\delta} \right)^{-2/m} \sigma_0 \quad (4)$$

$$\text{local Reynolds number } R_h = hu_h/\nu, \quad (5)$$

where  $\nu$  is the kinematic viscosity of the fluid. When  $\sigma_h$  for Benson's three-dimensional roughness elements is plotted as a function of  $R_h$  on log-log paper, straight lines can be fitted through the points for each roughness geometry. The curves in Fig. 1 are reproduced from his report (3).

If the ORL data for two-dimensional circular arc and triangle roughness elements (4) are recomputed in the same way, these data also show a linear variation with Reynolds number (Fig. 2). Despite the fact that these data were obtained in two water tunnels and four boundary layers, the points are well distributed about the mean line. Straight lines of the form

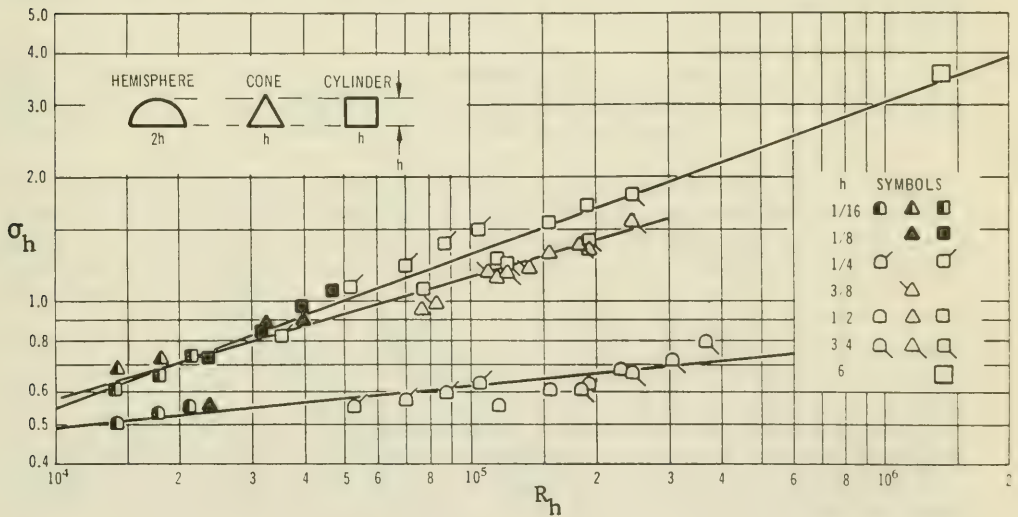


Fig. 1 - Local cavitation number as a function of the local Reynolds number (DTMB three-dimensional data based on  $u_h$ )

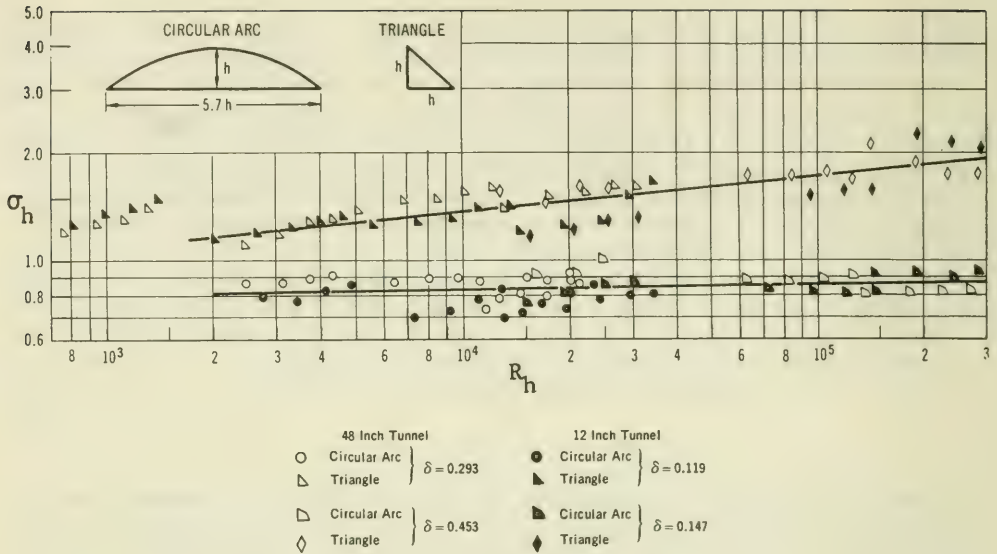


Fig. 2 - Local cavitation number as a function of the local Reynolds number (ORL two-dimensional data based on  $u_h$ )

$$\log \sigma_h = A + B \log R_h \quad (6)$$

were fitted to the data in Figs. 1 and 2 by the method of least squares. Values of the parameters  $A$  and  $B$  for the different roughness geometries are listed in Table 1.

Table 1  
Values of  $A$  and  $B$  For Two- and Three-Dimensional Roughness Elements

| Geometry     |     | $A$     | $B$     |
|--------------|-----|---------|---------|
| Hemisphere   | 3-D | -0.7011 | 0.09779 |
| Cone         | 3-D | -1.430  | 0.2962  |
| Cylinder     | 3-D | -1.725  | 0.3674  |
| Circular Arc | 2-D | -0.1398 | 0.01433 |
| Triangle     | 2-D | -0.2568 | 0.09867 |

In two-dimensional flow, all the streamlines in the boundary layer are deflected by a two-dimensional roughness element. Since the roughness creates a large disturbance in the local flow,  $u_h$  (computed from the power law) does not seem to be an appropriate local velocity for computing the cavitation parameters. Holl (4) tried to predict cavitation indices of his circular-arc roughness elements by frozen streamline theory. This computation was laborious and gave poor correlation with the experimental values. He later obtained

good results by assuming that the pressure minimum of the element is proportional to the average of the square of the velocity in the boundary layer out to the height of the roughness (5). If a power law boundary layer profile is assumed for the oncoming flow,

$$\frac{\overline{u^2}}{U^2} = \frac{1}{h} \int_0^h \left( \frac{y}{\delta} \right)^{2/m} dy . \quad (7)$$

For  $h \leq \delta$

$$\frac{\overline{u^2}}{U^2} = \frac{1}{H} \left( \frac{h}{\delta} \right)^{2/m} . \quad (8)$$

For  $h \geq \delta$

$$\frac{\overline{u^2}}{U^2} = 1 - \frac{2}{m+2} \frac{\delta}{h} . \quad (9)$$

If these expressions are used for the velocity at the tip of two-dimensional roughness elements, the local cavitation number and Reynolds number become

$$\overline{\sigma}_h = \frac{U^2}{u^2} \sigma_0 \quad (10)$$

$$\overline{R}_h = \overline{u}_h h / \nu , \quad (11)$$

where

$$\overline{u}_h = [\overline{u^2}]^{1/2} . \quad (12)$$

Figure 3 shows the ORL two-dimensional data plotted in terms of  $\overline{\sigma}_h$  and  $\overline{R}_h$ . The constants  $\overline{A}$  and  $\overline{B}$  of the straight lines fitted through the points are listed in Table 2:

A comparison of Figs. 2 and 3 shows that the points are shifted somewhat but that there is not much difference in the amount of scatter in the two sets of curves. In both figures, the triangular roughnesses show an increase in local cavitation number with Reynolds number, which is characteristic of cavitation in a separated flow region.

On both figures, cavitation inception on the circular-arc roughness elements shows very little dependence on Reynolds number. A least-square fit to the data shows a small positive slope to the line in Fig. 2 and a small negative slope in Fig. 3. A different method of computing the local velocity might have produced a horizontal line. Since there was considerable scatter in the data, no further analysis was warranted. An average value of all the circular arc cavitation numbers on Fig. 3 is 1.11. The pressure coefficient, computed by potential flow, has the value 1.083 (5). Although this remarkable agreement may be fortuitous, it appears that the computation of the local flow parameters on the basis of  $\overline{u}_h$  is justified for two-dimensional flows. The constancy of the local cavitation number



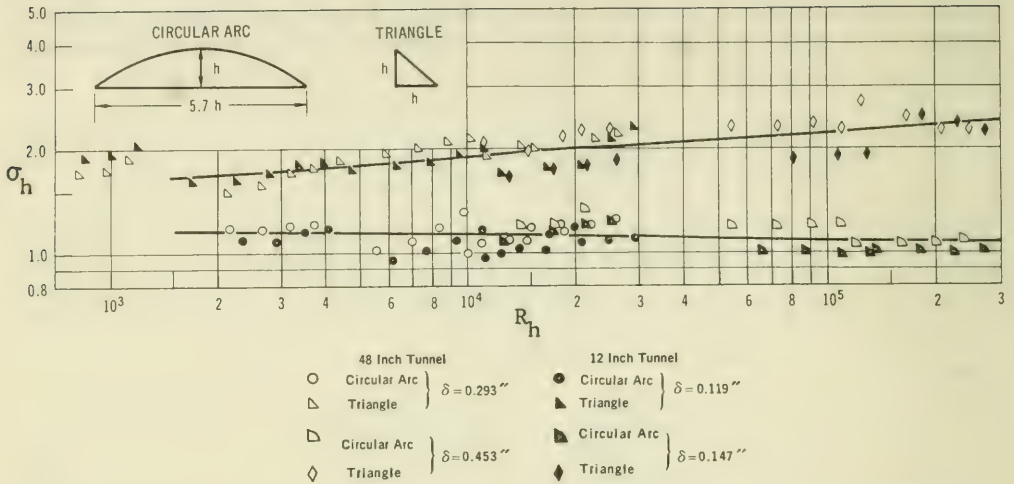


Fig. 3 - Local cavitation number as a function of the local Reynolds number (ORL two-dimensional data based on  $\bar{u}_h$ )

Table 2  
Values of  $\bar{A}$  and  $\bar{B}$  for Two-Dimensional  
Roughness Elements

| Geometry     | $\bar{A}$ | $\bar{B}$ |
|--------------|-----------|-----------|
| Circular arc | 0.1209    | -0.01782  |
| Triangle     | 0.003435  | 0.06745   |

with local Reynolds number implies that cavitation inception on circular-arc roughness elements is not occurring in a separated region but is produced when the pressure on the element is reduced to a value close to the vapor pressure.

The same averaging process did not seem appropriate for finding the local flow conditions over three-dimensional roughness elements. Here the flow lines are displaced to the side as well as over the tip of the element. Figure 1 includes a single datum point obtained for cavitation inception on a much larger cylinder in a uniform flow. These measurements were made on a cylinder, 8 in. in diameter and 6 in. high, mounted on a flat plate, and towed at a depth of 6.5 ft in the towing basin. Cavitation occurred at the tip of the cylinder at a towing speed of 16 knots. The boundary layer on the plate was a small fraction of the cylinder height, so that the oncoming flow over the tip was essentially uniform. Despite the different scale and flow conditions, the cavitation inception number falls on the same curve with the other cylinder data. Therefore the velocity  $u_h$  of the oncoming flow at the height of cavitation inception on the roughness

element appeared to be a reasonable value for computing the local flow parameters. With this choice of local flow velocity,  $u_h$  becomes equal to  $U$ , the velocity at the edge of the boundary layer, as soon as the roughness protrudes through the boundary layer. Unfortunately no data were obtained for roughness elements with heights of the same magnitude as the boundary layer thickness.

## COMPUTATION OF CRITICAL HEIGHTS OF ROUGHNESS ELEMENTS

Holl (4) has postulated that the cavitation inception data obtained in the boundary layer on a flat plate can be used to predict cavitation inception on an isolated roughness element embedded in the boundary layer of a smooth hydrodynamic body having an arbitrary pressure distribution. Although he recognized the importance of matching boundary layer profiles and roughness height to boundary layer thickness ratios, he failed to recognize the importance of Reynolds number scaling. With the use of local cavitation number and local Reynolds number scaling, the data can be used in a great variety of boundary layer conditions. In order to use the roughness data quantitatively for a given hydrodynamic body, it is necessary to know the pressure distribution, boundary layer thickness, and velocity profile on the parent body. On the other hand, an estimate of the effect of isolated roughness on a parent body can be obtained from a reasonable estimate of the pressure distribution and boundary layer development.

If a roughness element is placed on a smooth parent body at a point where the pressure coefficient is  $C_p$ , the cavitation inception number at the roughness is

$$\sigma_R = -C_p + (1 - C_p) \sigma_0, \quad (13)$$

where  $\sigma_0$  is the cavitation inception number of the roughness computed in terms of the velocity  $U$  at the edge of the flat plate boundary layer which has the same velocity profile as the parent body. Inasmuch as the boundary layer profiles are the same,  $U$  may also be interpreted as the velocity at the edge of the boundary layer of the parent body. Therefore, the factor

$$1 - C_p = U^2/U_\infty^2 \quad (14)$$

converts  $\sigma_0$  to free-stream flow conditions. If the isolated roughness is at a water depth  $H_w$  on the parent body,

$$\sigma_R = \frac{p_\infty - p_v}{\rho U_\infty^2 / 2} = \frac{2g (H_w + H_a - H_v)}{U_\infty^2}, \quad (15)$$

where  $H_a$  is the pressure head of the atmosphere and  $H_v$  is the pressure head of the vapor pressure.

Thus  $\sigma_0$  in Eq. (13) is determined as a function of the free-stream velocity  $U_\infty$ , the pressure coefficient  $C_p$ , the water depth  $H_w$ , the atmospheric pressure

head  $H_a$ , and the temperature or vapor pressure  $H_v$ . It now remains to examine the local flow conditions in the boundary layer and to determine the roughness height which is compatible with the Reynolds number condition. This can be done by a method of trial and error. For example, select a value of  $u_h$  (or  $\overline{u_h}$ ). Then obtain  $\sigma_h$  from Eq. (4) or (10). Find the local Reynolds number from Eq. (6), compute  $h$ , and compare  $h/\delta$  with  $y/\delta$  of the velocity profile. When the two are equal, the problem is solved.

## COMPUTED CAVITATION PARAMETERS

A computer program was written for calculating the minimum or critical roughness heights which would produce cavitation inception on a particular roughness geometry as functions of flow velocity, submergence depth, water temperature, boundary layer thickness and profile, and the pressure distribution of the parent body. Calculations have been made for each of the roughness geometries investigated. The trends of these calculations are shown in Figs. 4 through 12. The pressure coefficients and boundary parameters selected for

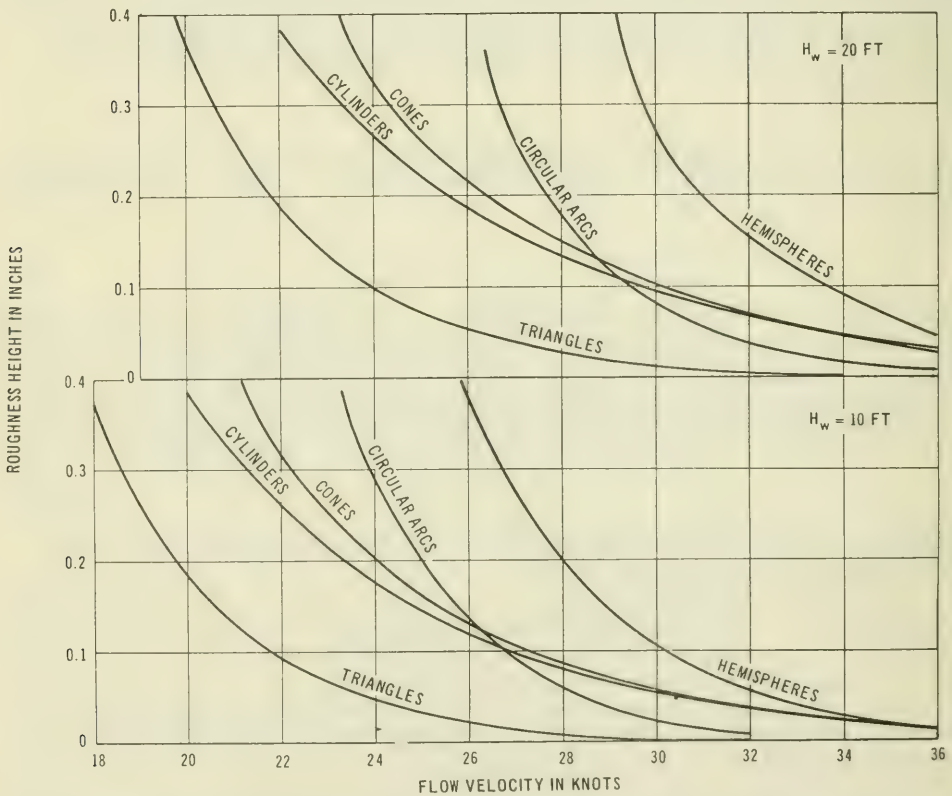


Fig. 4 - Effect of geometry of isolated roughness elements on cavitation inception ( $C_p = -0.5$ ,  $\delta = 0.4$  in.,  $m = 7$ ,  $T = 54^\circ \text{F}$ )

## Cavitation Inception on Rough Bodies

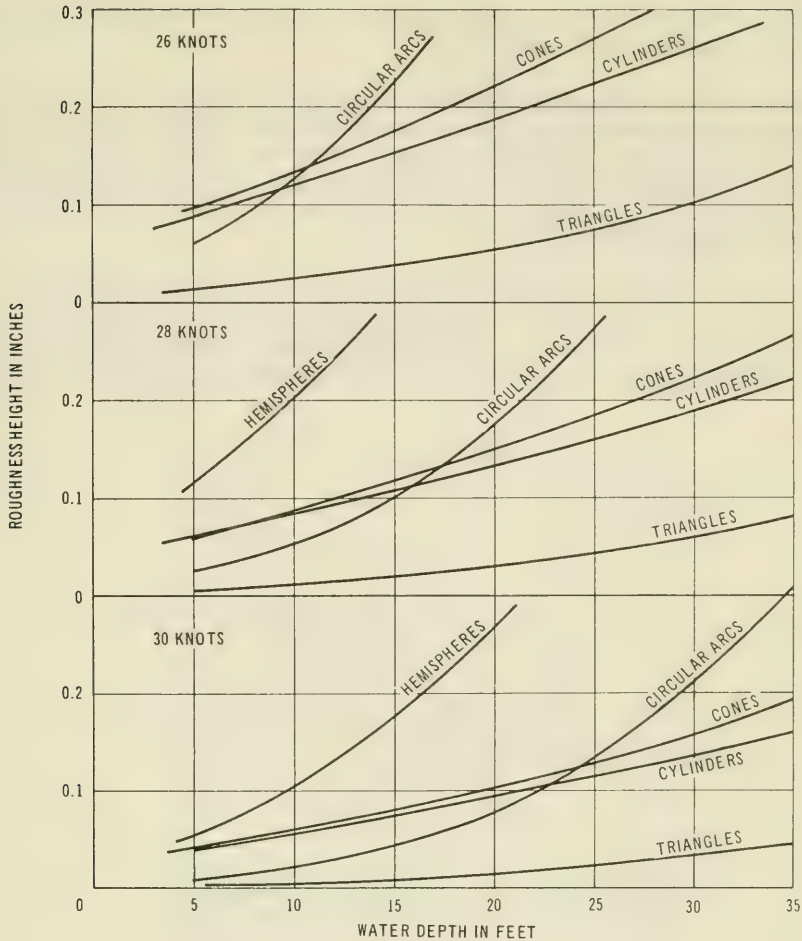


Fig. 5 - Effect of geometry of isolated roughness elements on cavitation inception ( $C_p = -0.5$ ,  $\delta = 0.4$  in.,  $m = 7$ ,  $T = 54^\circ\text{F}$ )

these curves might very well apply to large sonar domes on destroyer leaders or frigates. A temperature of  $54^\circ\text{F}$  was used as a representative temperature of the ocean.

Figures 4 and 5 compare the cavitation inception properties of the five different roughness geometries. The critical roughness heights for cavitation inception are plotted as functions of the flow velocity for two submergence depths in Fig. 4 and as functions of submergence depth for three flow velocities in Fig. 5. It is evident from these figures that two-dimensional triangular roughness elements are most critical for cavitation inception. The cylinders and cones, which also present sharp discontinuities to the flow, are the next



most critical. At flow velocities above 28 knots, all types of roughness must be kept extremely small to avoid cavitation.

Figures 6, 7, and 8 show the effect of the shape of the boundary layer profile on cavitation inception over three of the roughness geometries: triangles, circular arcs, and cylinders. Cavitation inception over the triangle and circular-arc roughnesses is particularly critical to changes in velocity profile because of the averaging process needed to represent the local flow velocity  $\bar{u}_h$  over the two-dimensional roughness elements.

An increase in water temperature from 35 to 75°F increases the vapor pressure head from 0.23 to 0.86 ft of water and decreases the kinematic viscosity from  $1.87$  to  $1.04 \times 10^{-5}$  ft<sup>2</sup>/sec. The change in vapor pressure is equivalent to a decrease in submergence depth of 0.63 ft, which makes the body only slightly

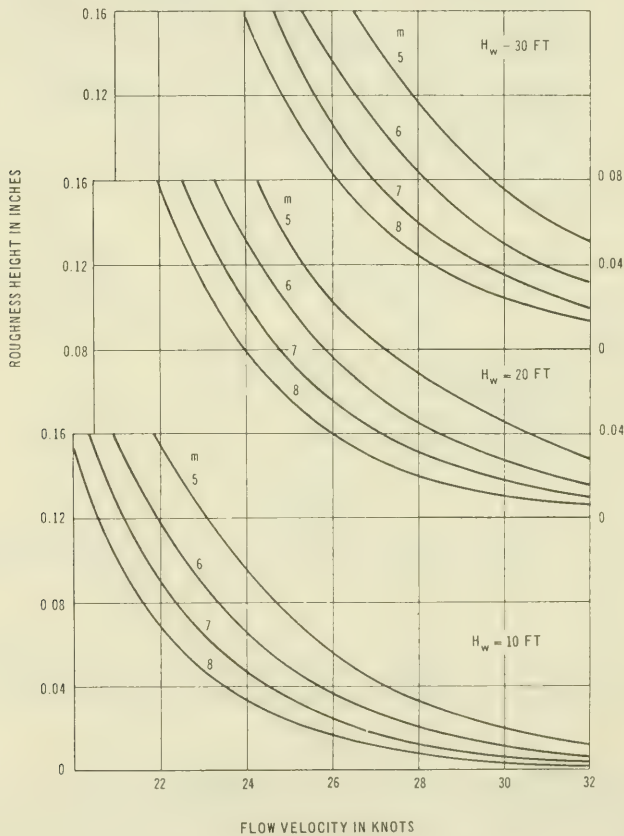


Fig. 6 - Effect of the velocity profile on cavitation inception on a two-dimensional triangular roughness ( $C_p = -0.5$ ,  $\delta = 0.4$  in.,  $T = 54^\circ\text{F}$ )

## Cavitation Inception on Rough Bodies

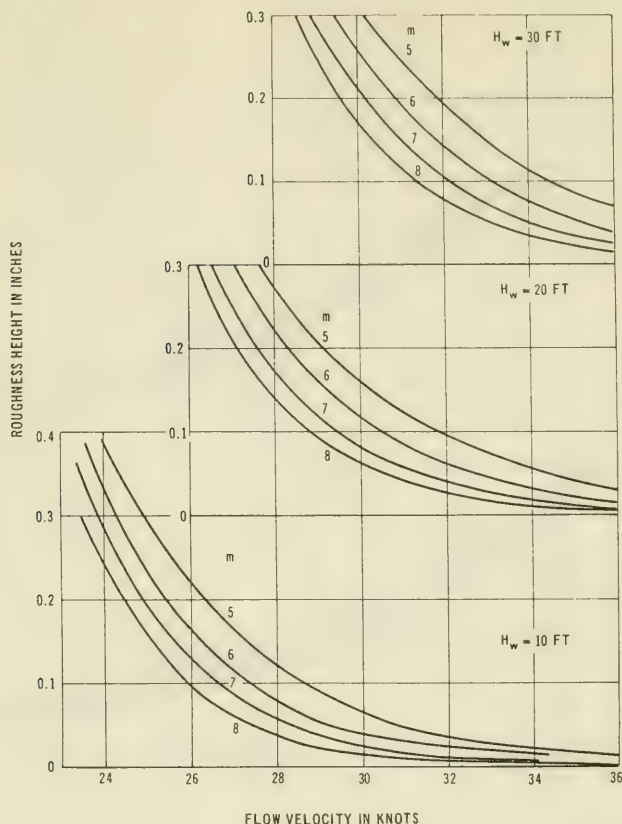


Fig. 7 - Effect of the velocity profile on cavitation inception on a two-dimensional circular-arc roughness ( $C_p = -0.5$ ,  $\delta = 0.4$  in.,  $T = 54^\circ\text{F}$ )

more susceptible to cavitation. The change in viscosity, however, makes a significant change in Reynolds number. As the temperature increases, the cavitation inception velocity decreases significantly on those roughness elements which are particularly sensitive to Reynolds number effects. Figures 9 and 10 show how the critical roughness heights of triangles and cylinders vary with flow velocity for the two extreme temperatures. Whereas a  $40^\circ\text{F}$  temperature change decreases the cavitation inception speed for two-dimensional triangular roughness elements by only about 0.5 knot, it decreases the inception speed for three-dimensional cylindrical roughnesses by about 2 knots.

Figures 11 and 12 show the relationships between roughness heights, the boundary layer thickness, and the pressure coefficient of the parent body. It can be seen in both figures that roughness is most critical in small boundary

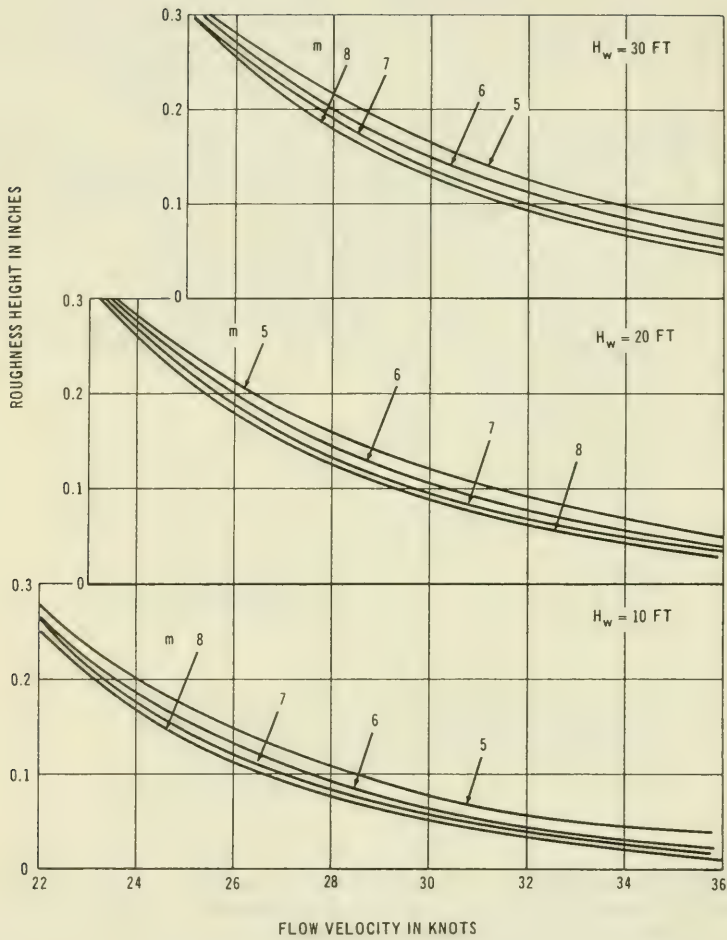


Fig. 8 - Effect of the velocity profile on cavitation inception on a three-dimensional cylindrical roughness ( $C_p = -0.5$ ,  $\delta = 0.4 \text{ in.}$ ,  $T = 54^\circ \text{F}$ )

# Cavitation Inception on Rough Bodies

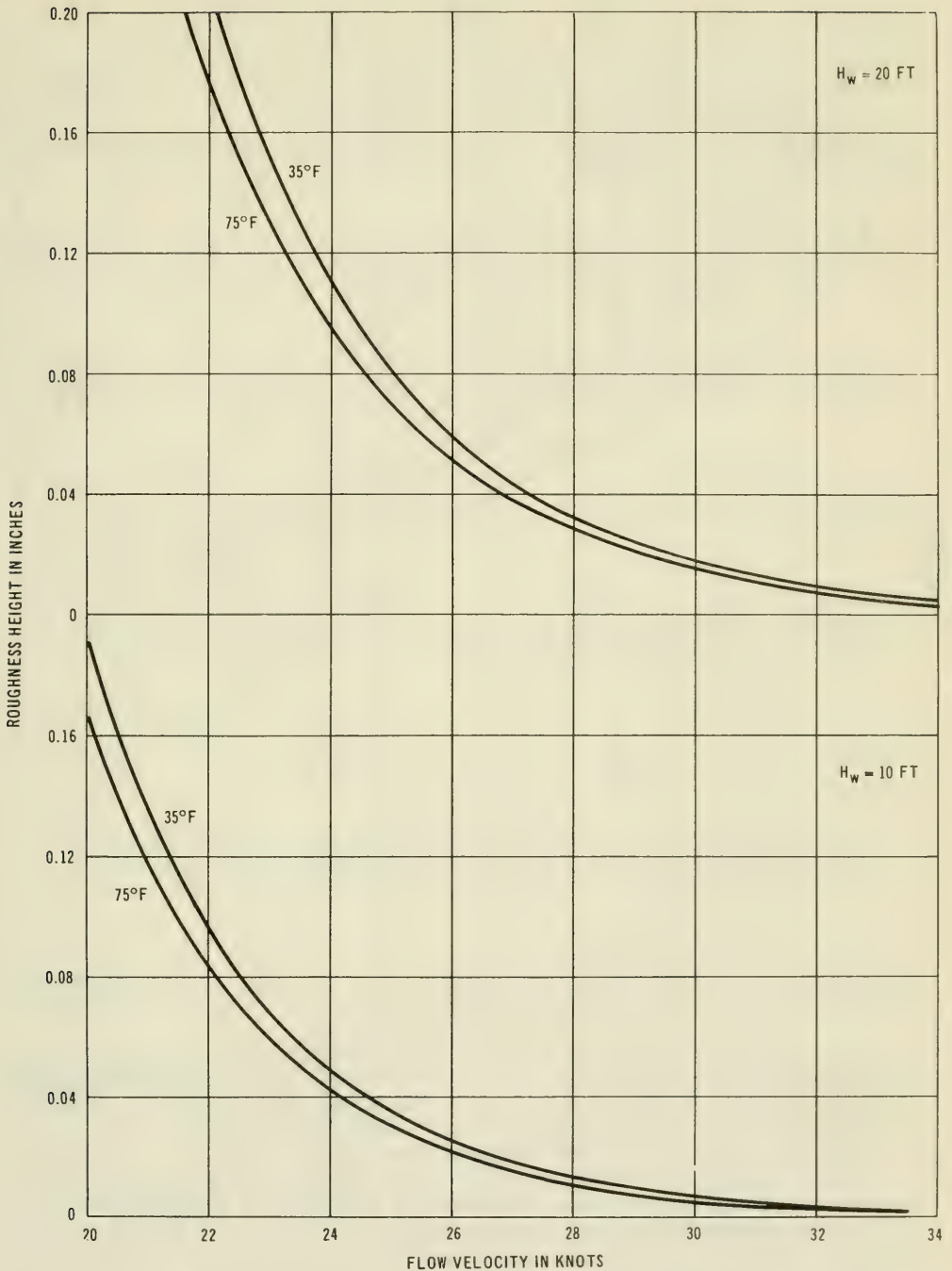


Fig. 9 - Effect of the water temperature on cavitation inception on a two-dimensional triangular roughness ( $C_p = -0.5$ ,  $\delta = 0.4$  in.,  $m = 7$ )



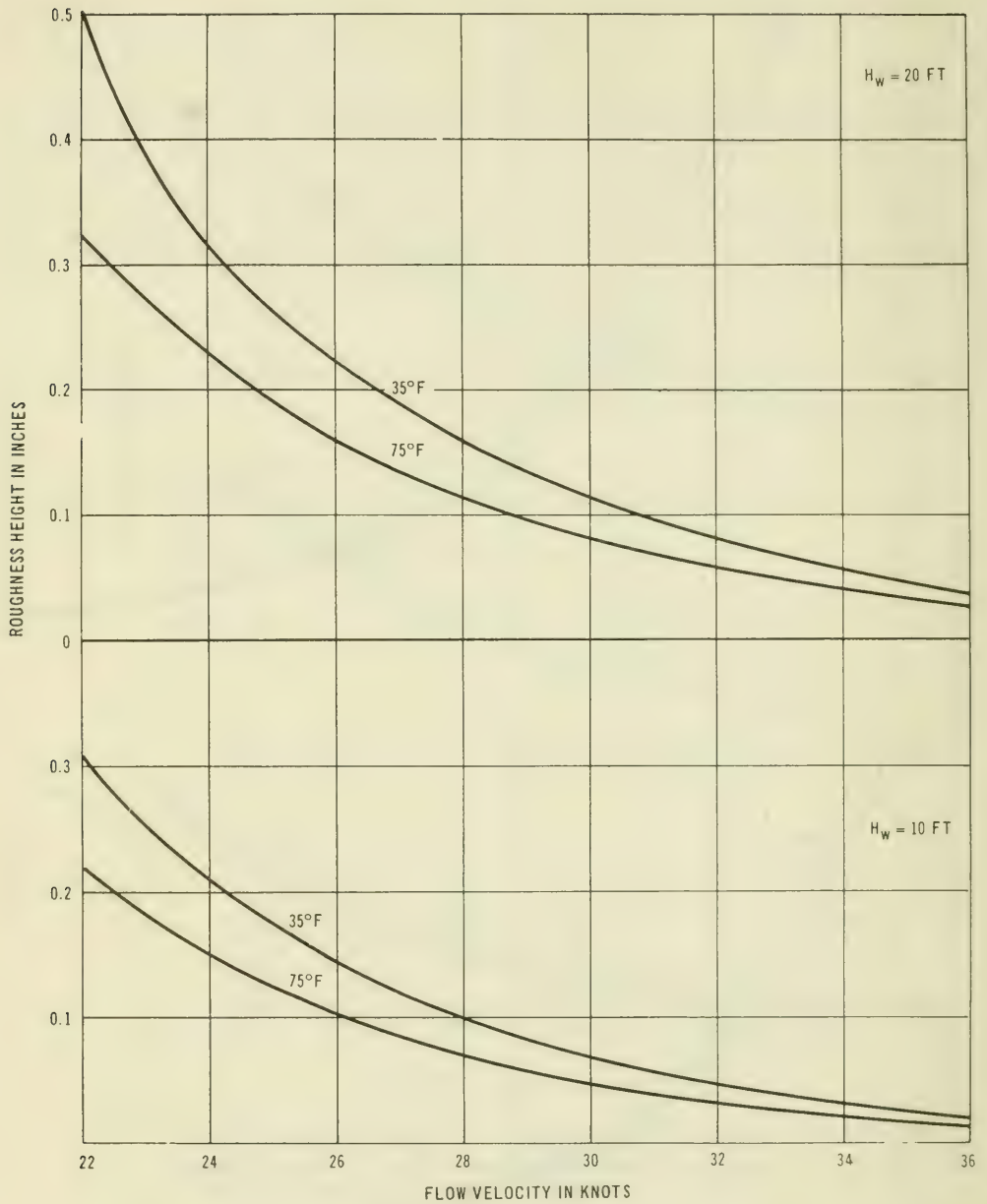


Fig. 10 - Effect of the water temperature on cavitation inception on a three-dimensional cylindrical roughness ( $C_p = -0.5$ ,  $\delta = 0.4 \text{ in.}$ ,  $m = 7$ )

## Cavitation Inception on Rough Bodies

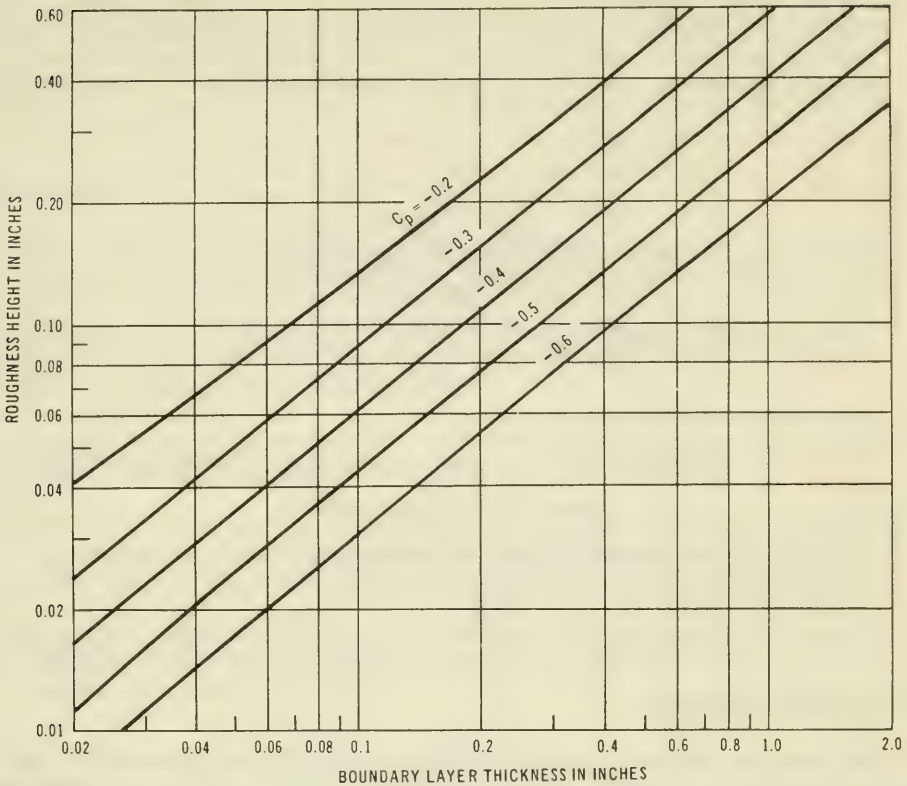


Fig. 11 - Effect of the pressure coefficient and the boundary layer thickness on cavitation inception on a two-dimensional roughness ( $U_\infty = 22$  knots,  $H_w = 15$  ft,  $m = 7$ ,  $T = 54^\circ\text{F}$ )

layers and on those areas of the body where the pressure coefficient is a minimum.

Figure 11 shows that the height of a triangular roughness needed to give cavitation inception at any particular point on a parent body increases with the boundary layer thickness. There are no breaks in the curves when the roughness element protrudes through the boundary layer. Figure 12 shows similar curves for the cylindrical roughness element. When the roughness height becomes greater than the boundary layer thickness, cavitation inception is no longer dependent on the boundary layer thickness. This follows from the original assumption that cavitation inception on a three-dimensional element is a function of the local velocity at the height of the roughness in the oncoming flow. Although the sharp discontinuities in the curves show a deficiency in the analysis, there are insufficient data for a suitable formulation of the local flow in the vicinity of the cavity on these three-dimensional roughness elements.

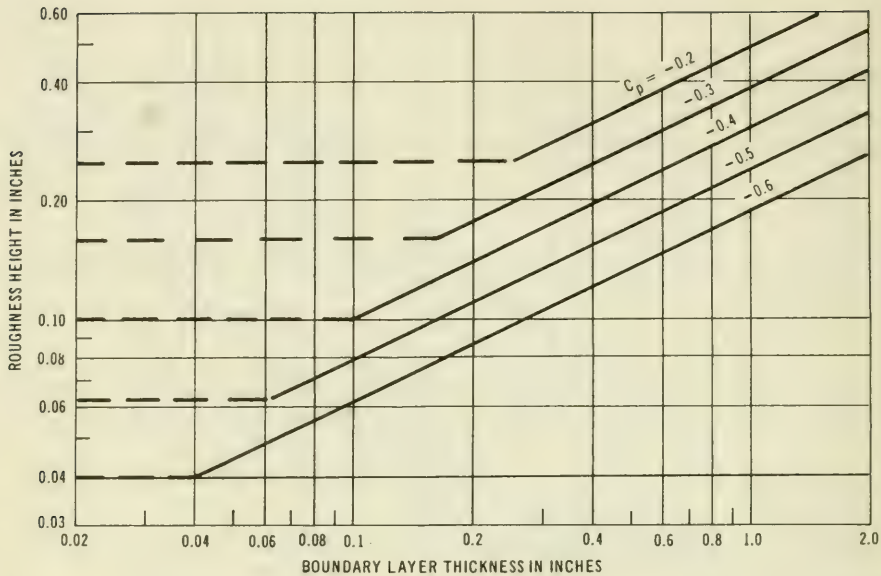


Fig. 12 - Effect of the pressure coefficient and the boundary layer thickness on a three-dimensional cylindrical roughness ( $U_\infty = 26$  knots,  $H_w = 15$  ft,  $m = 7$ ,  $T = 54^\circ\text{F}$ )

## CONCLUDING REMARKS

The foregoing analysis has not discussed such physical properties of the fluid as dissolved air content, air nuclei, surface tension, and various chemical and thermodynamic properties of the water. This paper is merely an analysis of two sets of experiments performed in different laboratories. In both sets of experiments, the data from cavitation inception on bluff bodies show a definite dependence on Reynolds number. The Reynolds number effect was particularly large on the cylinders and cones but was smaller on the hemispheres. The two-dimensional triangular roughness elements showed less dependence on Reynolds number than the bluff three-dimensional elements. The circular-arc roughnesses had a small height-to-length ratio and were not bluff enough to cause separation. Cavitation inception on these roughnesses was essentially independent of Reynolds number.

Although both studies indicate that small changes in air content did not affect cavitation inception speeds significantly, these physical parameters should be investigated further. The experiments should be repeated in other flow facilities, in other types of water, and at different air contents and temperatures.

Despite the limitations of the roughness data disclosed here, the results point up the need for considering viscous effects in the scaling of cavitation on bluff bodies or on roughness elements where flow separation may occur. Discontinuities from peeling paint or lapping seams can be likened to the

two-dimensional triangular roughness elements. Weld splatterings and barnacles are similar to the three-dimensional cylinders and cones. Plate buckling or poor forming of a hydrofoil or propeller blade would tend to change the pressure distribution and degrade the cavitation parameters of the hydrodynamic body. Small scratches and nicks on hydrofoil propeller blades can be very critical because these bodies operate at high speed and the boundary layers are thin.

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Friday, September 30, 1966

Morning Session

## **MANEUVERABILITY AND OCEAN PLATFORMS**

Chairman: R. Brard

Bassin d'Essais des Carènes  
Paris, France

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# SOME HYDRODYNAMIC ASPECTS OF SHIP MANEUVERABILITY

J. N. Newman  
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## ABSTRACT

Recent developments are discussed which further the understanding of the hydrodynamic force and moment acting on surface vessels during maneuvers. Photographs are shown of the flow past a model of a Mariner class hull with several drift angles, to support the validity of simplified mathematical models for the hydrodynamic force and moment analyses. It is shown that memory effects due to separation and free-surface wave motions, together with the non-analytic dependence of the nonlinear viscous forces on drift angle, require a more complicated mathematical model for the force and moment than the usual Taylor series expansion in powers of the velocity and acceleration. Methods for analyzing the added mass, circulation, free surface, viscous, and propeller influences are reviewed and discussed. Experimental and theoretical evidence of the existence of memory or frequency effects is noted. Several recommendations are made for detailed problems which could profit from further investigation.

## INTRODUCTION

The study of maneuverability and control of surface ships had its genesis when man first set forth with raft and paddle. In the ensuing millennium this field has become progressively more refined and scientific. Nevertheless much work remains to be done in order to advance our understanding of the hydrodynamic phenomena which are involved when a ship undergoes a maneuver. The purpose of this paper is to review recent accomplishments and also to indicate possible directions for future research and progress.

The history of ship maneuverability and control can be traced back at least as far as Euler (1749). Two centuries later, the paper of Davidson and Schiff (1946) organized the subject of ship dynamics within a modern engineering framework and served as a stimulus and basis for most of the subsequent research in this field. Six years ago the David Taylor Model Basin was host to a Symposium on Ship Maneuverability which included a broad survey of the field by Norrbin (1960).



Subsequently the International Towing Tank Conference has organized a large concerted experimental program in which several model basins have participated to compare experimental model and full-scale maneuvering data for a Mariner hull. This activity has led to extensive progress in experimental methods, particularly in the widespread employment of mechanical oscillators or "planar motion mechanisms" capable of exciting forced sinusoidal motions of ship models and measuring the associated force response. There is evidence to suggest that the results of this type of test can be applied in the prediction of non-oscillatory motions, such as the steady turning of ships, with the same degree of confidence as the prediction by a direct small-scale time-domain modeling of the maneuver and with substantially more generality.

In recent years there have also been a few investigators who dared to attack with more fundamental techniques the hydrodynamic aspects of the motions of ship hulls in the horizontal plane. Low aspect-ratio wing theory has been applied to yield predictions of the side force and moment on a yawed ship. Similar analytical methods have been used to estimate the free surface effects associated with wave motion at the air-water interface. And a parallel experimental investigation has been carried out to compare the side force on a yawed surface ship model with that of an immersed geosim double-body. As a result of these studies and of closely related developments in the field of aerodynamics, one can claim to understand qualitatively most of the separate mechanisms which contribute to the hydrodynamic force and moment acting on the hull. However, as in the field of speed and powering, we are less certain of the interactions between the separate aspects of the problem.

Reduced to its bare essentials, the analysis of ship maneuverability deals with the motion of a rigid body on the free surface of a real fluid, subject to the influence of the body's control surfaces and propellers. The dynamics of the rigid body itself (i.e., the inertial characteristics of the ship) can be readily treated with Newton's laws, but the dynamics of the surrounding fluid cannot be described quantitatively unless severe idealizations are made. The difficulty rests primarily with the complications of viscous and free surface effects. Thus the classical description of a rigid body in an ideal unbounded fluid, which is so elegantly developed in Lamb's "Hydrodynamics," is not applicable except as a guide in setting up the equations of motion. And in principle not even this is permissible since the generation both of vorticity and of surface waves will give rise to "memory" effects of the fluid motion which will fundamentally affect the form of the equations of motion.

In view of the complexity of the flow phenomena involved, it is not surprising that most of the work in this field is semi-empirical. Nevertheless, available analytic methods can be relied upon for qualitative predictions and these can serve the important role of supplementing and guiding the process of experimental investigation. It is hoped that this paper will help to advance that role by surveying and collating some of the fundamental studies in the field. We shall restrict ourselves to conventional ship hulls whose motion is primarily in the horizontal plane and in otherwise calm water of infinite depth and horizontal extent. Thus we will not discuss such topics as maneuverability in restricted waters or in a seaway (including broaching!) or the behavior of high-performance vessels such as planing boats, hydrofoils, or ground-effect-machines.

The remaining portion of this paper will be devoted to a detailed discussion of the relevant hydrodynamic force and moment which act on a ship hull during a maneuver, including analytical methods of description and those experimental results which help to further our understanding of the mechanisms involved. The Appendix includes a brief statement of the nomenclature and equations of motion which are the basis for most work in this field.

## QUALITATIVE DESCRIPTION OF THE FLUID FLOW

The practical description of the flow of water past a ship hull during a maneuver poses one of the most complex problems encountered in the field of ship hydrodynamics. Included to a significant extent are most of the phenomena which can be associated with an incompressible Newtonian fluid. Under the circumstances it is not surprising that literature on the subject of ship maneuverability is devoted primarily to experimental investigations and techniques. Fortunately at least a qualitative description of the flow and associated force and moment can be provided if the disturbance of the free surface is sufficiently small, the boundary layer is thin, and there is no large-scale separation or ventilation.

In order to evaluate some of the above assumptions, flow observations and photographs were made in the Circulating Water Channel of the David Taylor Model Basin. A model of the Mariner Class ship USS Compass Island was used. The results of this investigation are illustrated in Figs. 1-6. The model is 12.6 feet long and the flow velocity is 3.1 knots, corresponding to a full-scale speed of 20 knots. Steady drift angles of 0, 3, and 10 degrees are shown. Bilge keels and a rudder were installed on the model, but the propeller was not in place. The flow can be visualized by means of the wool tufts fastened on the port side of the model and also by means of dye streams, which were injected through the hull at stations nine and fifteen and through two external tubes at the bow. The wool tufts located on integer stations were fastened directly to the hull, whereas those located on half-stations were fastened to pins and separated from the hull surface by a distance of about one inch.

Over the forebody the perturbation of the basic flow by the drift angle is in agreement qualitatively with the expected cross-flow; that is the streamlines are curved down on the upstream side of the hull and up on the downstream side. However on the afterbody the situation is reversed, to an increasing extent with distance downstream, and the downward flow on the downstream side is especially pronounced immediately ahead of the propeller aperture. It should be noted that this downward flow on the downstream side (Fig. 1, bottom photograph) occurs first in the vicinity of stations 12 and 13, where it is confined to the immediate hull surface, whereas further aft the downward flow has thickened so as to be indicated on the outer (half-station) tufts as well. Thus it appears that this reversal of the perturbation crossflow is confined to a boundary layer which becomes increasingly thicker with distance downstream. A similar effect near the bow can be noted in the bottom views of the downstream side of the hull (Fig. 6) where there is a noticeable "phase difference" between the inner and outer tufts which disappears with increasing distance downstream. However this secondary flow occurs only close to the surface and at the bow, whereas

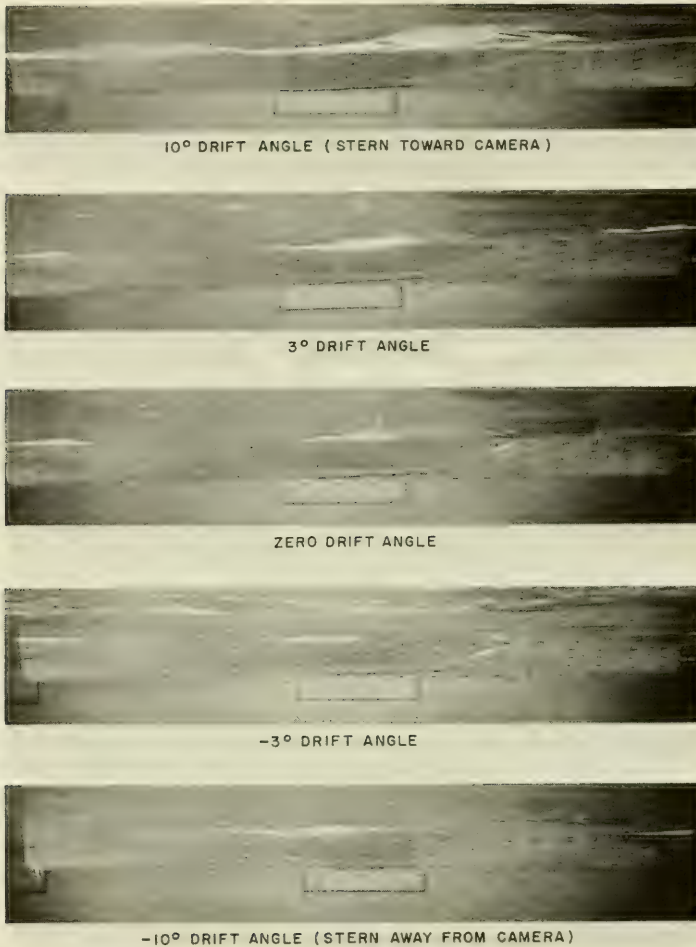


Fig. 1 - Side view of MARINER hull in the circulating water channel showing the flow pattern at various drift angles (model length 12.6 ft, model speed 3.1 knots)

further downstream the entire cross-flow perturbation is in the expected direction, so that the growth in thickness with distance downstream does not occur.

A pronounced asymmetry of the free surface can be detected in the vicinity of station 16 (Fig. 1). In the straight condition (zero drift angle) there is a wave trough centered between stations 15 and 16. In the yawed condition this wave disappears on the downstream side of the hull (bottom photograph) but increases slightly and moves aft to station 17 on the upstream side.



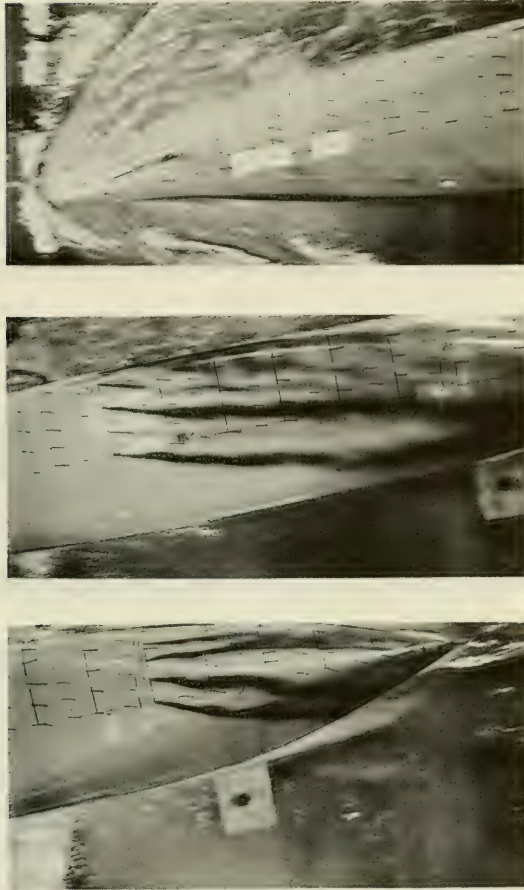


Fig. 2 - Bottom view of MARINER hull in the circulating water channel showing the flow pattern at a  $10^\circ$  drift angle



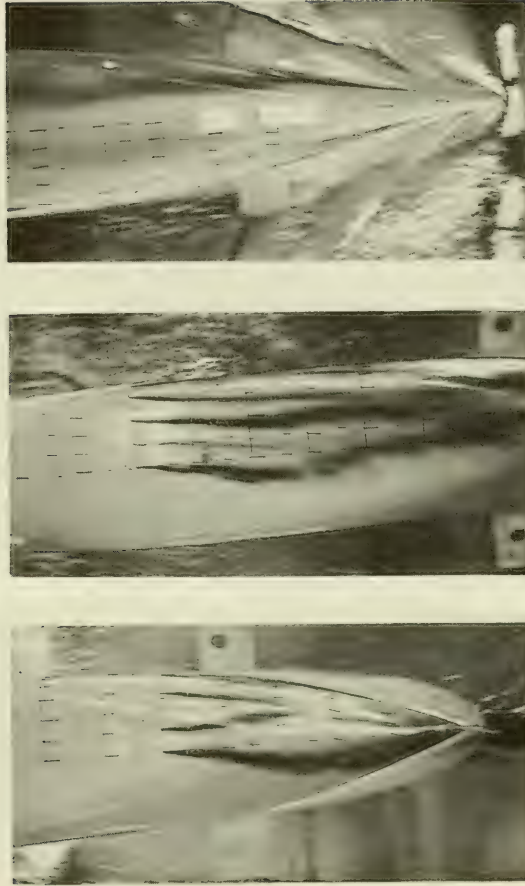


Fig. 3 - Bottom view of MARINER hull in the circulating water channel showing the flow pattern at a 3° drift angle

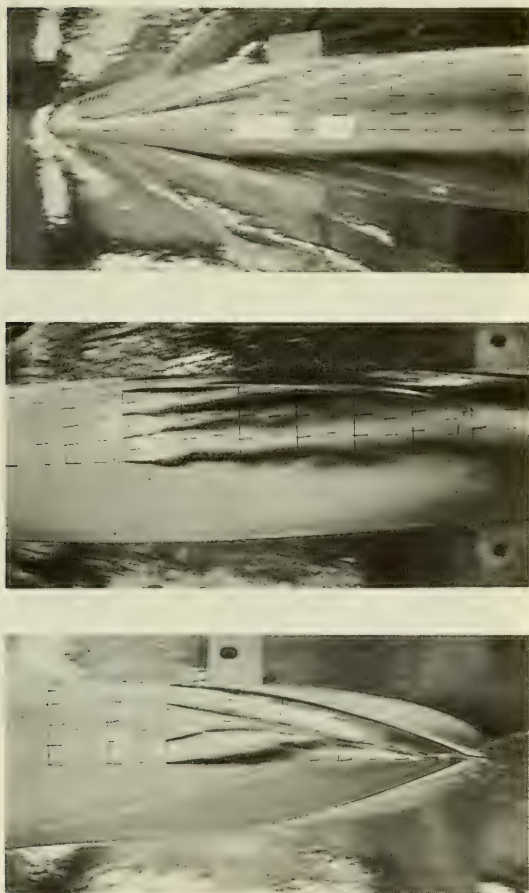


Fig. 4 - Bottom view of MARINER hull in the circulating water channel showing the flow pattern at zero drift angle

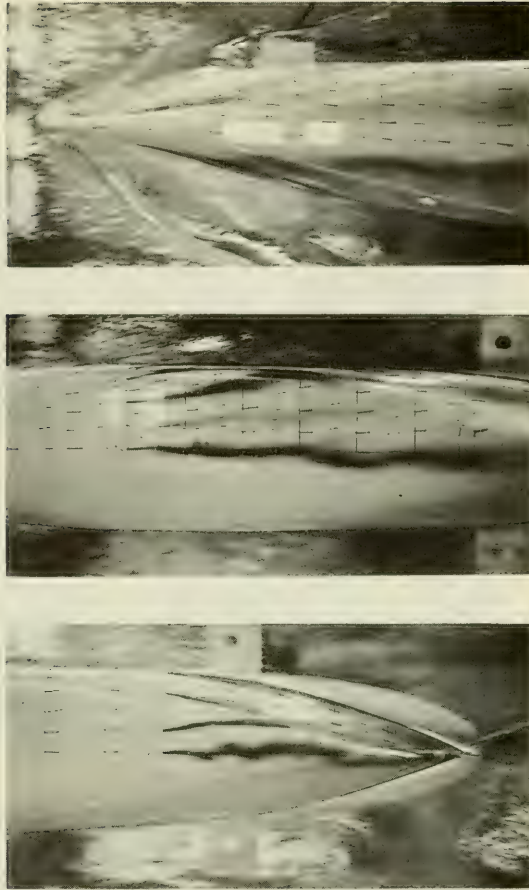


Fig. 5 - Bottom view of MARINER hull in the circulating water channel showing the flow pattern at a  $-3^\circ$  drift angle

The principal conclusion to be drawn from these flow photographs is that for a hull such as the Mariner there is no large scale separation up to drift angles of 10 degrees, if we assume that such a conclusion can be made from model results with Froude scaling. On the other hand there will be local areas of separation, especially at the bilge keels and stern, and occasional traces of ventilation at the bow, and there will be a rather complex boundary-layer flow on the afterbody which increases in thickness towards the stern. A consequence therefore is that a theoretical inviscid-fluid model based upon low aspect-ratio wing theory or slender body theory, preferably with free surface effects included, should give a satisfactory description of the flow over the forebody. It should also suffice for describing the large-scale flow over the entire hull, at least for steady-state drift angles up to ten degrees. Thus there is value in pursuing

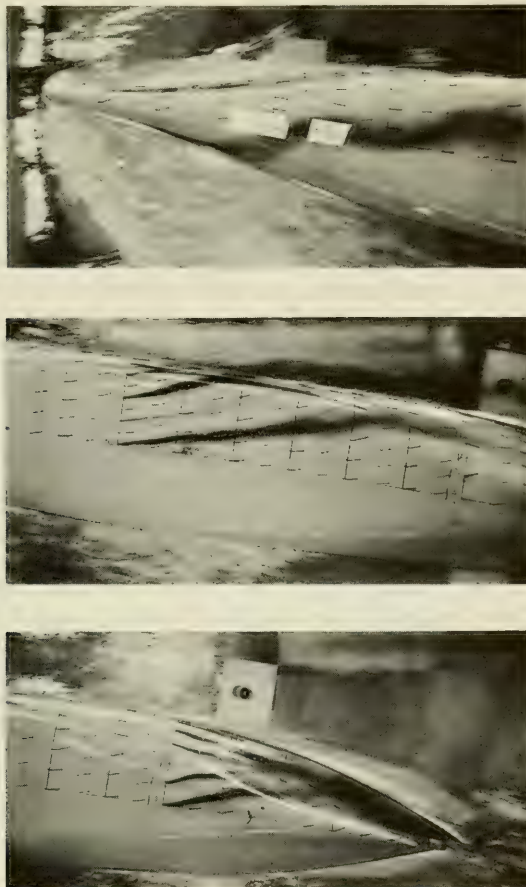


Fig. 6 - Bottom view of MARINER hull in the circulating water channel showing the flow pattern at a  $-10^\circ$  drift angle

idealized analytical solutions for the hydrodynamic force and moment, in spite of the complexity of the real physical flow.

#### ANALYTICAL EXPRESSIONS FOR THE FORCE AND MOMENT

As in the case of ship motions in waves, there are two common methods for obtaining in analytical form the hydrodynamic force and moment which act on a ship hull during a maneuver. We can use our physical understanding or empirical knowledge to formulate equations in which the force and moment have an assumed dependence on the motions of the ship, generally in the form of Taylor series expansions in powers of the displacements, velocities and accelerations



with undetermined coefficients of damping, added mass, etc. Alternatively we may formulate beforehand a physically realistic but mathematically tractable idealized model for the ship and fluid, and proceed in a systematic manner through the analysis thereof. In general the first approach is expedient but dangerous, while the second is elegant but less productive.

In the first approach, following Abkowitz (1964), one assumes that the hydrodynamic force and moment at any instant of time are analytic functions of the linear and angular acceleration, velocity and displacement of the hull at that same instant and that they are independent of any other details of the motion except for the geometrical properties of the ship and the physical properties of the water. It follows that the six components of the force and moment can be expanded as Taylor series in powers of the above variables, and this leads directly to a set of linear and nonlinear terms in the equations of motion. Of course this does not furnish immediately the desired solution, since the coefficients of these terms remain to be determined either through analytical or experimental techniques; usually the latter are employed at this point, so that the approach serves only as a method of curve-fitting. However it does offer the very practical advantage of a basis for generalizing captive model experimental results to free maneuvering problems.

In principle there are fundamental objections to the assumption that the forces and moments are analytical functions of the above mentioned variables. In recent years Cummins (1962) and Brard (1964) have called attention to the "memory" associated with the effects of the free surface and of vorticity, respectively, and the resulting necessity to represent the hydrodynamic forces arising from a transient ship motion in terms of a convolution integral over the entire time history of the motion. This situation has been recognized for many years in the field of unsteady aerodynamics and in fact it was also pointed out by earlier workers in ship hydrodynamics, e.g., Haskind (1946). Thus we should properly consider the six components of the hydrodynamic force and moment to be of the form

$$F_i(t) = \int_{-\infty}^t K_i[u_j(\tau), t - \tau] d\tau \quad (i = 1, 2, \dots, 6),$$

where  $K_i$  is a kernel function which depends in general on all six velocity components  $u_j$ , and on the retarded time  $t - \tau$ . In the linearized case we have the more familiar and simple form

$$F_i(t) = \sum_{j=1}^6 \int_{-\infty}^t u_j(\tau) K_{ij}(t - \tau) d\tau,$$

where the kernel  $K_{ij}$  depends only on the retarded time and on the geometrical properties of the ship and the physical properties of the fluid. As an example of the necessity for this representation, we note that in the case of a captive model which is given a short "pulse" disturbance and then returned to its original steady restrained condition, an unsteady fluid motion (visible especially in the free surface disturbance) and associated force and moment will persist thereafter, in principle *ad infinitum*.

An altogether different objection to the Taylor series approach arises from the existence of nonlinear effects associated with separation drag. This can be illustrated in the simplest case of steady motion at a drift angle  $\beta$ , where there is both theoretical and experimental evidence (Thwaites, 1960) that the side force is of the form\*

$$\begin{aligned} Y &= A \sin 2\beta + B \sin \beta |\sin \beta| \\ &= 2A\beta + B\beta |\beta| + O(\beta^3) \end{aligned}$$

for a slender body with transverse symmetry. (Here  $A$  and  $B$  are constants which depend on the body shape and Reynolds number.) In the Taylor series approach, on the other hand, second-order effects would of necessity be even functions of the drift angle and are ruled out by the transverse symmetry condition, so that the nonlinear effects are by assumption of third order in the drift angle. This particular point has led to some controversy regarding the correct mathematical model for curve-fitting of experimental data. Martin (1961) and Norrbin (1965b) lump all of the nonlinearities in a second-order term, whereas most other workers assume that the third-order correction is valid, unless second-order (even) terms are present as a result of the asymmetry introduced by the ship's propeller (Suarez, 1963).

It should be emphasized that those components of the hydrodynamic force and moment which can be regarded as inviscid in their origin can in fact be represented by finite Taylor series in powers of the drift angle, so that the nonlinearities of these forces will be more amenable to the Taylor series approach. These will include the higher order effects associated with both circulation (in the idealized lifting-surface sense) and with the free surface. In summary, both second- and third-order terms should be included in a nonlinear model, and it may be expected that these will not always scale simultaneously since the separation drag will generally depend on the Reynolds number.

## SOME SPECIFIC THEORETICAL MODELS

As stated in the Introduction, progress with theoretical predictions of ship maneuvers requires that the various hydrodynamic processes be treated separately, or at most in pairs, since the complete problem is intractable. Thus it is necessary to restrict ourselves successively to individual details of the hydrodynamics in order to discuss specific theoretical techniques for prediction. The present section will be devoted to the discussion of some of these, with emphasis on those aspects wherein theoretical predictions appear to be most promising.

### Classical Added Mass

The oldest and simplest mathematical model is formulated by assuming that the fluid is ideal and ignoring the wave effects of the free surface. The

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\*The subsequent notation is as given in the Appendix.

effects of the free surface are approximated by reflection of the underwater portion of the hull above the surface, either with a positive (in phase) image corresponding to very slow motions or a negative (opposite phase) image corresponding to very rapid motions. It follows that there will exist a hydrodynamic force and moment which are strictly linear functions of the acceleration of the body or ship, and which can be lumped with the inertia terms in Euler's rigid-body equations of motion, equivalent to an added mass and added moment of inertia of the body itself. Rough estimates of the added mass and added moment of inertia for specific hull forms can be inferred from the extensive data available for ellipsoids. For elongated bodies such as ship hulls quantitative calculations can be based upon slender body theory which for lateral motions is equivalent to the simple strip theory. The sway added mass and yaw added moment of inertia are given respectively by the integrals

$$\begin{pmatrix} Y_{\dot{v}} \\ N_{\dot{\psi}} \end{pmatrix} = \int \begin{pmatrix} 1 \\ x^2 \end{pmatrix} m(x) dx,$$

where  $m(x)$  is the added mass of the two-dimensional transverse section at  $x$ , and the integral is over the length of the ship hull. The corresponding cross-coupling force and moment are

$$Y_{\psi}^{\ddot{}} = N_{\dot{v}}^{\ddot{}} = \int x m(x) dx.$$

This last symmetry relation is confirmed by experiments only in the low-frequency domain (van Leeuwen, 1964). At higher frequencies the inequality of the cross-coupling coefficients is probably due to the shedding of unsteady vorticity in the wake.

## Circulation

The next degree of complexity is introduced because the classical added mass concept cannot account for the side force and yaw moment associated with a steady drift angle. These are analogous to the lift force and pitching moment on a wing or hydrofoil and can be analyzed by regarding the hull and its image above the free surface as a symmetrical (uncambered) lifting surface, with the drift angle taking the role of the conventional angle of attack. In view of the small draft-length ratio of ships, we confine ourselves to the theory of wings of small aspect-ratio, rather than considering lifting-surface theory in its full generality. Here we can choose to treat the simplest case of a flat plate, appropriate to the case of small beam-draft ratio, or account for the fullness of the ship's sections within the framework of slender-body theory.

The applicability of these theories, at least to the total force and moment, rests on the validity of a Kutta condition at the ship's stern, as opposed to the more physical boundary layer growth and separation which may be expected to occur. In this context it is important to recall that the low aspect-ratio wing theory is somewhat unreliable in the case of a wing with rectangular planform and especially in the case of a wing whose span is decreasing towards the



trailing edge. Thus its applicability to a conventional ship hull with a long horizontal keel profile is open to question, and it certainly would not be applicable to ships with cut-away stern profiles.

The principal feature of the slender-body theory and low aspect-ratio wing theory, in the context of the lateral force and moment, is that the differential side force acting on each transverse section of the body depends only on the geometrical characteristics of that section. This (steady) side force can be expressed in terms of the added mass of the same two-dimensional section for horizontal motions, in accordance with the formula

$$Y_{2D} = -V^2 \beta \frac{d}{dx} m(x)$$

which has been derived by Lighthill (1960). Here  $V$  is the forward velocity,  $\beta$  is the drift angle, and  $m(x)$  is the two-dimensional added mass of the section. Integration of this differential force over the length of the hull gives the total side force and moment in the form

$$\begin{aligned} Y &= -V^2 \beta \int_{-L/2}^{L/2} \frac{d}{dx} m(x) dx = V^2 \beta m \left( -\frac{L}{2} \right) \\ N &= -V^2 \beta \int_{-L/2}^{L/2} x \frac{dm(x)}{dx} dx \\ &= V^2 \beta \int_{-L/2}^{L/2} m(x) dx - V^2 \beta \frac{L}{2} m \left( -\frac{L}{2} \right) \\ &= V^2 \beta \left[ Y_v - \frac{L}{2} m \left( -\frac{L}{2} \right) \right]. \end{aligned}$$

Note that we have allowed for the possibility of a non-zero added mass at the stern, as would be the case for a finite trailing edge or vertical deadwood on the hull, whereas we have assumed that the bow is sufficiently pointed that the added mass vanishes at the forward end of the hull.

If the body is pointed at the stern,  $m(-L/2) = 0$  and the classical potential-flow results are obtained:

$$Y = 0$$

$$N = V^2 \beta Y_v$$



On the other hand for a flat low aspect-ratio wing of rectangular planform and span  $2T$  we obtain the well known results\*

$$Y = \pi \rho V^2 \beta T^2$$

$$N = \frac{1}{2} \pi \rho V^2 \beta T^2 L.$$

The practical application of these simple formulae is somewhat limited by the occurrence of separation. However if attention is focused upon the side force acting only upon the forebody, these results will be expected to furnish more realistic predictions. This situation is illustrated by the experiments and theoretical calculations of Norrbin (1965a), which are summarized in Table 1. Measurements were made on both the total force and the force on the forebody of a double-body model of a cargo ship. Experimental data are shown for the side force acting on the forebody alone, on the total hull without rudder, and on the total hull with rudder. Theoretical predictions are given for the forebody based upon slender-body theory as well as upon the low aspect-ratio flat plate theory which, since the draft is constant along the length, is applicable either to the forebody or to the entire hull. For the forebody alone both theories are in fair agreement with the experiments, the slender-body theory being within 5% and the low aspect-ratio wing theory within 10% of the measured force. As might be expected the comparison is less satisfactory for the entire hull. In this case the low aspect-ratio wing result is very close to the measured value for the bare hull but differs substantially from the measured value with the rudder fitted. Since the presence of the rudder may be expected to improve the validity of the theory, in the sense of providing a more substantial trailing edge, the good agreement in the case of the bare hull should be regarded with caution.

Table 1  
Comparison of Theoretical and Experimental  
Side Force Coefficients  $Y/(\frac{1}{2}\rho V^2 \beta LT)$  Obtained  
by Norrbin (1965a)

|                                      |       |
|--------------------------------------|-------|
| Low aspect-ratio theory              | 0.184 |
| Slender-body theory, forebody        | 0.193 |
| Forebody, experiment                 | 0.203 |
| Total hull, experiment (no rudder)   | 0.172 |
| Total hull, experiment (with rudder) | 0.232 |

Unsteady effects associated with non-zero yaw velocity also can be analyzed within the framework of low aspect-ratio wing theory or slender-body theory. Lighthill (1960) has shown that the two-dimensional unsteady side force is simply given by the expression

\*The formal extension of the slender-body theory results to this case is permissible for the lateral force and moment in spite of the fact that the leading edge violates the pointed-bow requirement (Ward, 1955).

$$Y_{2D} = - \left( \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right) \left\{ m(x) \left( \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right) h(x, t) \right\},$$

where  $h(x, t)$  denotes the transverse displacement of the ship's centerline from the original straight-ahead position. However, this approach does not produce different results from the usual pseudo-steady state added mass, damping, and restoring terms; its principal value is in showing that the pseudo-steady state approach can be rationalized if the hull is sufficiently slender and if there is no separation or shedding of vorticity along the length of the hull.

### Free-Surface Effects

Ship hydrodynamicists need no reminder that the motion of a ship on the surface of the water generates waves. Indeed the classical Kelvin wave system generated by the steady motion of a ship is one of the most beautiful and intriguing phenomena of our field. Nor do we need to be reminded that the hydrodynamic forces acting on the ship are influenced by the wave motion since wave resistance is an obvious example of this fact.

Michell's integral for the wave resistance of an idealized ship hull is so old and well studied that it is surprising to find, with only two exceptions, no parallel work in the case of the side force and moment on a yawed ship, even in the simplest case of a steady drift angle. The force and moment on a yawed thin ship or surface-piercing flat plate can be formulated in terms of a lifting-surface integral equation, but the resulting kernel is complicated and no calculations have been attempted. However the following important qualitative conclusions can be established by this means:

1. In the limit of low Froude numbers the kernel of the integral equation tends to that associated with the ship plus its simple mirror image above the free surface, corresponding to the lifting-surface problem of a flat wing of span equal to twice the draft of the ship and chord equal to the length of the ship.

2. In the limit of high Froude numbers the kernel of the integral equation tends to that associated with the reversed image (negative angle of attack) of the hull above the free surface, corresponding to the problem of a wing with a discontinuous asymmetrical twist.

3. For sufficiently small aspect-ratio (i.e., small draft-length ratio) the solution for finite Froude numbers tends to that associated with the low aspect-ratio wing and its simple mirror image in an infinite fluid. *Thus wave effects should be small if the draft-length ratio is sufficiently small.*

This problem has been treated in some detail by Hu (1961). He expanded the kernel of the integral equation in even powers of the Froude number thus obtaining a sequence of iterative equations with the relatively simple kernel corresponding to the zero Froude number limit. Subsequently he assumed that the aspect ratio is small so that the integral equations can be inverted in closed form. Computations on this basis show the side force and moment due to a steady drift angle to be increasing functions of the Froude number.

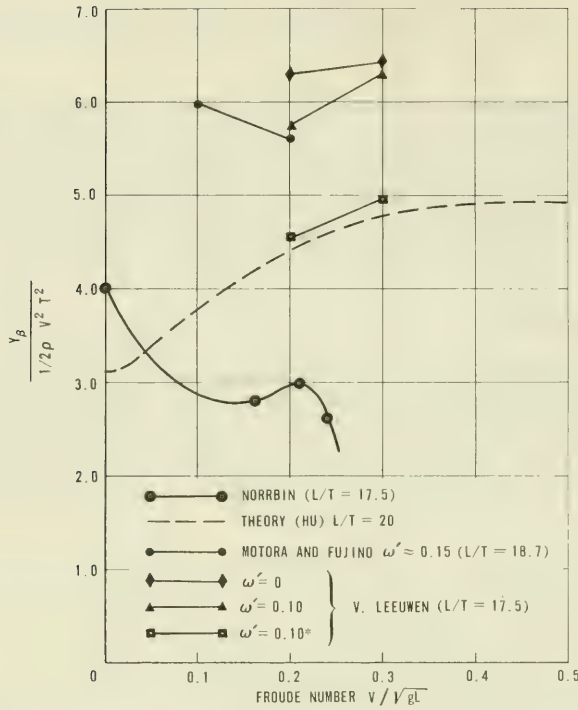


Fig. 7 - Comparison of theoretical side force with various experiments (an asterisk denotes experiments with bare hull; in all other cases the data is for base hull with rudder)

Figures 7 and 8 show a comparison of Hu's results with experiments. The theoretical curves in these figures are as computed by Hu for a rectangular flat plate of length-draft ratio  $L/T = 20$ . The experimental results are for somewhat different values of the length-draft ratio, and to account for this discrepancy the force and moment have been non-dimensionalized in terms of the square of the draft, so that according to the classical low aspect-ratio theory the results should then be independent of the length-draft ratio at zero Froude number. The side force is shown in Fig. 7, and it is seen that there is relatively poor agreement between the various results, both with respect to a comparison of the theory and experiments and with respect to a comparison of the experiments themselves. The corresponding situation for the moment, as shown in Fig. 8, is much better and here the agreement not only between the various experiments but also between the experiments and the theory is quite striking. It is remarkable that the moment, rather than the force, should be in a satisfactory state unless there is a substantial amount of separation confined to the midship area of the hull which influences the force but not the moment. With regard to the experimental results, both van Leeuwen and Motora and Fujino used a Series



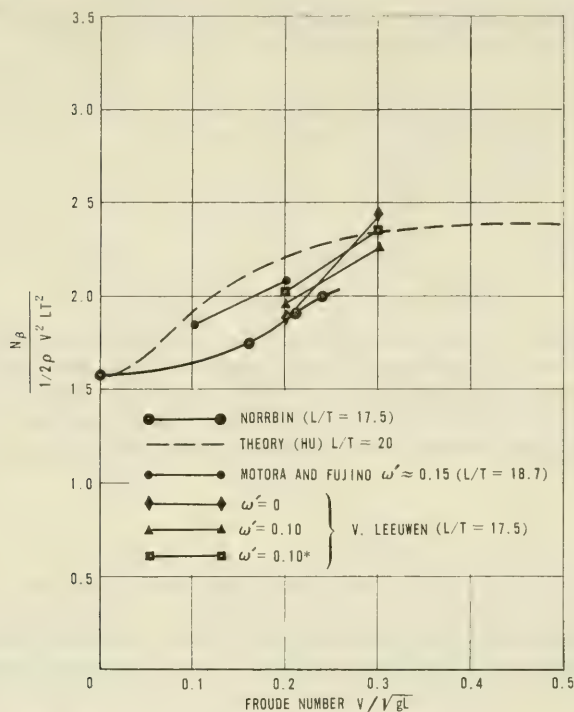


Fig. 8 - Comparison of theoretical drift moment with various experiments (an asterisk denotes experiments with bare hull; in all other cases the data is for base hull with rudder)

60 (Block .60) hull of length approximately 2 meters; Norrrbin tested a cargo liner model of length 5.5 meters, and at zero Froude number his results were obtained with a geosim double-body model of length 2.7 meters. One set of the results of van Leeuwen (denoted by \*) are for the bare hull alone, while the remaining experiments are for the bare hull with rudder. The experiments of Norrrbin, and one set of van Leeuwen's data, were obtained from steady towing tests with a static drift angle; the remaining tests of van Leeuwen and the results of Motora and Fujino are based upon the analysis of slow oscillatory drifting motions at the values of the nondimensional frequency parameter  $\omega' = \omega(L/g)^{1/2}$  shown.

It should be noted that the free surface can give rise to a steady side force without the existence of circulation, so that thickness effects as well as the angle of attack must be considered, in contrast to the analogous aerodynamic situation. These can be analyzed in much the same manner as the wave resistance. A convenient starting place for such a study is with Kochin's equations for the



longitudinal and transverse forces acting on an arbitrary floating body in steady translational motion (Wehausen and Laitone, 1960; Eq. 20.37):

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = - \frac{\nu^2 \rho}{2\pi} \int_{-\pi/2}^{\pi/2} |H(\theta)|^2 \sec^4 \theta \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} d\theta,$$

where

$$H(\theta) = \iint \left( \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \right) \exp \left[ -\nu \sec^2 \theta (ix_0 \cos \theta + iy_0 \sin \theta + z_0) \right] dS.$$

Here  $\nu = g/V^2$  where  $V$  is the forward velocity,  $\rho$  is the fluid density,  $\phi$  is the velocity potential, and the surface integral may be taken over any closed surface surrounding the body. We note that, as is customary in ship-wave theory, the reference frame is with respect to the forward velocity so that a rotation of the coordinates is required to obtain the following formula for the side force due to a drift angle  $\beta$ :

$$Y = \frac{\nu^2 \rho}{2\pi} \int_{-\pi/2}^{\pi/2} |H(\theta)|^2 \sec^3 \theta (\tan \beta - \tan \theta) d\theta.$$

For bodies with circulation this equation also can be applied provided the surface integral defining the Kochin function  $H(\theta)$  is taken over the body plus its vortex wake.

Apparently no equally general formula for the yaw moment has been derived; such an extension would appear to require the analysis of nonlinear free-surface effects.

The above equation is of limited value since generally the Kochin function is unknown, and it is more difficult to obtain rational approximations for idealized body shapes than in the parallel situation of wave-resistance theory, due to the fact that the body is yawed. For example, the Kochin function for a yawed thin ship will involve both source and dipole distributions, and it must be obtained from the solution of an integral equation analogous to lifting surface theory. However if the ship hull is assumed to be slender rather than thin, some progress can be made.

Restricting ourselves to the simplest case of a body of revolution, whose axis lies in the plane of the undisturbed free surface, the following results can be derived:

$$\begin{aligned} H(\theta) = & -V \int S'(x) \exp(-i\nu x \sec \theta) dx \\ & + \beta V \tan \theta \int [3S'(x) + xS''(x)] \exp(-i\nu x \sec \theta) dx \\ & + O(\beta^2, S^{3/2}), \end{aligned}$$

and thus the linearized side-force derivative, per unit drift angle, is given by the integral

$$Y_{\beta} = \frac{1}{\pi} \rho g \nu \int_0^{\pi/2} \sec^3 \theta \left\{ (6 \sec^2 \theta - 5) (P_S'^2 + Q_S'^2) + 2(\sec^2 \theta - 1) (P_S' P_{xS''} + Q_S' Q_{xS''}) \right\} d\theta,$$

where\*

$$P_S' + iQ_S' = \int S'(x) \exp(-i\nu x \sec \theta) dx$$

$$P_{xS''} + iQ_{xS''} = \int xS''(x) \exp(-i\nu x \sec \theta) dx.$$

The integrals with respect to  $x$  are taken over the body length and

$$S(x) = \frac{1}{2} \pi r^2(x)$$

is the sectional-area function of the submerged portion of the hull, with  $r(x)$  the local radius.  $S'(x)$  and  $S''(x)$  denote the first and second derivatives respectively. The practical value of these equations is limited by the fact that the side-force integral diverges unless the ends of the body are cusped, i.e., both the first and second derivatives of the sectional-area curve must vanish at the ends. However it seems likely that this non-uniformity can be accounted for (Tuck, 1964, Handelsman and Keller, 1966) and it would then be worthwhile to generalize the side-force integral to more general and realistic hull forms with full midbodies and a sharp trailing edge, to account analytically for the free-surface effects which have been observed in experiments.

Kochin's formulas have also been applied, but within the framework of thin-ship theory, by Reinov and Shen Tszu-In (1964). They decompose the potential into two terms, one due to thickness and one due to vorticity, and assume that the vorticity can be approximated locally by the corresponding value for a wing of low aspect-ratio in an infinite fluid. Integrals of the vortex distribution are carried out only over the surface of the hull, with no account taken of the vorticity in the wake. Unfortunately the results are presented graphically without sufficient identification to permit comparison with other theoretical or experimental data.

## Viscous Effects

In order to discuss the effects of viscosity we shall ignore the free surface, or, more precisely, consider the simple image of the hull above the free surface

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\*We note that the term with the factor 5 in this side-force integral may be identified with the wave resistance as determined from Michell's integral, and indeed there is a similarity in form between the complete side-force integral and Michell's integral.

so that the problem reduces to that of an elongated yawed body in an infinite real fluid. This is of course one of the fundamental unsolved problems of subsonic aerodynamics; if the body is basically flat a lifting-surface theory is appropriate and the circulation is prescribed by the Kutta condition at the trailing edge, but for a bluff body such as a body of revolution or ship hull there is no convenient way of prescribing either the location or strength of the shed vorticity. It is known that the vorticity and lift force arise from separation of the cross flow, with regions of large vorticity at the boundaries of the separated region which can be idealized in terms of vortex sheets. A detailed mathematical model based upon this idealization has been constructed by Brard (1964) for application to ship maneuvering problems, and in that work unsteady effects are included, but the final analytical results are limited to rather elaborate convolution integral representations for the forces, whose principal utility lies in indicating the proper form for the mathematical modeling and interpretation of experimental data. (Brard's experimental results will be discussed in a subsequent section.)

The lift on an oscillating body of revolution has been analyzed in considerable detail by Sevik (1965a,b). General momentum relations are derived for the lift in terms of the far-field circulation, which is determined by means of an unsteady laminar boundary-layer theory. The results are compared with experiments in the case of a slender spheroid and show substantial qualitative agreement, although the magnitude of the lift force is overpredicted in the theory by a factor of two. The moment is dominated by the inviscid potential flow result, and is reduced by only 16% due to viscous effects. A comparison is made between the pseudo-steady flow and that for high frequency oscillations, with substantial differences noted both in magnitude and phase. Comparison of the experimental pressure distribution with potential theory shows good agreement except over the after 20% of the spheroid. The unsteady results are limited to one value of the reduced frequency,  $\omega L/V = 9.68$ .

The steady lift and associated boundary-layer flow on a body of revolution at constant drift angle have been studied by Nonweiler (see Thwaites, 1960). Rather complicated flow patterns are described particularly in the case of large angles of incidence, but the form of the side-force coefficient is given very simply by the equation

$$Y'_\beta = (a_0 \cos \beta + C_d |\sin \beta|) \sin \beta,$$

where  $a_0$  is the lift-curve slope appropriate to small angles of incidence and  $C_d$  is the cross-flow drag coefficient. In practice these two coefficients must be determined experimentally, but the results appear to be valid with engineering accuracy throughout the range of angles up to and including normal incidence. It is pertinent to the correlation of model and full-scale data to note that there is a substantial Reynolds number dependence for the drag coefficient but not for the coefficient  $a_0$ , so that strictly the results from small-scale models should be used only within the linear regime.

### Propeller Influence

It has long been known that the action of a screw propeller or propellers during a maneuver was stabilizing or, in effect, that of a skeg (Davidson and



Schiff, 1946). Recent experimental confirmation of this has been given by van Leeuwen (1964) who performed oscillator tests both with and without the propeller. This stabilizing influence can be explained in terms of the velocity field incident upon a yawed propeller in an open stream; if the propeller and stern are swung, e.g., to port, then a stabilizing (positive to starboard) force reaction will be exerted by the propeller blades when they are in that part of their revolution moving to port, and vice versa. But the angle of attack of the blade will be increased by the drift angle when the blade is moving to port, and decreased when it is moving to starboard (if the propeller is operating in normal ahead conditions). Thus the net reaction on each blade over a complete cycle is a stabilizing force tending to return the stern to the original centerline. If the ship is backing or the propeller is located so as to pull at the bow, the opposite conclusion would result.

Quantitative data on the performance of a propeller in an oblique flow are available from the investigation of Gutsche (1964). The results show that for moderate values of the advance ratio  $J$  the force vector due to the resultant of the longitudinal and lateral propeller forces is rotated by approximately 50% of the drift angle beyond the ship's instantaneous  $x$  axis (in other words about 50% more than the rotation of the ship itself). However it is difficult to apply these results directly to normal ships, and especially to single screw vessels with large deadwood area, due to the interaction between the deadwood and the propeller. The deadwood may be expected to straighten the flow into the propeller, and thus the stabilizing side force due to the propeller will be reduced by a significant amount.

Similar interference phenomena must be considered in evaluating the important effects of the propeller slipstream on the rudder. This particular subject would be well suited to theoretical analysis.

## FREQUENCY EFFECTS IN GENERAL

Most of the discussion of the previous sections pertains to steady-state hydrodynamic forces and moments acting on a yawed ship, and these can only be useful if a pseudo-steady-state analysis is valid. Likewise the experimental results of captive model tests generally are analyzed on the assumption that the hydrodynamic force and moment depend only on the instantaneous velocity and acceleration of the ship, as stated earlier, whereas strictly speaking it is necessary to represent the force and moment as convolution integrals over the previous time history of the motion. These "memory" effects of the fluid appear in captive model tests as dependence of the force coefficients on the frequency of oscillation. They stem both from the vorticity which is shed from the oscillating hull and from the wave effects associated with the unsteady motion of the hull at the free surface. The characteristic nondimensional frequency parameter associated with the vortex wake is the reduced frequency  $\omega L/V$ , while for surface wave effects the corresponding parameter is  $\omega \sqrt{g/L}$ . Here  $\omega$  is the radian frequency of the oscillations,  $L$  is the ship length,  $V$  is the forward velocity and  $g$  is the gravitational acceleration. For sufficiently small frequencies these two parameters tend to zero and the pseudo-steady-state analysis is valid.



It is therefore important to determine the values of these parameters at which frequency effects become important and the characteristic frequencies which are significant in practical ship maneuvers. Here the assumption of calm water is important since it is well known from the field of seakeeping that significant frequency dependence exists in the practical range of ship speeds and frequencies of wave encounter, corresponding to values of the parameter  $\omega V/g$  of order one-half to one. But it is also known that conventional ships are relatively slow in their maneuvers, so that in this case frequency effects may be relatively unimportant.

In discussing this question we shall lump together both of the sources of frequency effects, viz., viscosity and the free surface, and examine the experimental and theoretical evidence available. This includes the oscillator tests at various establishments, both with surface ships and submerged bodies, the theoretical predictions which exist for unsteady forces as functions of the frequency, and full-scale data from ships' maneuvers which delineates the practical range of frequencies which are involved.

First we note that the two frequency parameters differ by the ratio of the square of the Froude number, or

$$\frac{\omega V}{g} = \frac{\omega L}{V} \left( \frac{V^2}{gL} \right)$$

so that for fast ships

$$\frac{\omega V}{g} \approx 0.1 \left( \frac{\omega L}{V} \right),$$

and for slow ships

$$\frac{\omega V}{g} \approx 0.03 \left( \frac{\omega L}{V} \right).$$

Thus a tentative conclusion is that the reduced frequency will be the controlling factor, or that unsteady viscous effects will become significant before those associated with the free surface, especially for slower vessels. The experimental results which have been obtained with oscillator techniques have been limited primarily to one Froude number in each case, or at best to a narrow range of Froude numbers; it would be of interest to vary the Froude number sufficiently in a given experiment to determine whether the frequency effects which are in fact noted therein will collapse with respect to one or the other of the two non-dimensional frequency parameters, thus giving a direct measure of the relative importance of the two physical mechanisms involved. To a limited extent this comparison can be made from the data of van Leeuwen (1964) which includes results for the Froude numbers 0.2 and 0.3, and from Matora and Fujino (1965) which includes results for 0.1 and 0.2. The preliminary conclusion is that such a collapse of the data takes place, if at all, only in the initial stages of frequency dependence. This is consistent with the conclusions that initially the viscous effects come into play rather than the free surface effects.

The maximum frequency at which these effects are unimportant can be inferred quantitatively from the above referenced oscillator tests, as well as from those of Paulling and Wood (1962), Brard (1964), and others. Such results are outlined in Table 2.

Table 2  
Limiting Values of the Frequency Parameters Below  
Which Frequency Effects are of Secondary Importance

| References                       | Type of Model | Froude Number | Phase of Force with Respect to Motion | Limiting Frequency |                      |
|----------------------------------|---------------|---------------|---------------------------------------|--------------------|----------------------|
|                                  |               |               |                                       | $\omega L/V$       | $\omega V/g$         |
| Motora and Couch (1961)          | Ore Carrier   | 0.1           | Both                                  | $\geq 2.5$         | $\geq 0.03$          |
| Paulling and Wood (1962)         | Mariner       | 0.285         | Both                                  | 1-2                | 0.1-0.2              |
| van Leeuwen (1964)               | Series 60     | 0.2, 0.3      | Both                                  | 2-4                | 0.1-0.2              |
| Chislett and Strøm-Tejsen (1965) | Mariner       | 0.2           | In<br>Out                             | $>4$<br>$<4$       | $>0.15$<br>$<0.15$   |
| Motora and Fujino (1965)         | Series 60     | 0.1, 0.2      | In<br>Out                             | $>6$<br>3          | $>0.12$<br>0.03-0.12 |
| Brard (1964)                     | Submarine     |               | In<br>Out                             | $>6$<br>$<1$       |                      |
| Sevik (1964)                     | Spheroid      |               | Both                                  | $<10$              |                      |

While the results of Table 2 are not entirely consistent with each other, it is clear that the general order of magnitude of the limiting reduced frequency is in the range between one and four, and for conventional Froude numbers the corresponding limiting values of the frequency parameter  $\omega V/g$  are in the range between 0.1 and 0.2. Moreover there is some evidence to suggest that the frequency effects play a more important role in the out-of-phase (i.e., damping) components of the force and moment than in the corresponding in-phase or added-mass components. This is to be expected since in an ideal unbounded fluid the force is entirely in phase with the acceleration.

It remains to estimate the range of frequencies which are encountered in practical ship maneuvers. This is somewhat less certain since in general ship maneuvers are transient rather than sinusoidal processes and will include harmonic content at all frequencies. If we arbitrarily select a  $20^\circ$ - $20^\circ$  zig-zag test as a "typical" ship maneuver, then the reduced frequency follows directly in terms of the period of time required for the maneuver, or simply

$$\frac{\omega L}{V} = \frac{2\pi L}{D},$$

where  $D$  is the distance run in one cycle. Typical values of this distance are six to ten ship lengths\* and it follows that the reduced frequency parameter is roughly in the range of from one-half to one. Thus a  $20^\circ$ - $20^\circ$  zig-zag maneuver is marginal from the viewpoint of frequency effects. Ship maneuvers which take place more rapidly than this can be expected to involve significant frequency effects, which can only be accounted for by Laplace transform techniques or the equivalent thereof, as outlined by Brard (1964). It remains to be determined, however, what magnitude of error is incurred by neglecting this complication, and it would seem desirable to carry out a numerical comparison of the two predictions for realistic maneuvers.

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## Appendix

## NOMENCLATURE AND EQUATIONS OF MOTION

A common notation has become almost universal in the field of maneuvering and control. The principal elements are given below, and further details are given in S.N.A.M.E. Bulletin 1-5 (anon., 1950) and in the Proceedings of the 10th I.T.T.C. Conference (Goodrich, ed., 1963).

Cartesian coordinates  $(x_0, y_0, z_0)$  are taken to be fixed in space or, nominally, with respect to the earth, such that  $x_0$  coincides with the general direction of the "initial" motion of the ship and  $z_0$  is vertically downward. The associated coordinates fixed with respect to the ship are denoted  $(x, y, z)$ , the right-hand convention is applied so that  $y$  is positive to starboard, and  $y = 0$  is taken to be the plane of symmetry of the ship. The three components of force  $(X, Y, Z)$ , moment  $(K, M, N)$ , linear velocity  $(u, v, w)$ , and angular velocity  $(p, q, r)$  are all defined in relation to the ship coordinates  $(x, y, z)$ . Angular orientation of the ship is defined by the symbols  $\phi$  (roll),  $\theta$  (trim), and  $\psi$  (yaw), in accordance with the following coordinate transformation:

$$\begin{aligned} x_0 &= x \cos \theta \cos \psi + y (\sin \theta \sin \phi \cos \psi - \cos \phi \sin \psi) \\ &\quad + z (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) \\ y_0 &= x \cos \theta \sin \psi + y (\cos \phi \cos \psi + \sin \theta \sin \psi \sin \phi) \\ &\quad + z (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) \\ z_0 &= -x \sin \theta + y \cos \theta \sin \phi + z \cos \theta \cos \phi. \end{aligned}$$

Alternatively, these (finite) angles can be defined by prescribing their order: if the  $(x, y, z)$  coordinates coincide initially with the fixed  $(x_0, y_0, z_0)$  system, then the final orientation is obtained by first a yaw angular displacement, secondly a trim, and lastly a roll, all with respect to the  $(x, y, z)$  axis and in the right-handed sense.

It will be noted that the above systems are not sufficiently general to describe translations as well as rotations between the fixed and moving coordinate systems. In fact, it is customary to consider that the spatially "fixed" coordinates  $(x_0, y_0, z_0)$  are in fact translating with the origin of the ship's  $(x, y, z)$  system but that the first system is fixed in space at each point in time to allow the application of Newton's laws in this system. Such a shortcut is expedient if free surface and viscous effects are ignored, but for a complete physical description it is necessary to consider the effects of translations as well as rotations between the two coordinate systems.

Since the translational velocity components  $(u, v, w)$  are defined in reference to the ship-fixed  $(x, y, z)$  coordinates it follows that a steady "yawed" motion must be associated not with the yaw angle  $\psi$  but with a constant value of the "drift" or "sideslip" angle  $\beta = \tan^{-1}(v/u)$ . The drift angle is used frequently

as a substitute for the dependent variable  $v$ , and plays a particularly important role in describing motions in the horizontal plane.

If the origin coincides with the ship's center of gravity the six equations of motion of the ship are as follows:

$$X = m(\dot{u} - rv + qw) + mg \sin \theta$$

$$Y = m(\dot{v} - pw + ru) - mg \cos \theta \sin \phi$$

$$Z = m(\dot{w} - qu + pv) - mg \cos \theta \cos \phi$$

$$K = I_{xx}\dot{p} + (I_{zz} - I_{yy})qr + I_{xy}q + I_{xz}r$$

$$M = I_{yy}\dot{q} + (I_{xx} - I_{zz})pr + I_{yz}r + I_{xy}p$$

$$N = I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + I_{xz}p + I_{yz}q$$

Here a dot denotes time differentiation,  $m$  is the mass of the ship,  $g$  is the gravitational acceleration,  $(I_{xx}, I_{yy}, I_{zz})$  are the moments of inertia of the ship's mass, and  $(I_{xy}, I_{xz}, I_{yz})$  are the corresponding products of inertia. If the mass distribution is symmetrical with respect to the plane  $y = 0$  (i.e., port and starboard) the products of inertia  $I_{xy}$  and  $I_{yz}$  will vanish, and in practice for conventional ships the fore-and-aft symmetry is sufficiently dominant that the remaining product  $I_{xz}$  can probably be ignored. However, this assumption does not appear to have been verified, and it should be emphasized that, in general, the coordinates cannot be taken to coincide with the principal axes of inertia unless the usual assumptions that  $x$  is horizontal and  $z$  is vertical are sacrificed.

Various manipulations can be performed with the above system of equations. Coordinate transformations can be made to a more convenient origin than the center of gravity, several additional terms thus being introduced to the above equations; these transformed equations can be found, for example, in Bulletin 1-5 (anon., 1950). The degrees of freedom can be limited to the horizontal plane, so that the number of terms in the equations of motion is substantially reduced. Also, these equations may be linearized, on the assumption that the unsteady motions are small perturbations of an initial steady longitudinal velocity; this step is justifiable in many instances, particularly in performing a dynamic stability analysis, but not necessarily in predicting maneuvering characteristics which may be inherently nonlinear, and certainly not in studying the motions of unstable ships.

The left sides of the equations of motion, the external force and moment applied to the ship, include all of the hydrodynamic effects which act on the hull. For a conventional self-propelled ship there are no other components of the external force and moment if we assume that aerodynamic effects on the above-water portion of the ship can be neglected and that there exist no internal changes of the mass or its distribution. On the other hand, towed or towing bodies such as barges or tugs and captive ship models will have additional



external force and moment contributions to the left side of the equations arising from their associated towing constraints, and in special cases, e.g., ships equipped with a flume stabilizer system, significant internal changes of the mass distribution will exist.

\* \* \*

## COMMENTS

M. Gertler  
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Although I have not had time to study this paper in any great detail, I would like to make a few comments with respect to the problem of "frequency effects."

The author divides the frequency effects into two categories: those that stem from vorticity shed from the oscillating hull and those that stem from the unsteady motion of the hull at the free surface. The characteristic nondimensional frequencies governing these two phenomena are given as  $\omega L/V$  and  $\omega V/g$ , respectively. By comparing the two frequency parameters, he concludes that the unsteady viscous effects will become significant before the unsteady free-surface effects, especially for slower ships. This interesting finding suggests that if we examine the case of a deeply submerged submarine where frequency effects, if any, can only be of viscous origin, then we can determine whether such frequency effects are apt to be important in the surface ship case.

Since 1958, the David Taylor Model Basin has carried out carefully conducted planar-motion-mechanism tests on well over a hundred models of submarines and other types of submerged bodies. In general, the experiments were carried out over a range of  $\omega L/V$  values at least up to about 4, which is well beyond the realm of practical interest for most submarine and submerged body applications. Except for a few isolated cases involving forms which were extremely poor from a hydrodynamic standpoint, the data from the oscillation tests showed no significant evidence which could be interpreted as a variation of the pertinent hydrodynamic force or moment coefficients with frequency within the range mentioned. This was true of both the out-of-phase quantities (rotary derivatives) and in-phase quantities (acceleration derivatives). Furthermore, the coefficients derived from planar-motion-mechanism tests have been used in conjunction with quasi steady-state equations of motion to make predictions of the trajectories of submarines in various types of maneuvers such as spirals, zigzags, and turns. These predictions, in a number of cases, have been compared with the results of free-running model tests and full-scale trials and the correlation has usually been excellent. Thus, we are firmly convinced that, in the case of deeply submerged submarines, frequency effects can be neglected both in predicting motions and in stability and control analyses.



On the other hand, the Model Basin has moved cautiously in applying the planar-motion-mechanism technique to the case of surface (displacement) ships. This was primarily due to concern over frequency effects which it was believed would be caused by the waves generated by the oscillating hull. However, in view of our experience with submarine forms and the authors assurances that the viscous effects are more to be feared than the free-surface effects, plus the recent experiences of other investigators, we should be able to proceed with planar-motion-mechanism tests on surface ship models with full confidence.

\* \* \*

## COMMENTS

Roger Brard  
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The two main purposes of Dr. Newman's paper can be summarized as follows:

1. He wants to show that, in spite of the complexity of the flow surrounding a maneuvering ship, it is possible to construct mathematical models taking into account some of the principal characteristics of the flow; these mathematical models lead to a better understanding of the phenomena involved in the maneuvering of ships.

2. He discusses the analytical methods corresponding to the mathematical models and he points out where further investigations are needed.

In order to support his first point, Dr. Newman describes the flow over the model of a Mariner class ship in yawed conditions. The observations were carried out in the circulating water channel of the David Taylor Model Basin. They show that the flow inside and outside the boundary layer differs substantially according as the model is in straight or in yawed conditions. However, in yawed condition, the areas of separation are of a limited extent. From that Dr. Newman concludes that a theoretical inviscid fluid model based on the low aspect ratio wing theory or on the slender body theory, should lead to an acceptable description of the flow over the entire hull for angles of attack less than about 10 degrees.

I personally believe that any application of theoretical views to the maneuvering of a surface ship or of a submarine requires preliminary observation of the flow over the model. So I think that it is not sufficient to watch only the distribution of the flow over the hull surface in yawed condition. It is also necessary to watch the flow in turning condition. For this reason, at the Bassin d'Essais des Carènes, we closely examine the flow pattern around ship models in yawed and in turning conditions. In Casal's thesis, which was prepared under

my direction at the Paris Tank in 1949-50, is included a photograph of a paint test. This photograph shows that in turning condition, a concentrated bound vortex may exist near the bow and that a free vortex originates as a consequence. The same phenomenon may occur in yawed condition but the field of velocities induced would be different.

Moreover, when a ship is turning, the local angle of attack varies along the length of the ship. The transverse flow, when the ship is turning to port comes from port on the forebody and from starboard amidship and on the aft body. When the radius of gyration is small, there are string eddies on the port side near the stern.

In spite of the above considerations, I wish to state that I am in agreement with Dr. Newman's opinion. But the reasons why I agree with him may be more pragmatic.

The phenomena being very intricate, we have to test various mathematical models to examine the predictions to which they lead, and to choose accordingly the model that we will finally adopt for the class of ships that we have under consideration. In particular, the structure of the expressions of the force and moment acting on the ship is likely to vary from one class of ship to another; in addition, the order of magnitude of the numerical values of the coefficients are bound to change substantially. Generally speaking, the flow can be idealized, but in a manner specific of each type of ship.

The ship herself must not be too much idealized if we want to obtain sufficiently accurate predictions for practical purposes. It is well known, for instance that the behavior of a ship is very sensitive to differences in the block coefficient, the angle of entrance, the extent of the deadwood, and so on.

The controversy that Dr. Newman mentions in page 213 of his paper is familiar to me. This controversy originates in the fact that, if the force and moment are considered as nonanalytic, the use of Taylor series expansions is not permissible. But limited expansions are polynomial, and it is always possible to use polynomial expressions provided the functions, which are to be approximated, are continuous. Another controversy concerns the low aspect ratio wing theory. This theory is useful, but it is necessary that it take into account the effect of the angular velocity.

In my opinion the real problem is to represent the force and moment by expressions which contain as few terms as possible, while ensuring the smallest discrepancy with respect to the results of the experiments, and which lead to the best understanding of the effects of the shape of the hull and of the appendages on the behavior of the maneuvering ship. For instance, Casal's thesis seems particularly interesting because it helps to understand why, in yawed condition, the force is acting near the bow and even ahead of the bow while, in turning condition, it is acting behind the middle transverse section. For analogous reasons, the dimensionless expressions of the force and moment must be able to account for their rapid variation when  $L/2R$ , that is the reduced angular velocity, is in the vicinity of the zero value.

So I think that it is necessary to study carefully the various components of the force and moment due respectively to: added mass, circulation, free-surface effect, separation, propeller. As for the personal views of the author about the free-surface effect, I think that emphasis must be put on them, since they are of very great interest.

The last part of Dr. Newman's paper is devoted to a discussion of the frequency effects, or to the "memory" effects. He explains why initially the viscous effects rather than the free-surface effects come into play.

My own paper read in Bergen in 1964 was aimed at drawing the attention to these viscous effects. The experiments which were carried out at the Bassin d'Essais des Carènes at the time, using a planar motion mechanism, showed that these effects could be of importance in certain cases. Since then, new experiments have been undertaken. They have shown that it is necessary to ensure a very accurate angular setting of the sine-cosine potentiometer used for the integration, with respect to the driving shaft of the struts, or else significant errors would affect the calculation of the out-and-in force and moment. This has been a problem for sometime, but it is now solved. And now when experiments are repeated with identical conditions, the results obtained are quite consistent. This was not the case before. The curves given as examples in my 1964 paper must be considered as erroneous. Now, as is correct, the curves giving the coefficients of the so-called quasi steady motion versus the reduced frequency always show a horizontal asymptotic line. For instance in the case of motions parallel to the horizontal plane, we have found that the asymptotic value of the derivative  $\partial \zeta_y / \partial \beta$  may be 50% greater than the value corresponding to the steady motion. For  $\partial \zeta_y / \partial \psi$  the variations are insignificant and this is in agreement with the fact that  $\partial \zeta_y / \partial \psi$  is very small in steady conditions. The variations of  $\partial \zeta_N / \partial \beta$  are small but  $\partial \zeta_N / \partial \psi$  decreases when the reduced frequency increases.

The techniques to be used in experiments whose purpose is to study the effect of the frequency in the case of a submerged body are rather critical because the small values of the reduced frequency are obtained for high speed values. The effects of the free surface may interfere with those of the viscosity. However, the experiments indicate that the influence of the free surface is small even for  $\omega L/U$  less than unity, the submergence of the model being about 2 meters and the greatest period being 5.69 sec.

The asymptotic values of the coefficients which are sensitive to the reduced frequency are reached for practical purposes at a reduced frequency of about ten. That leads to the feeling that the history of the motion may be of importance in the case of rapid transient motions. However the problem of determining the magnitude of the error when the effect of the history of the motion is neglected is not yet solved in a satisfactory manner. In any case, it is to be noted that, when a planar motion mechanism is used for obtaining the added mass, the exact values of the corresponding coefficients are the asymptotic values and not the limit values for  $\omega L/U \rightarrow 0$ .

Concluding my comments on Dr. Newman's paper, I should like to express the opinion that the author has reached his goal. Not only has he made a very



brilliant synthesis of the work already done, but concerning the work which is now to be undertaken, he has formulated numerous pertinent personal views which, in my opinion, will be very fruitful. For those reasons, I think that Dr. Newman fully deserves to be very warmly congratulated.

\* \* \*

## SUMMARY OF ORAL DISCUSSION

Nils H. Norrbin

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This paper offers interesting reading — still, I suppose, that will apply even more to the next paper by Dr. Newman on this subject.

It should be somewhat embarrassing to many of us that the mathematician was the one to bring with him so many instructive photographs illustrating basic events around a hull in oblique flow. These flow studies will help to judge the merits of the two simple analogies usually applied to explain the viscous lift on the submerged body, or the surface ship at low speeds where hull and image move together, i.e., the slender body of revolution and the low-aspect-ratio flat wing.

We remember the concepts of Nonweiler for the viscous flow around a body of revolution, touched upon by the author: the crossflow is decelerated in the boundary layer along the body, and at some distance aft of the bow the up-wash is reversed, on the leeward side of the afterbody the radial displacement flow will change into an in-flow, and only more near to the stern the rolling-up of two trailing vortices will be completed. This physical picture suggests that viscous ring vortices at the after part of the hull should be added to the ideal flow model, characterized by a double-vortex line along the axis in the parallel middle-body connecting ring vortices bound at bow and stern. These added ring vortices are shed away in trailing line vortices from positions rather far aft; there effect is to reduce the negative lift on the afterbody by say 50 or 60 percent. In our experiments with the divided double-body ship model, referred to by the author, this reduction amounts to about 85 percent, bringing the result quite close to that of the wing analogy.

The double-vortex line along the axis of the body of revolution induces an up-wash angle, which close to the hull is equal to the angle of attack. If we now look back to Fig. 2 or 3 of Dr. Newman's paper we do not find the additional up-wash associated with the ideal flow around the slender body, but we might note a local separation along the bilges.

If the low-aspect-ratio wing analogy does apply, then I think the normal force on the forebody will be very close to the force on the total hull, as indicated



by our experiments quoted on page 216 of the paper. The form of the tail is of minor importance in this case, and the addition of a rudder does not improve the trailing edge condition but means the addition of an isolated lift force, which in turn may give an unstable moment on the low side (Cf. Fig. 8).

As for the total normal force results displayed in Fig. 7, it must be realized that the experimental accuracy was less in case of the ship model — because of the relatively shorter length of the balance beam — and that the dip at the highest speed may partly be due to a local effect of the stern wave on the force on the rudder.

Finally I will make a brief comment on the formal higher order approximations. From physical considerations the cross-flow-drag lift suggests a square term, which we have found to agree with experimental results. Due to the finite time required for the development of the cross-flow boundary layer, however, the contribution from sections with small fullness might be of a higher order, and this contribution might in turn be reflected in the small nonlinear moment term.

\*       \*       \*

## REPLY TO DISCUSSION

J. N. Newman

In reply to Mr. Gertler I must first emphasize that my "conclusion" regarding the relative importance of viscous and free-surface effects should be regarded for the present time as a hypothesis. Mr. Gertler's summary of the extensive experiments carried out on submarines at the David Taylor Model Basin is a valuable supplement to the data in Table 2. Taken in conjunction with the revised data which Admiral Brard reports in his discussion, it suggests that submerged bodies of good hydrodynamic form will have a limiting reduced frequency, below which their motion can be regarded essentially as pseudo-steady-state, of between four and ten. This range of reduced frequency is substantially higher than the corresponding values for surface ship models as reported in Table 2, thus suggesting that frequency effects come into play sooner for surface ships than for submarines. At first glance this conclusion implies that, contrary to our hypothesis, the free surface-effects are dominant, but a likely alternative is that the hull form of surface ships is inferior to that of submarines. This situation could be clarified by performing planar-motion-type experiments with a submerged double-body, analogous to the steady-state experiments of Norrbin (1965a).

I am very grateful to Admiral Brard for amplifying upon the results of Casal's thesis and for bringing the results of his own research up to date. The observation by Casal of a horseshoe vortex system generated near the bow during a turn, as opposed to a steady yawed condition, is another indication of unsteady viscous effects which in this case is associated with the vertical stem of

a surface ship and would not be expected for a submarine. Thus, I would certainly agree with Admiral Brard's statement that "generally speaking, the flow can be idealized, but in a manner specific to each type of ship." As regards the approximation of the lift force and moment by Taylor series or polynomials, I feel that the important distinction from the mathematical viewpoint is that the resulting expansions must be one-sided, with discontinuities in the even-order derivatives, in order to account properly for the cross-flow drag effects. The recent experimental results quoted by Admiral Brard, in modification of his paper at the last Naval Hydrodynamics Symposium, are extremely interesting and I look forward to the opportunity to study them in more detail when they are published.

\* \* \*



# RESEARCH ON THE MANOEUVRABILITY AND PROPULSION OF VERY LARGE TANKERS

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## ABSTRACT

This paper deals with the results of investigations into the manoeuvrability and propulsion of large tankers, equipped with an advanced stern arrangement for propulsion and ship-control. Authors suggest to eliminate partly or completely the rudder, provide the ship with an extremely cigar-shaped afterbody having a shrouded propeller with a large hub-to-diameter ratio and fit both bow and stern thrusters to the ship. The application of cargo-pump-driven lateral thrusters working on the ejector principle is discussed.

## 1. INTRODUCTION

The requirements for the manoeuvrability and propulsion of very large tankers, to be built in the near future, open the question whether conventional solutions for the stern arrangement still have to be maintained.

The increasing displacement of tankers results in an increase of the required shaft horsepower. The nonuniformity of the flow at the propeller and the high required shaft horsepower may lead in many cases to vibration troubles and cavitation-erosion damage on the propeller blades. The increasing size of tankers and the difficulty of developing port areas large enough to accommodate them will lead to higher requirements with respect to the effectiveness of the ship-control devices. The effectiveness of a rudder strongly depends on the ship speed. As speed decreases, the effectiveness of the rudder becomes less and less.

In an attempt to provide large tankers with propulsion devices with superior cavitation and propeller-induced vibration characteristics in addition to a high propulsive efficiency and to provide tankers with a greater degree of manoeuvrability, the authors suggest to eliminate the rudder, provide the afterbody of an extremely cigar-shaped stern with a shrouded propeller having a large hub-to-diameter ratio and fit both bow and stern thrusters to the ship. In Fig. 1, the installation of the propeller in a clear plastic nozzle on the afterbody of a tanker



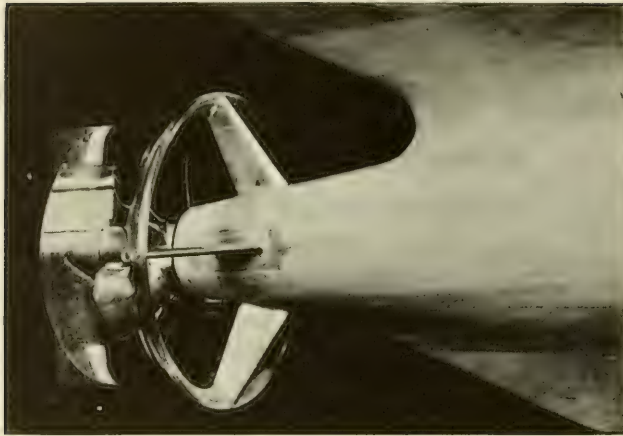


Fig. 1 - Arrangement of propeller  
in nozzle for a tanker model

model as suggested by the authors is shown. Figure 2, shows the bow and stern thruster fitted to the model.

The draft, propeller diameter, efficiency, propeller cavitation, propeller-induced vibration, construction of the rudder and so on, are all design factors, influenced favourably by the solution indicated by Figs. 1 and 2. To obtain a comparison between the propulsive performance and the manoeuvrability of a tanker with a conventional stern arrangement and the one according to Figs. 1 and 2, tests were carried out with models representing a 90,000-ton deadweight tanker. The results of these experiments are presented and discussed in this paper.

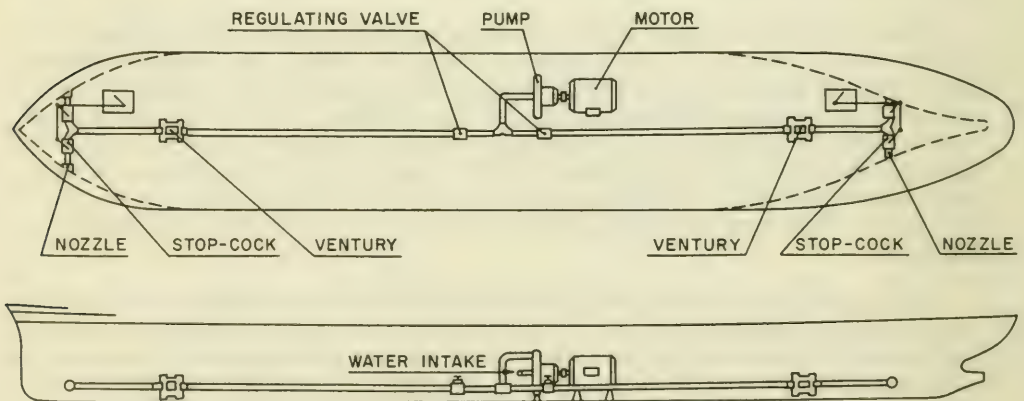


Fig. 2 - Bow and stern thruster arrangements fitted to the model

Fixed and controllable-pitch impellers are the most common prime moving devices located in transverse thrusters. In recent literature the use of cargo-oil pumps of tankers as bow thruster devices has been suggested. Such a system has the drawback that the efficiency is low due to the large kinetic energy losses in the slipstream. These losses can be diminished by using an ejector in which a high-velocity jet with a low mass flow rate can be converted into a lower velocity jet with a higher mass flow. Therefore, the authors suggest to use bow and stern thrusters working according to this ejector principle. Figure 3 shows schematically an ejector bow thruster.

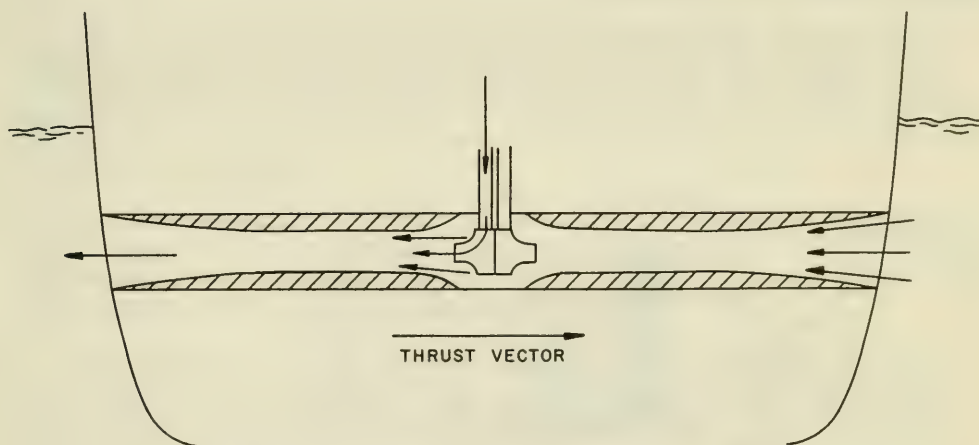


Fig. 3 - Arrangement of ejector bow thruster

## 2. RESISTANCE AND PROPULSION

The trend in the design of tankers has been toward large dimensions at nearly constant speed and therefore high-powered ships. The high required shaft horsepower and the nonuniformity of the flow at the propeller disk may lead to vibration troubles and cavitation-erosion damage on the propeller blades.

In order to minimize the chance of propeller-induced vibration and cavitation damage, the inequality of the flow at the propeller disk must be reduced. Investigations performed by van Manen and Kamps (1), Nitzki (2) and Hadler and Cheng (3) have shown that extreme shapes of afterbody for obtaining a uniform circumferential velocity field in way of the propeller offer favourable prospects.

As a consequence of the decreasing propulsive efficiency with increasing shaft horsepower of tankers with conventional screws in addition to propeller cavitation and vibration problems, the application of ducted propellers (1,4) tandem propellers, contrarotating propellers (5) and twin-screw devices (1,5,6) has been suggested. From these propulsion devices, the ducted propeller and the tandem propellers will be less expensive and complicated than contrarotating

and twin-screw installations. In this case of high-powered tankers, the ducted propeller is, according to the experience of the N.S.M.B., superior in propulsive efficiency compared to all other propellers.

First, the considerations will be given which have led to the choice of the shape of afterbody and propulsor of the tanker discussed in this paper. Then, the results of resistance and self-propulsion model tests will be presented for each of the conventional and the suggested versions of the tanker.

### Effect of Afterbody on Propulsion

Investigations into the effect of the afterbody shape on propulsion of a 39,000-ton deadweight tanker model carried out at the N.S.M.B. were reported in Refs. 1 and 4. The following variations in afterbody were studied (Fig. 4).

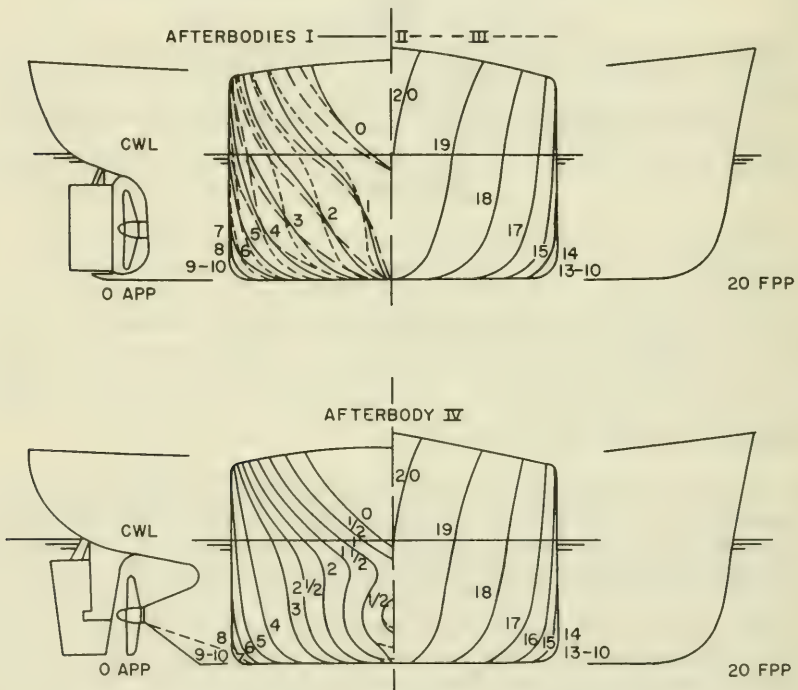


Fig. 4 - Body plan and propeller arrangement of afterbodies I, II, III and IV

- Model I, having an afterbody with moderately U-shaped sections and a rudder shoe.



- Model II, having an afterbody with extremely V-shaped sections and a rudder shoe.
- Model III, having an afterbody with extremely U-shaped sections and a rudder shoe.
- Model IV, having a cigar-shaped stern (Hogner-type of afterbody) and a Mariner rudder arrangement (clear-water stern).

From the experimental investigations, it was concluded that model II, having an afterbody with extremely V-shaped sections is to be recommended from a viewpoint of resistance. The improvement in resistance qualities, however, did not include a reduction of the required shaft horsepower. The reduction in resistance is counterbalanced by a change in interaction effects between hull and propeller. Investigations performed by Nichols et al. (6) have led to the same result. Model I which is optimum according to the statistical data of the N.S.M.B. is still optimum with respect to the delivered shaft horsepower at the design speed of 16 knots.

The cigar-shaped stern of model IV is bad from a viewpoint of resistance. However, the interaction effects between hull and propeller are such that in the loaded condition as well as in the light condition, model IV requires 2-3 percent more power than the optimum hull form (model I).

Measurements of the longitudinal component of the velocity in the plane of the propeller were made with the four models. From these measurements it was concluded that the circumferential inequalities behind model II is worse than that behind the models I and III. Model IV shows the most homogeneous wake pattern of all afterbody variations.

Recently, the results of wake measurements made on a large number of ship models at the David Taylor Model Basin have been published by Hadler and Cheng (3). Both the longitudinal and the tangential velocity components of the wake were measured. From experiments with a series of tanker models with systematically varied stern shapes it was concluded that the models with the more U-shaped stern tended to generate a more uniform longitudinal velocity and a large downward velocity in the propeller plane near the propeller hub. The more V-shaped stern tended to give a large fluctuation in the longitudinal velocity, while the tangential velocity is relatively smaller in magnitude as is the downward flow near the hub. From an analysis of the effect of the nonuniform velocity field on propeller cavitation and propeller-induced vibration, it was concluded that the moderate U-shaped stern would provide best vibration and cavitation characteristics.

Wake measurements on a ship model with a Hogner-type stern have also been presented by Hadler and Cheng (3). From a comparison of the results of the Hogner-shape stern design and the conventional stern design from the standpoint of minimizing vibration and cavitation problems in terms of uniformity in flow, the superiority of the Hogner-type stern was evident. This conclusion is in accordance with the results of observations of the cavitation patterns



and measurements of thrust and torque variations of the screws, performed at the N.S.M.B. with the Models I and IV.

### Choice of Propulsion System

Among the special types of propulsors which have as their object an improvement in the propulsive quality of the ship, the ducted propeller takes an important place. The ducted propeller is now extensively used in those cases where the ship screw is heavily loaded or where the screw is limited in diameter. The possibility of using the ducted propeller for the propulsion of tugs, towboats, and trawlers has been adequately demonstrated in practice in the course of the last thirty years.

In previous publications of the N.S.M.B. (7-10), the results of systematic experiments with screw series in nozzles were given. These experiments supplied design data for propeller-nozzle systems with optimum efficiency. The standard nozzle profile applied by the N.S.M.B. from the viewpoint of efficiency is nozzle No. 19a, having an angle between nose-tail line of nozzle profile and propeller shaft axis  $\alpha_i = 9.2^\circ$ , a camber ratio  $f/e = 0.069$ , a maximum thickness ratio  $s/\ell = 0.15$  and a nozzle length-diameter ratio  $\ell/D = 0.5$ . The ordinates of this profile are given in Fig. 5.

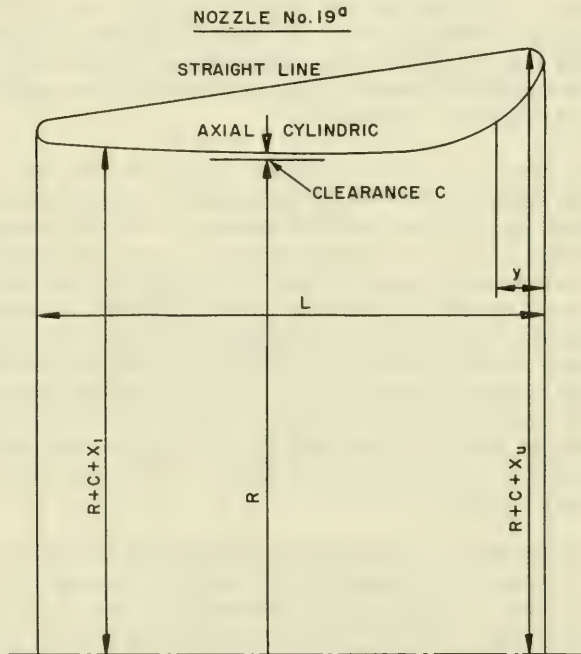


Fig. 5 - Profile and ordinates of nozzle No. 19A

## Manoeuvrability and Propulsion of Very large Tankers

A method used frequently for expressing the characteristics of a propeller type is the relationship between  $B_p$ ,  $\delta$ , and  $\eta_p$  for optimum propeller diameter. The design coefficient  $B_p$  and the speed ratio  $\delta$  are defined as

$$B_p = \frac{NP^{1/2}}{V_e^{5/2}}$$

and

$$\delta = \frac{ND}{V_e},$$

where  $N$  is the number of revolutions per minute, the power  $P$  is in horsepower, the speed of advance  $V_e$  is in knots and the screw diameter  $D$  is in feet. The optimum relationship between  $B_p$ ,  $\delta$ , and  $\eta_p$  is given in Fig. 6 for the Ka 3-65, Ka 4-70, and Ka 5-75 screw series in nozzle No. 19a and for the B 4-70 screw series. At the top of the figure the ranges of  $B_p$ -values typical for different ship types are indicated. The lightly-loaded screws of fast ships are at the left-hand side, while the heavily loaded propellers of towing vessels are at the right.

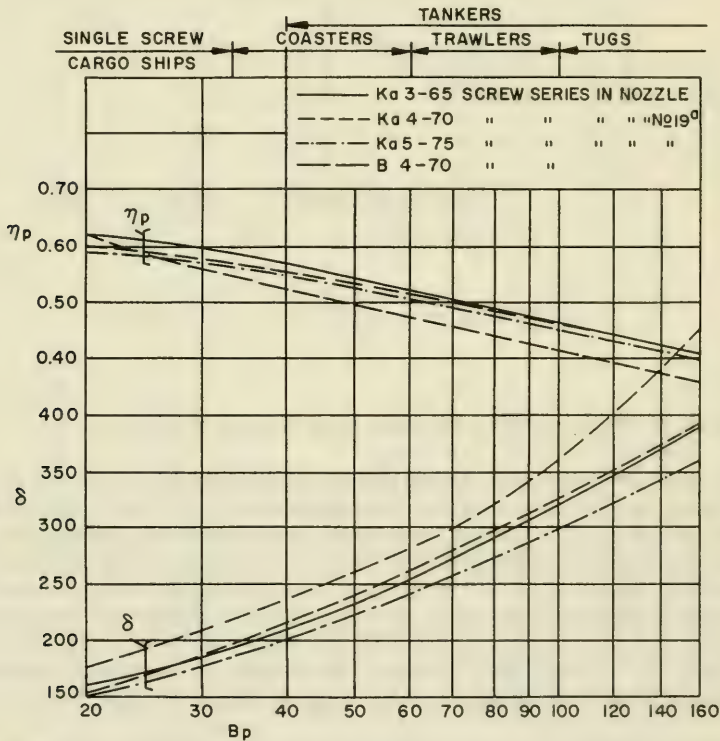


Fig. 6 - Comparison of the Ka-screw series in nozzle No. 19A and B4-70 screw series

Figure 6 shows clearly the reasons why ducted propellers are used at heavy propeller-loadings occurring in tugs, trawlers, and large tankers. As tanker sizes continue to grow, the advantage of ducted propellers, higher efficiency, and reduced optimum diameter, will become greater.

Analysis of the reduction in SHP due to the use of a nozzle has been made for tankers with an installed power of 30,000 SHP and different deadweight. The rotative speed of the impeller has been fixed at 100 RPM. The result is given in Fig. 7 which clearly shows the great reduction in SHP which can be obtained. For a tanker of 100,000 tons deadweight, a reduction of 8 percent in SHP is attainable.

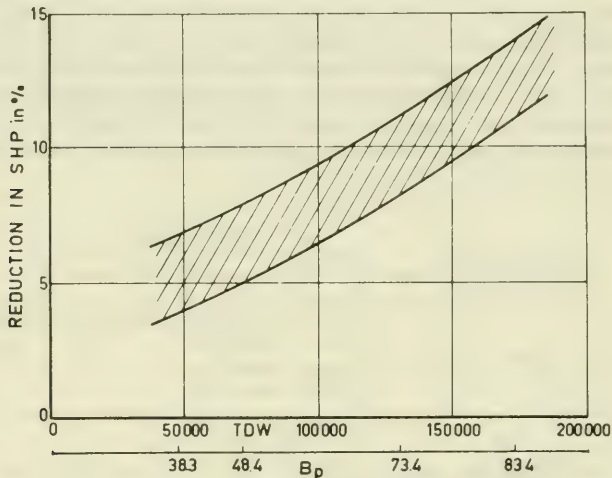


Fig. 7 - Reduction in SHP, due to the application of a nozzle for large tankers

#### Applications of Hogner-type Afterbody with Ducted Propeller

From the results of the investigations discussed so far it can be concluded that for large tankers, the Hogner-type afterbody provided with a complete ring nozzle propeller system offers favourable prospects with respect to minimizing vibration and cavitation problems in addition to a high propulsive efficiency. In order to obtain data on the differences in resistance and propulsive efficiency of tankers with a conventional stern arrangement and a Hogner-type stern with nozzle fitted, tests were carried out at the N.S.M.B. with models representing 32,500-TDW and 48,500-TDW tankers. Both tankers had a conventionally shaped bow. The various stern arrangements considered were as follows:

32,500-TDW tanker: conventional stern (moderately U-shaped sections and rudder shoe)

Hogner-type clearwater stern fitted with a nozzle

48,500-TDW tanker: conventional stern (clearwater stern with moderately U-shaped sections)

Hogner-type stern with rudder shoe and nozzle

Details of the stern arrangements are presented in Figs. 8 and 9. Resistance and self-propulsion tests were run for the light and loaded condition. Table 1 compares the results of the EHP and SHP tests of the 32,500-TDW tanker model. The results for the 48,500-TDW tanker are shown in Table 2.

Table 1  
EHP and SHP Tests of 32,500-TDW Tanker  
(design speed 17 knots at 16,000 SHP)

| Speed<br>in<br>Knots | Hogner-Type Stern Better (+) or Worse (-)<br>Than Conventional Stern Arrangement |                    |                     |                    |
|----------------------|----------------------------------------------------------------------------------|--------------------|---------------------|--------------------|
|                      | EHP                                                                              |                    | SHP                 |                    |
|                      | Loaded<br>Condition                                                              | Light<br>Condition | Loaded<br>Condition | Light<br>Condition |
| 14                   | -6.2%                                                                            | -                  | +13.1%              | -                  |
| 15                   | -6.3%                                                                            | -5.1%              | +11.9%              | +0.0%              |
| 16                   | -6.0%                                                                            | -6.5%              | + 8.6%              | +0.3%              |
| 17                   | -5.1%                                                                            | -8.0%              | + 4.8%              | -0.3%              |

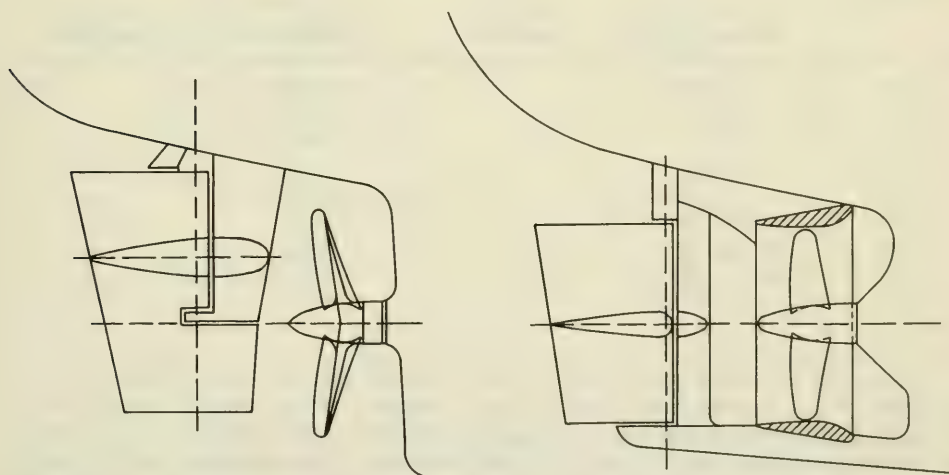


Fig. 8 - Details of stern arrangements for 32,500-TDW tanker model



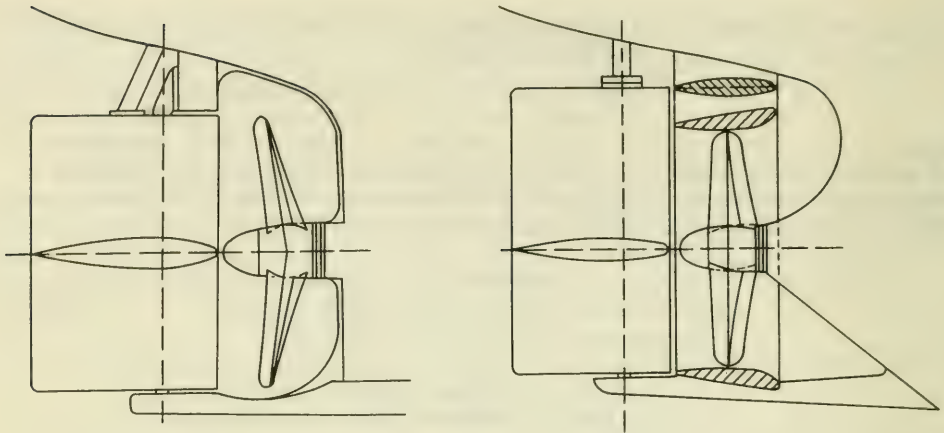


Fig. 9 - Details of stern arrangements for 48,500-TDW tanker model

Table 2  
EHP and SHP Tests with 48,500-TDW Tanker  
(design speed 16 knots at 16,000 SHP)

| Speed<br>in<br>Knots | Hogner-Type Stern, Better (+) or Worse (-)<br>Than Conventional Stern Arrangement |                    |                     |                    |
|----------------------|-----------------------------------------------------------------------------------|--------------------|---------------------|--------------------|
|                      | EHP                                                                               |                    | SHP                 |                    |
|                      | Loaded<br>Condition                                                               | Light<br>Condition | Loaded<br>Condition | Light<br>Condition |
| 13                   | -5.3%                                                                             | -                  | +4.4%               | -                  |
| 14                   | -7.7%                                                                             | -                  | +4.1%               | -                  |
| 15                   | -7.2%                                                                             | -                  | +3.8%               | +3.6%              |
| 16                   | -6.7%                                                                             | -                  | +2.7%               | +3.7%              |
| 17                   | -9.6%                                                                             | -                  | +2.8%               | +4.5%              |
| 18                   | -9.6%                                                                             | -                  | +2.9%               | +5.1%              |

From the tables, it may be seen again that the Hogner-type afterbody is really bad from a viewpoint of resistance; however, in the loaded as well as in the light condition, the tankers with the Hogner-type stern and nozzle require less power than the conventionally shaped tankers. The reduction in SHP is 4.8 percent for the 32,500-TDW tanker in the loaded condition at the design speed of 17 knots. A reduction in SHP of 2.7 percent has been found for the 48,500-TDW tanker in the loaded condition at the design speed of 16 knots.

### Advanced Propulsion for a High-Powered Tanker with Bow and Stern Thruster Control

In the introduction the elimination of the rudder was suggested by providing large tankers with bow and stern thrusters for transverse control. The elimination of the rudder opens the possibility of extremely cigar-shaped sterns. A complete nozzle ring can be fitted then, attached with four brackets to this afterbody.

The presence of a rudder on a single-screw ship tends to improve the propulsive efficiency. In addition, the tailpiece may be attractive from a viewpoint of course stability. Therefore, a fixed tailpiece must be fitted to the nozzle ring. In Fig. 1, such an afterbody arrangement is shown.

Model tests have been carried out to obtain a comparison between the propulsive quality of tankers with a conventional stern arrangement and with the stern arrangement indicated in Fig. 1. The different stern designs have been made for a 90,000-ton deadweight tanker, having an engine power of 18,000 SHP and a speed of 15.75 knots. The principal particulars of the tanker are given in Table 3.

Table 3  
Principal Particulars of Tanker

| <u>Loaded Condition</u>       |         |                |
|-------------------------------|---------|----------------|
| Length between perpendiculars | 253.0   | m              |
| Breadth                       | 40.0    | m              |
| Draft                         | 14.30   | m              |
| Wetted surface                | 14,980  | m <sup>2</sup> |
| Moulded volume                | 120,171 | m <sup>3</sup> |
| Half angle of entrance        | 15° 48' |                |
| Block coefficient             | 0.831   |                |
| Midship area coefficient      | 0.996   |                |
| Prismatic coefficient         | 0.834   |                |

| <u>Light Condition</u> |        |                |
|------------------------|--------|----------------|
| Draft                  | 8.13   | m              |
| Moulded volume         | 65,664 | m <sup>3</sup> |

The body plans and the screw arrangements of the afterbody variations are presented in Fig. 10. The scale of the model is 38.

The characteristics of the various propellers are given in Fig. 11. The screw for the conventional model was designed according to the circulation theory for wake-adapted propellers.

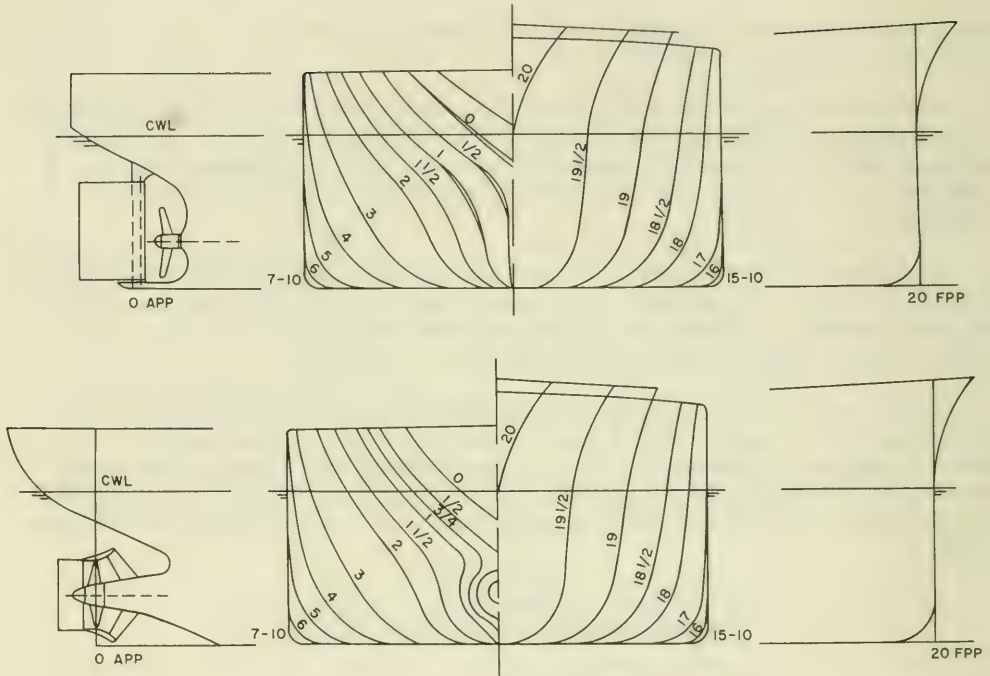


Fig. 10 - Body plans and propeller arrangements  
for the 90,000-TDW tanker model

The model with Hogner-type stern has been tested with three different nozzles. Nozzle No. 1 is derived from nozzle No. 19a by taking into account the effect of the hub shape of the flow. The taper of the hub is accounted for by locating the nozzle profile with reference to the direction of the undisturbed flow. The direction can be determined if one assumes frictional effects and static pressure variations to be negligible over the axial distance of the nozzle. It is then a matter of satisfying the law of continuity. This method is only permitted if the shape and the diameter of the hub are not too extreme. The design calculations for the ducted screw propeller were based on the method given in Refs. 9 and 10. The nozzles No. 2 and No. 3 have in comparison with nozzle No. 1, a more pronounced converging part before the screw. This may be attractive from a viewpoint of a more homogeneous flow in the nozzle. The particulars of the nozzle profiles Nos. 1, 2, and 3 are given in Fig. 12.

Model resistance and self-propulsion tests were conducted in the deep-water basin in accordance with established procedures. All model data were extrapolated to the full-scale ship values using Schoenherr's friction coefficients with an addition of 0.00035 for correlation allowances. A trip wire of 1 mm diameter, was fitted around each model at station 5 percent LBP for turbulence stimulation. The results of the resistance and self-propulsion tests in the loaded and in the light condition are given in Tables 4 and 5.

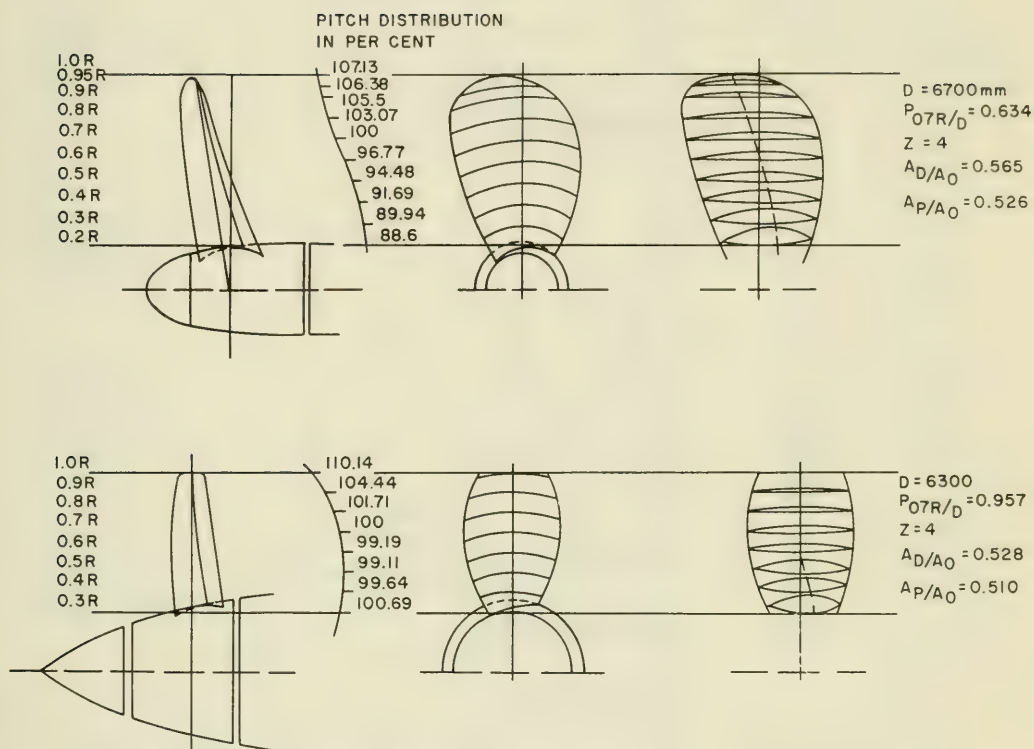


Fig. 11 - General plans of screws



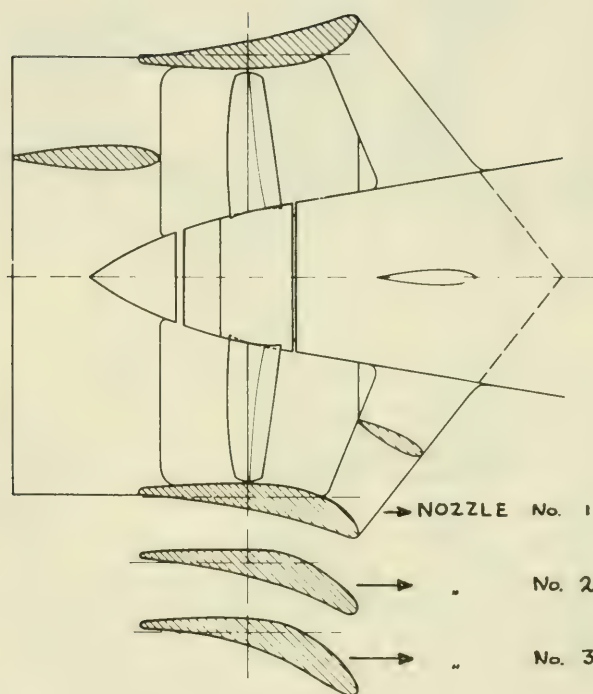


Fig. 12 - Profiles for nozzles No. 1, 2, and 3

Table 4  
Results of Resistance and Self-Propulsion Tests in Loaded Condition

|                                           | Speed<br>in<br>Knots | Resistance<br>Tests EHP<br>Metric | Self-Propulsion Tests |              |               |
|-------------------------------------------|----------------------|-----------------------------------|-----------------------|--------------|---------------|
|                                           |                      |                                   | SHP<br>Metric         | RPM<br>Screw | $\eta_{tot.}$ |
| Conventional<br>model                     | 13                   | 9,002                             | 13,721                | 98.9         | 0.656         |
|                                           | 14                   | 11,164                            | 17,079                | 106.7        | 0.654         |
|                                           | 15                   | 13,790                            | 21,023                | 114.5        | 0.656         |
|                                           | 16                   | 17,198                            | 26,417                | 123.8        | 0.651         |
|                                           | 17                   | 21,826                            | 34,399                | 134.9        | 0.634         |
| Hogner-type<br>stern with<br>nozzle No. 1 | 13                   | 8,431                             | 12,626                | 99.7         | 0.668         |
|                                           | 14                   | 11,035                            | 15,780                | 107.3        | 0.699         |
|                                           | 15                   | 13,877                            | 19,824                | 115.6        | 0.700         |
|                                           | 16                   | 17,074                            | 25,137                | 125.0        | 0.679         |
|                                           | 17                   | 21,203                            | 32,164                | 135.5        | 0.650         |
| Hogner-type<br>stern with<br>nozzle No. 2 | 13                   | 8,431                             | 13,463                | 100.3        | 0.626         |
|                                           | 14                   | 11,035                            | 16,669                | 108.0        | 0.662         |
|                                           | 15                   | 13,877                            | 20,756                | 116.4        | 0.669         |
|                                           | 16                   | 17,074                            | 26,157                | 125.8        | 0.648         |
|                                           | 17                   | 21,203                            | 33,324                | 136.3        | 0.636         |
| Hogner-type<br>stern with<br>nozzle No. 3 | 13                   | 8,431                             | 14,367                | 104.8        | 0.587         |
|                                           | 14                   | 11,035                            | 17,839                | 112.9        | 0.619         |
|                                           | 15                   | 13,877                            | 22,241                | 121.5        | 0.624         |
|                                           | 16                   | 17,074                            | 28,034                | 131.2        | 0.609         |
|                                           | 17                   | 21,203                            | 34,862                | 141.6        | 0.608         |

Table 6 compares the results of the EHP and SHP tests run at the loaded condition of the conventional model and the model with the Hogner-type stern with nozzles Nos. 1, 2, or 3. Table 7 compares the results at the light condition.

From the tables, it may be concluded that the extremely cigar-shaped afterbody is favourable from a viewpoint of resistance. The EHP of the model with the extremely cigar-shaped afterbody is about 1 percent less in the loaded condition and 5 percent less in the light condition, compared with the conventionally shaped model.

It is worth noting that this conclusion is contrary to cigar-shaped sterns investigated up till now. The tanker with the Hogner-type stern and nozzle No. 1 fitted, requires less power than the Hogner-type stern with nozzles No. 2 and No. 3 and the conventional stern. The Hogner-type stern with nozzle No. 1 fitted requires about 6 percent less SHP in the loaded condition and 9 percent less SHP in the light condition, than the conventionally shaped stern.

Table 5  
Results of Resistance and Self-Propulsion Tests in Light Condition

|                                           | Speed<br>in<br>Knots | Resistance<br>Tests EHP<br>Metric | Self-Propulsion Tests |              |               |
|-------------------------------------------|----------------------|-----------------------------------|-----------------------|--------------|---------------|
|                                           |                      |                                   | SHP<br>Metric         | RPM<br>Screw | $\eta_{tot.}$ |
| Conventional<br>model                     | 14                   | 9,014                             | 11,851                | 94.2         | 0.761         |
|                                           | 15                   | 11,348                            | 14,862                | 101.2        | 0.764         |
|                                           | 16                   | 14,154                            | 18,673                | 109.0        | 0.758         |
|                                           | 17                   | 17,806                            | 23,855                | 118.2        | 0.746         |
|                                           | 18                   | 22,576                            | 31,131                | 128.9        | 0.725         |
| Hogner-type<br>stern with<br>nozzle No. 1 | 14                   | 8,302                             | 10,069                | 94.0         | 0.750         |
|                                           | 15                   | 10,549                            | 13,805                | 101.2        | 0.764         |
|                                           | 16                   | 13,605                            | 17,085                | 108.4        | 0.796         |
|                                           | 17                   | 17,052                            | 21,754                | 117.6        | 0.784         |
|                                           | 18                   | 21,778                            | 28,645                | 128.7        | 0.760         |
| Hogner-type<br>stern with<br>nozzle No. 2 | 14                   | 8,302                             | 11,477                | 94.6         | 0.723         |
|                                           | 15                   | 10,549                            | 14,246                | 102.0        | 0.740         |
|                                           | 16                   | 13,605                            | 17,646                | 109.4        | 0.771         |
|                                           | 17                   | 17,052                            | 22,394                | 118.6        | 0.761         |
|                                           | 18                   | 21,778                            | 29,409                | 129.8        | 0.741         |
| Hogner-type<br>stern with<br>nozzle No. 3 | 14                   | 8,302                             | 12,531                | 99.1         | 0.663         |
|                                           | 15                   | 10,549                            | 15,614                | 106.8        | 0.676         |
|                                           | 16                   | 13,605                            | 19,338                | 114.9        | 0.704         |
|                                           | 17                   | 17,052                            | 24,096                | 123.5        | 0.708         |
|                                           | 18                   | 21,778                            | 30,803                | 134.1        | 0.707         |

Table 6  
Hogner-Type Stern Better (+) or Worse (-) Than Conventional  
Stern Arrangement at Loaded Condition

| Speed<br>in<br>Knots | Resistance<br>Tests<br>EHP | Self-Propulsion Tests<br>SHP |              |              |
|----------------------|----------------------------|------------------------------|--------------|--------------|
|                      |                            | Nozzle No. 1                 | Nozzle No. 2 | Nozzle No. 3 |
| 13                   | +6.3%                      | +8.0%                        | +1.9%        | -4.7%        |
| 13.5                 | +4.4%                      | +8.2%                        | +2.8%        | -3.7%        |
| 14                   | +1.2%                      | +7.6%                        | +2.4%        | -4.4%        |
| 14.5                 | -0.5%                      | +6.4%                        | +1.9%        | -5.2%        |
| 15                   | -0.6%                      | +5.7%                        | +1.3%        | -5.8%        |
| 15.5                 | +0 %                       | +5.4%                        | +1.2%        | -6.1%        |
| 16                   | +0.7%                      | +4.8%                        | +1.0%        | -6.1%        |
| 16.5                 | +1.9%                      | +5.3%                        | +1.9%        | -5.0%        |
| 17                   | +2.9%                      | +6.5%                        | +3.1%        | -1.3%        |

Table 7  
Hogner-Type Stern Better (+) or Worse (-) Than Conventional  
Stern Arrangement at Light Condition

| Speed<br>in<br>Knots | Resistance<br>Tests<br>EHP | Self-Propulsion Tests<br>SHP |              |              |
|----------------------|----------------------------|------------------------------|--------------|--------------|
|                      |                            | Nozzle No. 1                 | Nozzle No. 2 | Nozzle No. 3 |
| 13.5                 | +6.6%                      | +6.5%                        | +2.3%        | -5.7%        |
| 14                   | +7.9%                      | +6.6%                        | +3.2%        | -5.7%        |
| 14.5                 | +7.9%                      | +6.9%                        | +3.3%        | -5.5%        |
| 15                   | +7.0%                      | +7.1%                        | +4.1%        | -5.1%        |
| 15.5                 | +5.3%                      | +7.6%                        | +4.6%        | -4.4%        |
| 16                   | +3.9%                      | +8.5%                        | +5.5%        | -3.6%        |
| 16.5                 | +3.4%                      | +8.9%                        | +6.1%        | -2.0%        |
| 17                   | +4.2%                      | +8.8%                        | +6.1%        | -1.0%        |
| 17.5                 | +4.0%                      | +8.0%                        | +5.6%        | -0.2%        |
| 18                   | +3.5%                      | +8.0%                        | +5.5%        | +1.1%        |
| 18.5                 | +3.4%                      | +8.0%                        | +5.9%        | +2.3%        |

Finally, it may be concluded that an afterbody arrangement as suggested in the introduction offers a means of improving the propulsive efficiency for large tankers in addition to minimizing vibration and cavitation problems. Economical considerations with respect to the increase in building costs and the decrease in costs as a consequence of the smaller required SHP and probably lower costs of maintenance will give the final answer to the shipowner on the question in how far application of this advanced propeller arrangement for high-powered tankers is not only a dream of the hydrodynamist.

### 3. A COMPARATIVE STUDY OF THE STEERING OF TANKERS WITH LATERAL JETS AND WITH A RUDDER

As stated in the introduction of this paper, the technical merits of steering a tanker with lateral thrust units instead of with a rudder are twofold. Firstly by eliminating the rudder a more favourable afterbody shape-propulsor combination can be designed resulting in a substantial decrease in required shaft horsepower and improved vibration and cavitation characteristics; secondly a better manoeuvrability at low speeds in confined areas, harbours, and river estuaries can be expected resulting in partly or completely eliminating tug boat assistance, the economic benefit of which will be important to tanker owners. The main problem of the jet steering units are their horsepower requirements at the higher ship speeds.

English (11) and Chislett-Björheden (13) have measured the side force created by a lateral thrust unit as a function of the ratio ship speed to jet speed  $V/U$ . English found a 50 percent decrease in side force for a typical screw-driven lateral thrust unit when the ship speed was increased from 0 to 2 knots. This



effect was also measured by the authors of Ref. 13; they went to higher  $V/U$  values than English and established that the total side force and effective moment acting on their model both had a minimum for  $V/U$  at about 0.4 to 0.5.

This peculiar behaviour can be understood from the work of Keffer and Baines (14). They found that for  $V/U$  values higher than 0.2, a jet which was initially blown through a hole in a wall perpendicular to the mainstream turns  $90^\circ$  and tends to cling to the wall creating a region of low pressure downstream of the nozzle. This low pressure region creates a suction force which tends to diminish the effect of the thrust unit. For ships with jet steering this is an important effect since the center of action of the suction force moves aft for higher  $V/U$  values, hence the turning effect on the ship is no longer proportional to the side force on the ship and must be assessed in terms of both side force and turning moment.

A qualitative view of the effect of the forces and moments involved is given by Fig. 13.

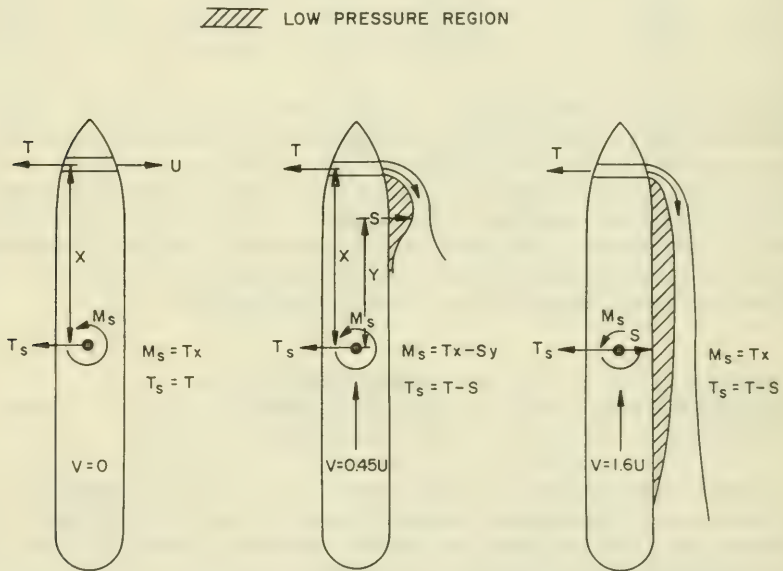


Fig. 13 - Forces and moments on a moving ship with a bow thruster

When  $v$  is increased, the center of the suction force  $S$  moves downstream decreasing the effective turning moment  $M_s$  on the ship. This turning moment passes a minimum and then increases again. When  $S$  passes the center of the ship,  $M_s$  has the same value as when  $v = 0$ . In Fig. 14, taken from Ref. 13, the dimensionless moment  $M_s / \rho A U^2 x$  is plotted against the velocity ratio  $V/U$ , and the curve clearly displays the effect. The line  $T_s / \rho A U^2$  against  $V/U$  is also

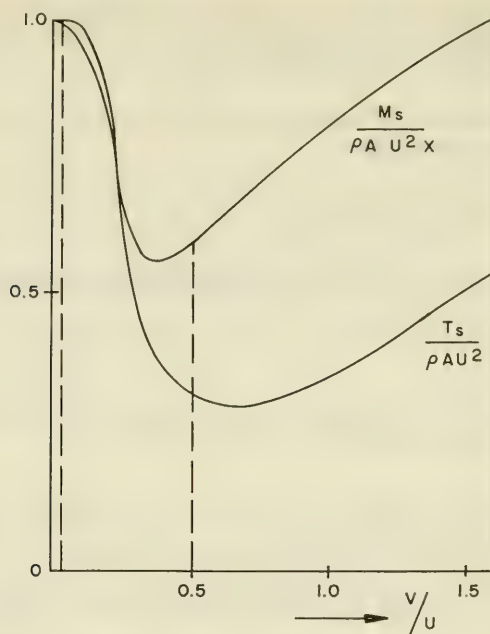


Fig. 14 - The dimensionless turning moment and side force as a function of  $V/U$  according to Ref. 13

shown. Since  $T = \rho A U^2$  and  $T = T_s + S$  signifies the constant thrust of the lateral thrust unit, it appears that  $T_s$  increases, passes a maximum and then decreases with increasing ship speed. From these considerations it is clear that bow thruster ship interaction is very important when studying the manoeuvrability capacities of various bow thrusting systems.

### Description of the Experiments

As mentioned in the preceding sections, two tanker models were built; one model with a conventional screw rudder arrangement and one model fitted with a Hogner afterbody, accelerating nozzle, bow and stern thrusters but without a rudder. In order to compare jet steering with conventional rudder steering it was intended to carry out Z manoeuvring as proposed by Kempf in the shallow water basin of the N.S.M.B. For the Z manoeuvring, the conventional model was equipped with a servo motor for the rudder angle adjustment. The model with jet steering was supplied with a centrifugal water pump and piping leading to the four exhaust nozzles situated at the bow and the stern of the model. In the water supply pipes, two venturi mass flow meters were mounted for measuring the water mass flow to the bow and stern thrusters. Servo motor-driven valves built in the water delivery pipes directed the water flow either to the port

or the starboard nozzles in such a manner that always a left- or right-hand torque acted on the ship. The time lag of the valve action was very small compared with the swinging period of the ship (Fig. 2).

During the Z manoeuvring experiments the relevant quantities were measured using the following methods:

*Water mass flow* from two mercury filled U-tubes connected with the venturi nozzles.

*Rudder angle*  $\delta$  was adjusted on the servo motor moving the rudder.

*Yaw angle*  $\psi$  and the drift angle  $\beta$  were measured with a gyroscope mounted on the model.

*Yaw angle rate*  $\dot{\psi} = d\psi/dt$  was derived by electronic differentiation of the gyroscope yaw angle signal with respect to the time  $t$ .

*The nozzle exhaust velocity*  $U$  was calculated with the aid of the continuity equation from the measured water mass flow rate and the known nozzle exhaust diameter. During the experiments the following quantities were changed: the draft  $H$ , the ship velocity  $V$ , the nozzle diameter  $D_n$ , the nozzle exhaust velocity  $U$  for the jet steering model, and the rudder angle  $\delta$  for the conventional model.

The following experimental program was started:

#### *Jet Steering*

$$H = 8.13, 14.3 \text{ m}$$

$$V = 5, 9, 13, 17 \text{ knots}$$

$$D_n = 1.10, 0.357 \text{ m}$$

$$U = 8.95, 12.1, 15, 17.9, 20.8 \text{ m/sec for } D_n = 1.10 \text{ m}$$

$$U = 18.6, 38.4, 58.5, 77, 97 \text{ m/sec for } D_n = 0.357 \text{ m}$$

#### *Rudder Steering*

$$H = 8.13 \text{ m}, 14.3 \text{ m}$$

$$V = 5, 9, 13, 17 \text{ knots}$$

$$\delta = 5, 10, 15, 20, 25 \text{ degrees}$$

For the adjustment of the nozzle exhaust velocity at model scale, Froude scaling was used. Thus the velocity ratio  $V/U$  for model scale and true scale remained the same.

The experiments were done in such a way that the model propelled itself with a certain desired model speed before rudder or jet action was initiated, after which the response of the model to alternating rudder movement or jet

action was measured as a function of time. A qualitative diagram of the different quantities such as  $\psi$ ,  $\dot{\psi}$ ,  $U$ ,  $\delta$  and the lateral displacement of the ship is shown in Fig. 15 whilst Fig. 16 defines the quantities used.

### The Results of the Comparative Z Manoeuvring Tests

One of the main reasons for this investigation was to devise criteria and the possible regions of application for conventional rudder steering and jet steering. It was clear that at service speed it would be difficult to improve on the conventional system since the power requirements for jet steering might become excessive.

A common yardstick for both steering systems was considered to be the maximum value of the yaw angle rate  $\dot{\psi}_{\max}$ . Figure 16 shows the behaviour of  $\dot{\psi}$  as a function of time during the Z manoeuvring tests. During the experiments

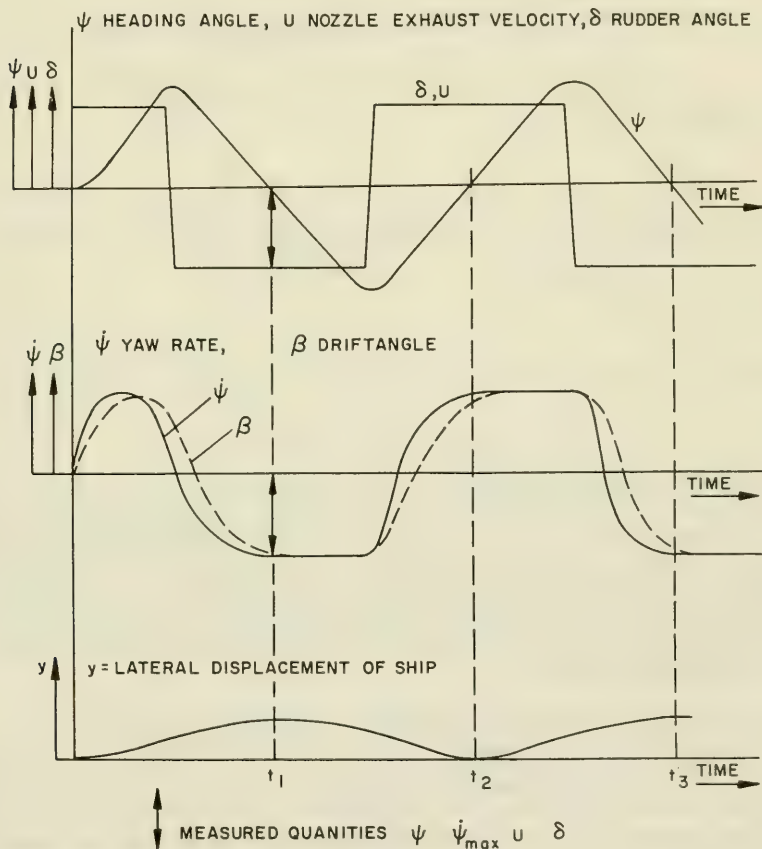


Fig. 15 - Path of some important steering quantities as a function of time



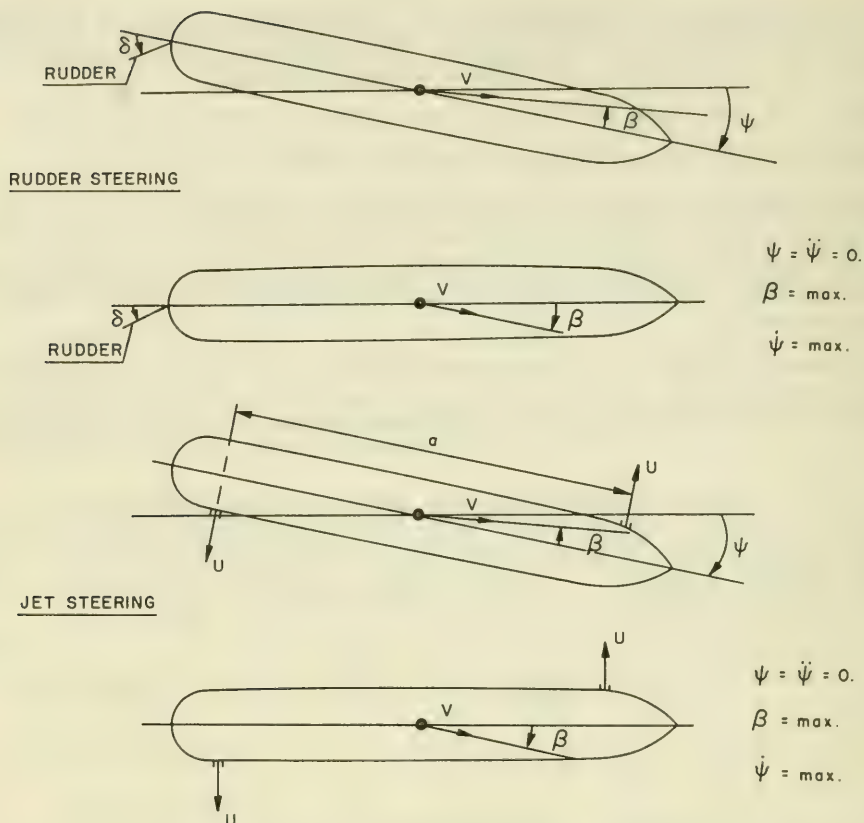


Fig. 16 - Definition sketch of some quantities given in Fig. 15

with the model with rudder steering,  $\dot{\psi}_{\max}$  could be measured as a function of the rudder angle  $\delta$  for a given ship speed  $V$ . When the model with jet steering was used, the dependence of  $\dot{\psi}_{\max}$  on the jet velocity  $U$  was measured for a certain ship speed. For a known  $U$ , the cross flow parameter  $V/U$  and the jet power given by the expression  $N_j = \frac{1}{2}(\rho AU^3) \cdot 75 \text{ HP}$ , could be computed. The experimental data are shown in Figs. 17-20 and Tables 9-11. The diagrams illustrate the fact that the  $\dot{\psi}_{\max}$  vs  $\delta$  relation has a positive gradient and the  $\dot{\psi}_{\max}$  vs  $V/U$  line a negative gradient.

The steering moment acting on the rudder of a ship decreases with  $V$  squared when the speed of the ship is lowered. For jet steering,  $U$  and hence  $N_j$  can be kept constant when the speed of the ship is lowered. This results in an increase of the turning moment on the ship since decreasing speed results in a lowering of the detrimental interaction effect between jet and hull as described in the preceding pages. So it can be said that when jet steering generates the same  $\dot{\psi}_{\max}$  as rudder steering with maximum rudder deflection, in the range of

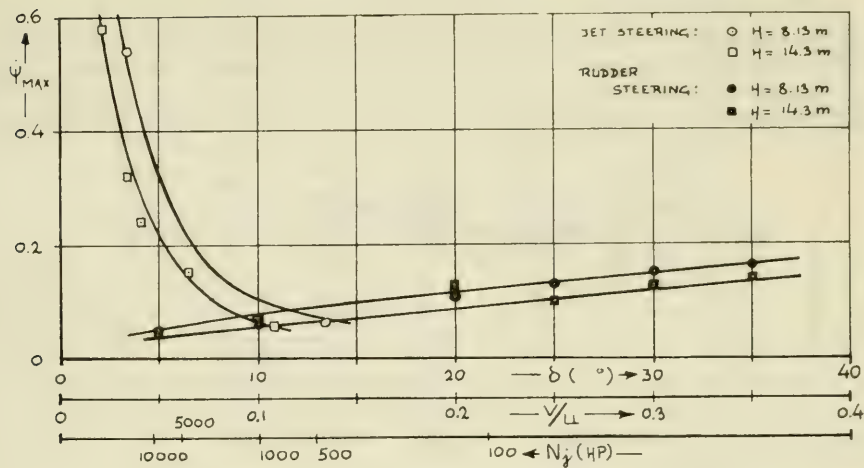


Fig. 17 - Yaw rate  $\dot{\psi}$  against rudder angle  $\delta$  and velocity ratio  $V/U$ ;  $V = 5$  knots,  $D_n = 0.357$  m

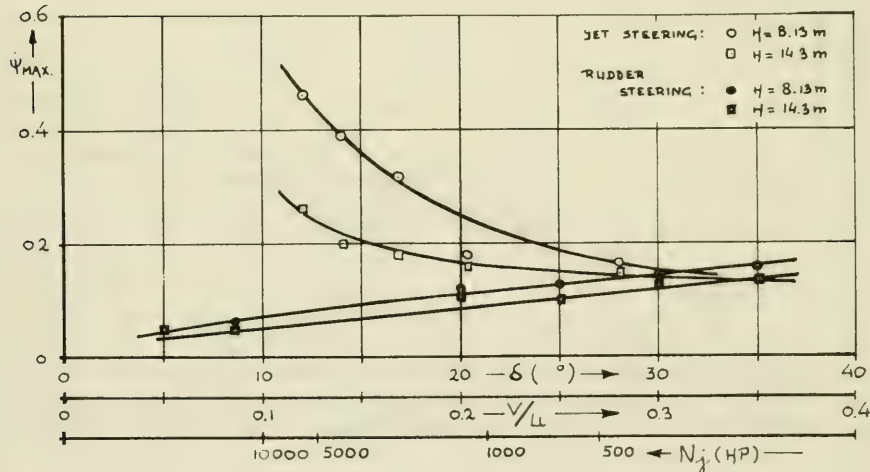


Fig. 18 - Yaw rate  $\dot{\psi}$  against rudder angle  $\delta$  and velocity ratio  $V/U$ ;  $V = 5$  knots,  $D_n = 1.10$  m

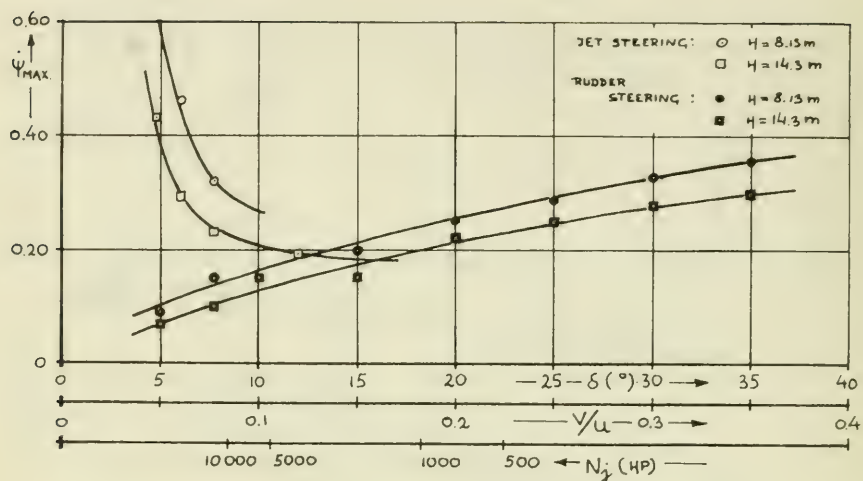


Fig. 19 - Yaw rate  $\dot{\psi}$  against rudder angle  $\delta$  and velocity ratio  $V/U$ ;  $V = 9$  knots,  $D_n = 0.357$  m

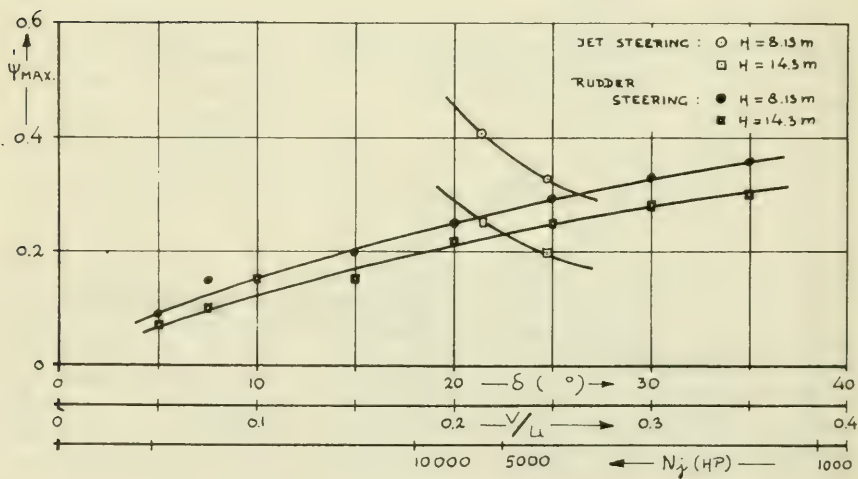


Fig. 20 - Yaw rate  $\dot{\psi}$  against rudder angle  $\delta$  and velocity ratio  $V/U$ ;  $V = 9$  knots,  $D_n = 1.10$  m

ship speeds between zero and  $V$ , jet steering will produce larger turning moments than the rudder system. This will result in better manoeuvring capabilities.

From the preceding discussion we conclude that for an evaluation of the jet steering system it is necessary to compare  $\dot{\psi}_{\max}$  at maximum rudder deflection of  $35^\circ$  with the same  $\dot{\psi}_{\max}$  generated by a certain nozzle with nozzle exit velocity  $U$ . Table 8 shows the results which are obtained from Figs. 17, 18, 19, and 20.

Table 8  
Points where  $\dot{\psi}_{\max}$  for Rudder Steering with a Rudder  
Deflection  $\delta = 35^\circ$  is Equal to  $\dot{\psi}_{\max}$  for Jet Steering

| $D_n = 1.10 \text{ m}$ | $V \text{ (knots)}$ | $N_j \text{ (hp)}$ | $U \text{ (m/s)}$ | $\dot{\psi}_{\max}$ | $V/U$ |
|------------------------|---------------------|--------------------|-------------------|---------------------|-------|
| $H = 14.3 \text{ m}$   | 5                   | 630                | 10.0              | 0.140               | 0.252 |
|                        | 9                   | 8,400              | 23.7              | 0.320               | 0.190 |
| $H = 8.13 \text{ m}$   | 5                   | 490                | 9.20              | 0.160               | 0.270 |
|                        | 9                   | 4,700              | 19.6              | 0.370               | 0.230 |

| $D_n = 0.357 \text{ m}$ | $V \text{ (knots)}$ | $N_j \text{ (hp)}$ | $U \text{ (m/s)}$ | $\dot{\psi}_{\max}$ | $V/U$ |
|-------------------------|---------------------|--------------------|-------------------|---------------------|-------|
| $H = 14.3 \text{ m}$    | 5                   | 3,100              | 36                | 0.165               | 0.070 |
|                         | 9                   | 27,600             | 75                | 0.300               | 0.060 |
| $H = 8.13 \text{ m}$    | 5                   | 1,980              | 31                | 0.175               | 0.080 |
|                         | 9                   | 17,300             | 64                | 0.360               | 0.070 |

Considering the data given in Table 8, we observe that only the large nozzle,  $D_n = 1.10 \text{ m}$ , is competitive with rudder steering in the 0-5 knots range. Here the power to drive the jets in order to be competitive with the rudder for rudder angles between  $0^\circ$  and  $35^\circ$  is too large. The same holds good for the small nozzle  $D_n = 0.357 \text{ m}$  at 5 knots as well as at 9 knots. However it can be seen in Fig. 19 that the small nozzle needs about 1500 hp to be competitive with the rudder at rudder angles between  $0^\circ$  and  $16^\circ$  at  $V = 9 \text{ knots}$ .

If we summarize the results, it can be said that the large nozzle can completely take over steering action from the rudder in the 0-5 knots range. For ship speeds ranging from 5 to 9 knots, jet steering with the small nozzle can be made only to assist and not to eliminate rudder action. Above 9 knots, rudder steering is definitely superior; the power needed to drive the jets becoming prohibitive. This is clear after considering the results in Tables 9 and 10 for  $V = 13 \text{ knots}$ .



Table 9  
Jet Steering ( $H = 8.13$  m)

| $v = 5$ Knots, $D_n = 1.10$ m |            |       |                    |
|-------------------------------|------------|-------|--------------------|
| U (m/sec)                     | $N_j$ (PK) | V/U   | $\dot{\psi}_{max}$ |
| 8.95                          | 450        | 0.260 | 0.16               |
| 12.1                          | 1,120      | 0.206 | 0.18               |
| 15                            | 2,150      | 0.168 | 0.32               |
| 17.9                          | 3,600      | 0.140 | 0.39               |
| 20.8                          | 5,650      | 0.120 | 0.46               |

| $v = 9$ Knots, $D_n = 1.10$ m |            |       |                    |
|-------------------------------|------------|-------|--------------------|
| U (m/sec)                     | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 17.9                          | 3,600      | 0.247 | 0.33               |
| 20.8                          | 5,650      | 0.214 | 0.41               |

| $v = 5$ Knots, $D_n = 0.357$ m |            |       |                    |
|--------------------------------|------------|-------|--------------------|
| U (m/sec)                      | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 18.6                           | 420        | 0.134 | 0.06               |
| 38.4                           | 3,750      | 0.065 | 0.19               |
| 76.8                           | 44,000     | 0.033 | 0.54               |
| 97                             | 63,000     | 0.026 | 0.70               |

| $v = 9$ Knots, $D_n = 0.357$ m |            |       |                    |
|--------------------------------|------------|-------|--------------------|
| U (m/sec)                      | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 58.2                           | 13,200     | 0.077 | 0.32               |
| 76.8                           | 44,000     | 0.059 | 0.46               |
| 97                             | 63,000     | 0.047 | 0.62               |

| $v = 13$ Knots, $D_n = 0.357$ m |            |       |                    |
|---------------------------------|------------|-------|--------------------|
| U (m/sec)                       | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 76.8                            | 44,000     | 0.085 | 0.48               |
| 97                              | 63,000     | 0.067 | 0.62               |

| $v = 0$ Knot, $D_n = 1.10$ m |            |     |                    |
|------------------------------|------------|-----|--------------------|
| U (m/sec)                    | $N_j$ (HP) | V/U | $\dot{\psi}_{max}$ |
| 20.8                         | 5,650      | 0   | 0.445              |

Table 10  
Jet Steering ( $H = 14.3$  m)

| $v = 5$ Knots, $D_n = 1.10$ m |            |       |                    |
|-------------------------------|------------|-------|--------------------|
| U (m/sec)                     | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 8.95                          | 450        | 0.280 | 0.15               |
| 12.1                          | 1,120      | 0.206 | 0.16               |
| 15                            | 2,150      | 0.168 | 0.18               |
| 17.9                          | 3,600      | 0.140 | 0.20               |
| 20.8                          | 5,650      | 0.120 | 0.26               |

| $v = 9$ Knots, $D_n = 1.10$ m |            |       |                    |
|-------------------------------|------------|-------|--------------------|
| U (m/sec)                     | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 17.9                          | 3,600      | 0.247 | 0.20               |
| 20.8                          | 5,650      | 0.214 | 0.25               |

| $v = 5$ Knots, $D_n = 0.357$ m |            |       |                    |
|--------------------------------|------------|-------|--------------------|
| U (m/sec)                      | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 38.4                           | 3,750      | 0.065 | 0.15               |
| 58.5                           | 13,200     | 0.043 | 0.24               |
| 77.0                           | 44,000     | 0.033 | 0.32               |
| 97.0                           | 63,000     | 0.026 | 0.48               |

| $v = 9$ Knots, $D_n = 0.357$ m |            |       |                    |
|--------------------------------|------------|-------|--------------------|
| U (m/sec)                      | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 38.4                           | 3,750      | 0.118 | 0.19               |
| 58.5                           | 13,200     | 0.077 | 0.25               |
| 77.0                           | 44,000     | 0.059 | 0.29               |
| 97.0                           | 63,000     | 0.047 | 0.43               |

| $v = 13$ Knots, $D_n = 0.357$ m |            |       |                    |
|---------------------------------|------------|-------|--------------------|
| U (m/sec)                       | $N_j$ (HP) | V/U   | $\dot{\psi}_{max}$ |
| 58.5                            | 13,200     | 0.111 | 0.25               |
| 77.0                            | 44,000     | 0.084 | 0.33               |
| 97.0                            | 63,000     | 0.067 | 0.45               |

| $v = 0$ Knot, $D_n = 1.10$ m |            |     |                    |
|------------------------------|------------|-----|--------------------|
| U (m/sec)                    | $N_j$ (HP) | V/U | $\dot{\psi}_{max}$ |
| 20.8                         | 5,650      | 0   | 0.705              |
| 17.9                         | 3,600      | 0   | 0.600              |
| 15                           | 2,150      | 0   | 0.547              |

Table 11  
Rudder Steering

| H = 8.13 m | V = 5 Knots         | V = 9 Knots         | V = 13 Knots        |
|------------|---------------------|---------------------|---------------------|
| $\delta$   | $\dot{\psi}_{\max}$ | $\dot{\psi}_{\max}$ | $\dot{\psi}_{\max}$ |
| 5          | 0.05                | 0.09                | 0.12                |
| 7.5        | —                   | 0.15                | 0.17                |
| 10         | 0.07                | 0.15                | 0.20                |
| 15         | —                   | 0.20                | 0.29                |
| 20         | 0.11                | 0.25                | 0.37                |
| 25         | 0.13                | 0.29                | 0.41                |
| 30         | 0.15                | 0.33                | 0.46                |
| 35         | 0.16                | 0.36                | 0.54                |

| H = 14.3 m | V = 5 Knots         | V = 9 Knots         | V = 13 Knots        |
|------------|---------------------|---------------------|---------------------|
| $\delta$   | $\dot{\psi}_{\max}$ | $\dot{\psi}_{\max}$ | $\dot{\psi}_{\max}$ |
| 5          | 0.05                | 0.07                | 0.15                |
| 7.5        | —                   | 0.10                | 0.16                |
| 10         | 0.06                | 0.15                | 0.22                |
| 15         | —                   | 0.15                | 0.30                |
| 20         | 0.12                | 0.22                | 0.33                |
| 25         | 0.10                | 0.25                | 0.37                |
| 30         | 0.13                | 0.28                | 0.41                |
| 35         | 0.14                | 0.30                | 0.47                |

We point out that  $\dot{\psi}_{\max}$  was not corrected for the increasing effectivity of the rudder due to propeller overload on model scale during the Z manoeuvring tests. This should decrease the average  $\dot{\psi}_{\max}$  value for rudder steering by 10 percent. So the above given result for jet steering superiority is on the conservative side. After contemplating the results of the experiments, it was felt that the nozzles tested were chosen too small. Whether the ship can be manoeuvred up till the design speed of 15.75 knots, with a larger nozzle,  $D_n = 2\text{--}4$  m say, remains to be seen. However nozzles with these small exhaust diameters are attractive in connection with main cargo pump-driven steering systems. These installations will be described in the following section.

#### 4. MAIN-CARGO-PUMP-DRIVEN JET-STEERING SYSTEMS

##### Simple Nozzle System

For a description of conventional lateral thrust units we refer to the publication of Jastram (12) and of Stuntz and Taylor (15).

Recently the idea of using the main cargo pumps of tankers for steering purposes has been put forward. It is intended to deliver sea water with these pumps to nozzles situated at stern and bow, the efflux of these nozzles creating a lateral force and a turning moment on the ship. The economic incentives for using this system are that expensive machinery which is otherwise idle is put to work for manoeuvring the ship without any additional rotating machinery. The possible elimination of tugboat assistance is also an important consideration.

A typical simple thrust generating system is illustrated in Fig. 21. The main cargo pumps deliver a certain volume flow rate with a certain head. The pressure increase across the pumps is needed to cancel the pressure drop in the system caused by friction in the pipelines and appendages, the height difference between sea chest and nozzle exhaust, and the pressure drop in the nozzle. Generally the last pressure decrease is the largest, the friction losses in the pipeline can become prohibitive only when the pipe diameter is chosen too small. The thrusters can be modulated with a throttle-valve in the pipeline. The pressure drop in the various pipes, valves, bends, etc., can be computed from data given in engineering textbooks. In a smoothly rounded exhaust nozzle (Fig. 22), the friction losses are very low compared with the friction losses in the pipeline system; these losses can be neglected when designing these systems. In that case Bernoulli's law results in

$$\Delta P_{\text{nozzle}} = \frac{1}{2} \rho U_{\text{nozzle}}^2 - \frac{1}{2} \rho U_{\text{pipe}}^2,$$

or using the law of continuity,

$$\Delta P_{\text{nozzle}} = \frac{8\rho Q^2}{\pi^2} \left( \frac{1}{D_{\text{nozzle}}^4} - \frac{1}{D_{\text{pipe}}^4} \right),$$

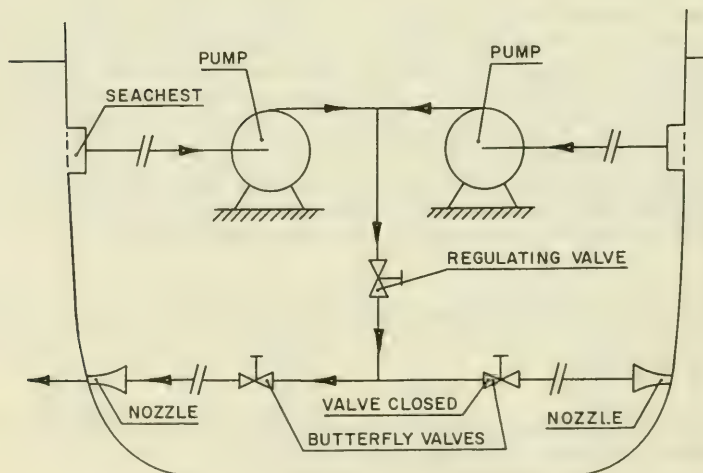


Fig. 21 - Arrangement of simple nozzle system

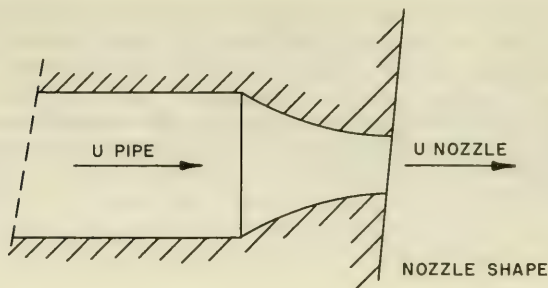


Fig. 22 - Exhaust nozzle

$Q$  being the volume flow rate through the nozzle. The generated thrust at zero speed amounts to

$$T = \frac{\pi}{4} \rho D_{\text{nozzle}}^2 U_{\text{nozzle}}^2$$

For the manoeuvring experiments described in the preceding section, the capabilities of an installation supplied with a nozzle exit diameter of  $D_n = 0.357$  m must be seen as representative.

The plain nozzle system has the drawback that cargo pumps have an unfavorable head-capacity relationship (high head, low volume flow rate) which results in a large kinetic energy loss in the jet leaving the nozzle. This is the reason why the plain nozzle system has a low thrust per installed pump horsepower of the order of 2-3 kg/hp at zero ship speed compared to 10 kg/hp for a typical tugboat at zero speed.

### The Ejector System

By placing the exhaust nozzle described above in a tunnel with open ends at both sides of the ship, an ejector system is obtained (Fig. 3). The jet leaving the nozzle mixes with the surrounding water and creates a flow in the tunnel. Since this installation increases the water mass flow rate and reduces the exhaust velocity a better thrust-horsepower ratio compared with the plain nozzle installation is obtained. From experiments carried out on a special 100-hp ejector test bench at the N.S.M.B., it was found that a thrust increase factor of two over the plain exhaust nozzle can be achieved corresponding to a thrust of 4-6 kg/hp for a practical installation at zero ship speed. The main advantage of this system is that a more favorable thrust-horsepower ratio is obtained without sacrificing the basic simplicity and absence of moving parts of the plain nozzle installation. The results of the manoeuvring tests with a nozzle diameter  $D_n = 1.10$  m should be seen as typical for the behaviour of such an ejector-driven steering system.



A further increase of thruster efficiency is to be expected from a two-stage ejector design (Fig. 23). By using the exhaust of a small primary ejector as the nozzle for a big secondary ejector, the mass flow is increased further. It is expected that a thrust of 6-8 kb/hp can be attained at zero ship speed with a cargo-pump-driven system of this configuration. Since the exhaust velocity of the primary nozzle is of the order of 30-40 m/sec, cavitation in the primary ejector may create problems.

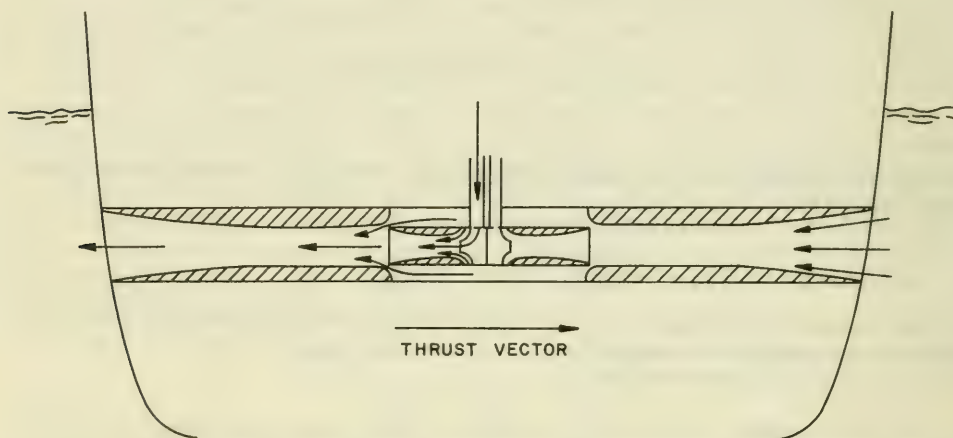


Fig. 23 - Arrangement of two-stage ejector system

## 5. CONCLUSIONS

The main conclusions of this feasibility study are:

The extremely cigar-shaped afterbody is favourable from a viewpoint of resistance. This is contrary to cigar-shaped sterns investigated up till now. The EHP of the tanker with the extremely cigar-shaped afterbody is about 1 percent less in the loaded condition and 5 percent less in the light condition, compared to the conventionally shaped tanker.

The extremely cigar-shaped stern with shrouded propeller having a large hub-to-diameter ratio, as suggested in the introduction, offers a means of improving the propulsive efficiency for large tankers in addition to minimizing vibration and cavitation problems. The Hogner-type stern with nozzle No. 1 requires about 6 percent less SHP in the loaded condition and 9 percent less SHP in the light condition than the conventionally shaped stern. This improvement is of the same magnitude of an extreme bulbous bow, which leads to a 15 percent SHP reduction in the light condition.

The experiments show a definite advantage of the lateral thrust arrangement over the conventional rudder for ship speeds up to 5 knots. Tests using

jet nozzles with larger diameters than 1.10 m may lead to an extension of the speed range where this type of ship control has a superior manoeuvrability, lowering at the same time the required pump power.

Comparing the two types of cargo-pump-driven lateral thrust units, it is found that at zero forward speed the ejector type doubles the specific thrust of 2-3 kg per installed pump horsepower of the simple nozzle system. At 5 knots ship speed, the nozzle with an exit diameter of 1.10 m, which is representative for the ejector system, requires about half the power of the plain nozzle system with a manoeuvrability which, when expressed in maximum yaw angle rate, is the same.

Since it was found that at speeds above 5 knots the jet steering system can only assist the rudder but not make it superfluous, the tail piece just behind the propeller nozzle should be used as a rudder. Therefore it should be made movable.

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\* \* \*

## DISCUSSION

H. Lackenby  
*British Ship Research Association*  
*London, England*

I was very interested to see the statement in the conclusions that the 9 percent improvement in SHP with the Hogner-type stern was of the same magnitude as an extreme bulbous bow which leads to a 15 percent reduction in SHP in the light condition.

In the first place, I would be glad to have some clarification on this matter, that is, whether the 15 percent is the figure for the bulbous bow alone or whether it refers to the bulbous bow and Hogner-stern together.

In this connection, I should like to mention an experience we had in carrying out bow and stern variations to a tanker model. First we did the bow variations in conjunction with a conventional stern and obtained a small improvement of the order of 3 percent (this was not a ram bow, by the way). We then tried a cigar-shaped stern in conjunction with a conventional bow and obtained an improvement, here of about 3 percent also. We then tried the bulbous bow and bulbous stern together, hoping of course that the two improvements would be additive, but this did not turn out to be the case. In fact the improvement was barely significant. In other words, the effect of either the bow or stern bulb alone was slightly better than both together.

I should like to underline here that the bulbs I am talking about were very much less pronounced than those dealt with by the authors, and to some extent we were probably flapping around in the experimental scatter band.



In some analysis work we have been carrying out recently, however, involving more pronounced ram bows, there is a fairly clear indication that the bow modification is affecting the propulsive efficiency, and one has to be very careful about considering the two ends of the ship separately and then assuming the effects to be additive.

Perhaps the authors would comment on this question of bow and stern interaction.

\* \* \*

## DISCUSSION

J. Strom-Tejsen

*David Taylor Model Basin*

*(after March 31, 1967 renamed the Naval  
Ship Research and Development Center)  
Washington, D.C.*

It is certainly refreshing to read a paper like this in which the authors have studied the application of an unconventional control system to very large tankers. It would have been more interesting had the proposed jet control system shown definite merits as an alternative to the traditional rudder configuration. Unfortunately the jet system alone according to the paper would require excessive power in order to produce forces comparable to those obtained by a conventional rudder. Also the authors' alternate proposal, combining the jet system and the small tail piece as a movable rudder, is questionable in its present configuration as it is unlikely that the small tail piece tested would suffice in area to produce the desired maneuvering forces. It would be of interest to know if the authors actually have explored this alternative further.

The extreme cigar-shaped afterbody having a shrouded propeller is apparently an attractive solution from a resistance and propulsion point of view since critical problems due to vibration and cavitation are reduced. But let us for a moment consider the merits of this form with respect to some of the problems encountered in the field of controllability of very large tankers. These ships, for instance, are known to have very poor inherent course stability qualities. As deadwood has been reduced and a big conventional rudder eliminated in the proposed stern arrangement, it is possible that the dynamic stability problem would become more critical in this case, and the proposal thus would be unrealistic unless additional fin area is added. I think many of us would appreciate the authors commenting on this problem and possibly adding to the paper the results from a course stability study.

The stopping and backing are other examples of maneuvers which by many are considered critical for the very large tankers. As shrouded propellers are known to be inferior to conventional propellers with respect to the retarding and backing force which can be produced in a stopping maneuver, it is necessary



in the case of the nozzle propeller propulsion to pay attention to this problem, and it might be necessary to introduce special braking arrangements in order to reduce the stopping length.

Finally, a short question. Has the comparison between the Hogner-type and conventional stern given in Tables 5 through 8 been based on experiments with jet tunnels installed in the model, thus actually taking into account the small increase of resistance due to the tunnel opening?

\* \* \*

## DISCUSSION

S. Bindel  
*Bassin d'Essais des Careènes*  
*Paris, France*

I am a little skeptical of the elimination of the rudder from such full ships as tankers. In fact, the rudder has two separate functions — the first one is, of course, to make the ship turn but it also constitutes, when amidship, a stabilizing fin to maintain the ship on a straight line. I would like to ask the authors if they have measured the course stability of the tanker without rudder.

Besides, it may be added that the rudder has a beneficial influence on the propulsive efficiency, in reducing the energy loss due to the rotation of the flow behind the propeller. Have the authors compared the supplementary resistance due to the hull openings of the stern lateral thruster with the effective resistance of the rudder?

\* \* \*

## DISCUSSION

Orvar Björheden  
*KMW Maine Laboratory*  
*Kristinehamn, Sweden*

Let me first compliment the authors for bringing up a most interesting paper within this very important field. Indisputably, a cigar-shaped stern as once suggested by the late Professor Hogner will, to a considerable degree, reduce the problems involved in cavitation erosion and propeller-induced ship hull vibrations. Among the most interesting results from the present N.S.M.B. tests is that such a stern arrangement also seems to give a considerable

reduction in propeller horsepower compared with a conventional stern arrangement. From the basic concepts it appears that a reduction in propeller horsepower is possible either by reducing ship resistance or by improving the propulsive efficiency. The present tests with a 90,000-TDW tanker model show both these improvements, the resistance being reduced by 1 and 5 and the efficiency improved by 5 and 4 percent at the loaded and light condition, respectively. According to another well-known equation, the propulsive efficiency may be expressed as the product of the propeller, hull, and relative rotative efficiencies. Unfortunately the paper does not give any information as to the way in which these propulsive coefficients are affected by the proposed nozzle propeller arrangement, and I would like to ask the authors if any such information is available. It would also have been interesting to know what efficiency one may obtain by using an unducted propeller in combination with the Hogner stern. From the related work on a 39,000-TDW tanker model (Section 2 of the authors' paper), one gets the impression that the Hogner stern permits the propulsive efficiency to be improved also with a normal free propeller.

The idea of using an ejector type of thruster is interesting as it seems to offer a solution to the problem of combining a high-pressure cargo pump with an effective low-speed transverse jet. However, I would like to draw your attention to the fact that it might occasionally be desirable to have the possibility of using the thrusters and the cargo pumps simultaneously, for instance when loading or unloading at mooring, for trimming purposes when getting out of harbors, etc. Obviously there are several other aspects, mainly of a practical nature, which should be regarded when considering a cargo-pump-driven thruster. With respect to the human factor, one might expect for instance an increased risk of accidental discharge of cargo oil. In my opinion, a cargo-pump thruster should not be considered unless it shows marked economical benefits over conventional thrusters. The present paper does not give any clear indication of this point. The power figures given represent only the jet power. I would like to know if the authors have made any estimates of frictional and bend losses in the rather extensive pipeline they are proposing, and if a normally sized pump equipment is capable of producing a sufficient thrust. It would also be interesting to see a comparison between the costs of a cargo-pump thruster including valves, nozzles, pipes, governors, etc., and a corresponding conventional thruster.

Among the possible merits of a thruster steering system that the authors mention are the reduced tug assistance costs. To illustrate the rate of these savings I would like to submit some statistical data given to us by a well-known shipping company in Europe. The figures apply to two 12,000-TDW sister cargo liners which have been in service between Europe and the Far East for about two years. One of them is equipped with a 600-hp bow thruster plus a controllable-pitch main propeller; the other has a conventional fixed-pitch propeller. It appears that for the ship with bow thruster, tug costs are reduced by about 80 percent compared with her sister ship without bow thruster. For some of the ports along the trade the following numbers of tugs are needed including both arrival and departure:

|                 | With<br>Bow Thruster | Without<br>Bow Thruster |
|-----------------|----------------------|-------------------------|
| Aden            | 1                    | 2                       |
| Hamburg         | 2                    | 5                       |
| Rottendam       | 1                    | 5                       |
| Antwerp         | 3                    | 8                       |
| Gottenburg      | 0                    | 4                       |
| Oslo            | 0                    | 1                       |
| Aarhus, Denmark | 1                    | 4                       |
| Copenhagen      | 0                    | 2                       |

The above figures have also been verified by several later installations.

An important matter which has not been treated by the authors is the stopping characteristics of these very large vessels. With a conventional stern arrangement and a free fixed-pitch propeller, the stopping time for tankers between 50 and 100,000 TDW is normally between 8 and 10 minutes corresponding to a stopping distance of 7 to 9000 feet. One may expect these figures to be even higher when using a nozzle propeller as proposed by the authors. A considerable reduction in stopping time and distance may be achieved by using a controllable pitch propeller. I would like to finish with a few figures obtained with some of our own propellers.

For a 50,000-TDW twin-screw tanker with an output of 8900 HP each shaft at 115 RPM and equipped with two 4-bladed 17-1/2-foot controllable-pitch propellers, a stopping time of 4.5 minutes from full speed, 15.75 knots, was received. For a 72,000-TDW single-screw ore carrier with an output 17,600 HP at 115 RPM, fitted with a 4-bladed 21-1/2-foot CP propeller, a stopping time of 5 minutes was obtained from full speed, 16.75 knots.

Compared with similar ships with normal fixed-pitch propellers, this implies a reduction in the order of 50 percent.

\* \* \*

## REPLY TO DISCUSSION

J. D. van Manen, M. W. C. Oosterveld, and J. H. Witte

We wish to thank Messrs. Lackenby, Stron-Tejsen, Bindel, and Björheden for their interesting comments.

In regard to Mr. Lackenby's question about the improvement in SHP with the Hogner-type stern and nozzle and the improvement in SHP with an extreme bulbous bow, it must be remarked that our experiments were performed with ships without bulbous bows. It was found that the Hogner stern with nozzle



requires about 6 percent less SHP in the loaded and 9 percent less SHP in the light condition. The application of an extreme bulbous bow for this type of ship will possibly lead to a 15-percent SHP reduction in the light and no reduction in the loaded condition. The time this type of ship will sail loaded and in ballast will be about equal, so the improvements in SHP with the Hogner stern and nozzle fitted and with an extreme bulbous bow are of the same magnitude. At this moment we do not have experience with respect to the resistance and the required shaft horsepower of ships with both an extreme bulbous bow and a Hogner stern with nozzle fitted.

We agree with Mr. Strom-Tejsen that the jet system according to the paper would require excessive power in order to produce at service speed forces comparable to those obtained by a conventional rudder, due to the relatively small nozzle diameters. The comparison between the EHP of the model with the Hogner-type and the conventional stern has been based on experiments without jet tunnels installed in the model. The increase in resistance due to the tunnel openings (for these small tunnel diameters less than 0.5-1.0 percent) was not taken into account.

We agree with Mr. Bindel that special attention must be paid to the problem of course stability. Some remarks can be made already. Firstly, we are aware that the Hogner stern will not affect favorably the course stability. On the other hand we expect that the nozzle itself will form a positive element with respect to course stability. Finally the tail piece will be an addition and forms a means for further affecting (by increasing the lateral area) the course stability of the tanker.

In regard to Mr. Björheden's questions, it must be remarked that for the determination of the propeller, hull, and relative rotative efficiencies, the thrust of both screw and nozzle must be measured. Only the thrust of the screw was measured here.

The data given by Mr. Björheden with respect to the reduction in tug assistance costs due to the application of bow thrusters and his remarks with respect to the stopping characteristics of large vessels are very interesting. Recently, investigations were performed at the N.S.M.B. with respect to the stopping abilities of a 100,000-TDW single-screw turbine tanker with an output of 28,000 HP at 85 RPM and equipped with different propeller types. The stopping abilities have been derived from model test results by means of a quasi-stationary method. Stopping times of about 32, 28, 23, and 18 min from full speed of 16 knots were found for, respectively, a conventional screw, a screw and nozzle system, a contra-rotating propeller system, and a controllable-pitch propeller. The head reaches were, respectively, 5100, 4800, 4450, and 2900 meters. For an extensive discussion of this matter, we refer to the original paper of Hooft and van Manen, to be read at the Spring Meeting 1967 of the RINA. From these results it can be seen that the stopping characteristics of the ducted propeller are better than those of the conventional screw.

\* \* \*





# CRUISING AND HOVERING RESPONSE OF A TAIL-STABILIZED SUBMERSIBLE

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## ABSTRACT

Equations of motion are used as a basis for analyzing the inherent dynamic stability and limit maneuver response of a tail-stabilized submersible in cruising and hovering vertical plane motions. It is shown that this type of vessel can perform adequately in cruising, but is subject to highly coupled, unstable hovering motion, especially in stern-to-bow ocean currents. The stability of a submersible with both bow and stern stabilizers, having fore-aft symmetry, also is treated. This latter type of design has inherent hovering stability, and its symmetry would have a salutary effect on the coupled motions of the vessel.

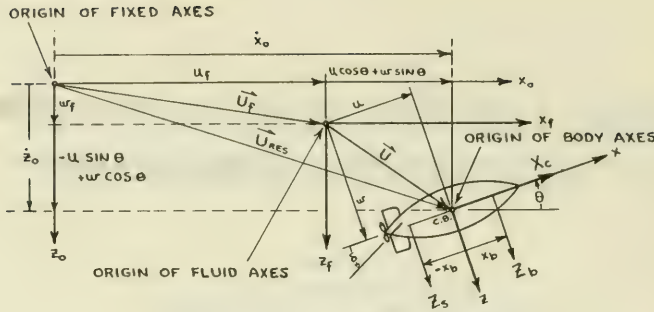
## NOMENCLATURE

The basic nomenclature is adapted from SNAME Technical and Research Bulletin No. 1-5. Three different right-handed rectangular coordinate frames are used:

1. Body axes  $(x, y, z)$  with origin at the center of buoyancy (CB)
2. An inertial frame  $(x_0, y_0, z_0)$  fixed in the earth
3. Fluid axes  $(x_f, y_f, z_f)$  which move with a constant velocity  $U_f$  relative to  $(x_0, y_0, z_0)$

A sketch of the axes systems and the positive directions of the various angles, velocities, and forces is shown.

Hydrodynamic forces in the  $x$ ,  $y$ , and  $z$  directions are designated by  $X$ ,  $Y$ ,  $Z$  and moments by  $K$ ,  $M$ , and  $N$ . Velocity components  $u$ ,  $v$ , and  $w$  of the CB are measured relative to the fluid axes in directions of the  $x$ ,  $y$ , and  $z$  axes, respectively. Angular velocity components of body axes are designated by  $p$ ,  $q$ , and  $r$ . Velocity components  $u_f, v_f, w_f$  of the fluid current relative to earth are measured in the directions of the  $x_f, y_f, z_f$  axes, respectively, which are always parallel to the  $x_0, y_0, z_0$  axes.



The weight  $W$  of the submersible is positive and includes internal water. The buoyancy  $B$ , also positive, is the weight of water displaced by the volume enclosed in the external boundary of the submersible. The coordinates  $x_G, y_G, z_G$  of the center of gravity (CG) are measured in the body axes frame. The pitch moment of inertia of the body is  $I_{yy}$  and is referred to the  $y$  axis, and the mass  $m$  is  $W/g$ .

Derivatives with respect to time  $t$  are denoted by a dot over the symbol, e.g.,  $\dot{u}$  is the  $x$ -component of acceleration. Hydrodynamic force and moment derivatives are evaluated at zero angular velocity, zero acceleration, and zero angles of attack  $\alpha$ , sideslip  $\beta$ , and elevator  $\delta$ . These are denoted by the force or moment with a velocity, acceleration, or angle symbol as subscript, e.g.,  $\partial Z/\partial w = Z_w$  and  $\partial M/\partial \delta = M_\delta$ . Second derivatives are denoted in a similar manner, e.g.,  $\partial^2 X/\partial w \partial q = X_{wq}$ .

Stability indices are given by  $\sigma_i$ , subscript  $i$  being an integer.  $n_x$  denotes the angular velocity of the longitudinal propeller.

Subscripts used include  $a$  for the change in a value due to addition of an appendage,  $b$  and  $s$  for bow and stern, respectively,  $c$  for thruster control forces and moments,  $e$  for the equilibrium value of a quantity,  $h$  for bare hull value, and  $m$  for maximum value of a quantity.

The length  $\ell$  and the frontal area  $A$  of the hull are used in forming dimensionless coefficients, which are denoted by the prime of a symbol. Some examples of these are

$$x'_G = \frac{x_G}{\ell}, \quad Z' = Z/\frac{\rho}{2} AU^2, \quad M' = M/\frac{\rho}{2} A \ell U^2, \quad m' = m/\frac{\rho}{2} A \ell, \quad q' = \frac{q \ell}{U},$$

$$Z'_w = Z_w/\frac{\rho}{2} AU, \quad Z'_q = Z_q/\frac{\rho}{2} A \ell U, \quad \dot{q}' = \frac{\dot{q} \ell^2}{U^2}, \quad I'_{yy} = I_{yy}/\frac{\rho}{2} A \ell^3,$$

and

$$\sigma' = \sigma \ell / U,$$

where  $\rho$  is the mass density of water and  $U$  is the magnitude of  $U$ .

## I. INTRODUCTION

In the present paper, motion equations are utilized to investigate some of the stability and control problems that may be encountered in the operation of a tail-stabilized submersible which must cruise and hover in an ocean current environment. Most of the examples given to illustrate the behavior of the vessel were obtained from a variation of parameter, digital computer study of the vertical plane cruising and hovering limit maneuvers of a 43-ft-long, 8-ft-diameter rescue submersible. The latter work was sponsored by the Special Projects Office of the U.S. Navy, and the results were presented recently in a Davidson Laboratory report (1).

## II. THE HYDRODYNAMIC SYSTEM

The basic hull form is a streamlined body of revolution with a screw propeller for forward thrust, bow and stern thrusters for hovering pitch, heave, yaw, and sway control, a mercury flow system for roll control, and either a movable ring tail or fixed tail fins with movable rudders and elevators for stabilization and control in cruising operations. In the case of the rescue submersible (2), a mating bell is near the center and on the bottom of the vessel. This is to attach to the access hatch of a disabled submarine to permit the transfer of twelve survivors at a time to the rescue submersible, and then to another operational submarine. Thus, the need for both precise hovering control during mating operations and the cruising maneuver capabilities.

## III. THE MATHEMATICAL MODEL

Although a mathematical model representing the dynamic behavior of the submersible in six-degrees-of-freedom was developed, this is simplified for treatment of the three-degree-of-freedom motions in the vertical plane, and the automatic control system is replaced by a series of simple control rules which describe the limit overshoot maneuvers in cruising and hovering flight. This simplified model is complete enough to show the underlying factors which can cause difficulties with the motions of the submersible.

During any given maneuver, it is assumed that the mass  $m$ , the pitch moment of inertia  $I_{yy}$ , and the fluid current velocity  $U_f$  are constant. It also is assumed that the submersible has geometric symmetry relative to its vertical  $xz$ -plane and that the main hull is a body of revolution with fore-aft symmetry. The latter assumption permits a relatively simple representation of the hydrodynamic forces and moments in hovering conditions where the angle of attack can vary from 0 to  $2\pi$ . Extensive experimental data are lacking for these conditions.

The equations of motion in the vertical plane are written using the body axes  $(x, y, z)$  and an inertial frame  $(x_f, y_f, z_f)$  which is fixed in the fluid. The fluid axes form an inertial frame because  $U = (u_f, 0, w_f)$  is constant relative to an inertial frame  $(x_0, y_0, z_0)$  fixed in the earth.



The longitudinal propulsive force  $X_c$  is used in both cruising and hovering operations, and the vertical bow and stern thrust forces  $Z_b$  and  $Z_s$  with longitudinal coordinates  $x_b$  and  $-x_b$ , respectively, are used in hovering operations. The movable elevators are used for control in cruising flight.

Under the above assumptions, the motion equations are as follows:

$$\begin{aligned}
 (a) \quad & A_1 \dot{u} + A_2 \dot{w} + A_3 \ddot{\theta} + A_4 u \dot{\theta} + A_5 |u| \dot{\theta} + A_6 w \dot{\theta} + A_7 uU + A_8 wU + A_9 wu + A_{10} \dot{\theta}^2 \\
 & + A_{11} |\dot{\theta}| \dot{\theta} + A_{12} \sin \theta + A_{13} \cos \theta + A_{14} \delta |u|u = X_c \\
 (b) \quad & B_1 \dot{u} + B_2 \dot{w} + B_3 \ddot{\theta} + B_4 u \dot{\theta} + B_5 |u| \dot{\theta} + B_6 w \dot{\theta} + B_7 uU + B_8 wU + B_9 wu + B_{10} \dot{\theta}^2 \\
 & + B_{11} |\dot{\theta}| \dot{\theta} + B_{12} \sin \theta + B_{13} \cos \theta + B_{14} \delta |u|u = Z_c \\
 (c) \quad & C_1 \dot{u} + C_2 \dot{w} + C_3 \ddot{\theta} + C_4 u \dot{\theta} + C_5 |u| \dot{\theta} + C_6 w \dot{\theta} + C_7 uU + C_8 wU + C_9 wu + C_{10} \dot{\theta}^2 \\
 & + C_{11} |\dot{\theta}| \dot{\theta} + C_{12} \sin \theta + C_{13} \cos \theta + C_{14} \delta |u|u = M_c
 \end{aligned} \tag{1}$$

where

$$\begin{aligned}
 X_c &= a_x |n_x| n_x + b_x |n_x| u & (a) \\
 Z_c &= Z_s + Z_b & (b) \\
 M_c &= x_b (Z_s - Z_b) & (c) \\
 \dot{x}_o &= u \cos \theta + w \sin \theta + u_f & (a) \\
 \dot{z}_o &= -u \sin \theta + w \cos \theta + w_f & (b) \\
 x_o &= \int_0^t \frac{dx_o}{d\tau} d\tau + x_o(0) & (c) \\
 z_o &= \int_0^t \frac{dz_o}{d\tau} d\tau + z_o(0) & (d) \\
 \dot{u} &= (u^2 + w^2)^{1/2} & (e)
 \end{aligned} \tag{3}$$

and

$$\begin{aligned}
 A_1 &= m - \frac{\rho}{2} A \ell X'_u & B_2 &= m - \frac{\rho}{2} A \ell Z'_w & C_1 &= C_6 = m z_G \\
 A_3 &= m z_G & B_3 &= -m x_G & C_2 &= -C_4 = -m x_G \\
 A_6 &= m - \frac{\rho}{2} A \ell X'_{wq} & B_4 &= - \left( m + \frac{\rho}{2} A \ell Z'_{qh} \right) & C_3 &= I_{yy} - \frac{\rho}{2} A \ell^3 M'_{q}
 \end{aligned} \tag{4}$$

(Cont.)

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$$\begin{aligned}
 A_7 &= -\frac{\rho}{2} AX'_0 & B_5 &= -\frac{\rho}{2} AlZ'_{qa} & C_5 &= -\frac{\rho}{2} Al^2M'_{q} \\
 A_{10} &= -mx_G & B_8 &= -\frac{\rho}{2} AZ'_w & C_8 &= -\frac{\rho}{2} AlM'_{w_a} \\
 A_{12} &= W-B & B_{10} &= -mz_G & C_9 &= -\frac{\rho}{2} AlM'_{w_h} \\
 A_2 &= A_4 = A_5 = A_8 = 0 & B_{13} &= -(W-B) & C_{11} &= -\frac{\rho}{4} Al^3M'_{qq} \quad (4) \\
 A_9 &= A_{11} = A_{13} = A_{14} = 0 & B_{14} &= -\frac{\rho}{2} AZ'_s & C_{12} &= Wz_G \\
 & & B_1 &= B_6 = B_7 = 0 & C_{13} &= Wx_G \\
 & & B_9 &= B_{11} = B_{12} = 0 & C_{14} &= -\frac{\rho}{2} AlM'_s \\
 & & & & C_7 &= C_{10} = 0
 \end{aligned}$$

Equation (2a) for the propeller thrust  $X_c$  was obtained from momentum theory. Although absolute values of velocity and angular velocity components do not ordinarily appear in motion equations for submerged bodies, they are necessary here because in hovering motions there exist physical discontinuities in the derivatives of the forces and moments with respect to these variables when the variable changes sign.

The hydrodynamic pitch moment arising from the velocity component  $w$  of the submersible is separated into two parts in Eq. (1c). One part, due to the main hull itself, is given by the term

$$-C_9 uw = \frac{\rho}{2} AlM'_{w_h} uw, \quad (5a)$$

and the other part, due to the appendage, is given by

$$-C_8 Uw = \frac{\rho}{2} AlM'_{w_a} Uw. \quad (5b)$$

This separation is dictated by conditions of symmetry and the need to determine the pitch moment for all angles of attack  $\alpha$  of the hovering submersible. Since  $u = U \cos \alpha$  and  $w = U \sin \alpha$ , the use of the products  $uw$  and  $Uw$  in connection with the hull and appendage moments, respectively, is of little consequence in cruising flight where  $\alpha$  is small.

Because the main hull is considered to be symmetric relative to both the  $yz$  and  $xz$  planes, its pitch moment must be zero when  $\alpha = \pm k\pi/2$ ,  $k$  being an integer. The product  $uw = (U^2/2) \sin 2\alpha$  has the required properties of symmetry, and therefore is suitable for estimating the hull moment in hovering flight.

The appendage moment must be zero when  $\alpha = \pm k\pi$ , and the product  $U_w = U^2 \sin \alpha$  has this property of symmetry, so it is suitable for estimating the contribution of the appendage to the pitch moment in hovering flight.

The only experimental results which are available to check the validity of this representation are those obtained at the Davidson Laboratory under sponsorship of the Lockheed Missile and Space Company (3). The hull was elliptical in cross-section, and the hydrodynamic yaw moment coefficient  $N'$  was obtained experimentally as a function of sideslip angle  $\beta$  in the range  $-15^\circ \leq \beta \leq 90^\circ$  for the fully appendaged condition. Analogously to Eqs. (5a,b) the yaw moment  $N$  due to sideslip velocity  $v$  is represented as

$$N = \frac{\rho}{2} A l \left[ N'_{v_h} uv + N'_{v_a} Uv \right]. \quad (6)$$

Setting

$$u = U \cos \beta \quad \text{and} \quad v = -U \sin \beta \quad (7)$$

in Eq. (6) gives

$$N = -\frac{\rho}{2} A l U^2 \left[ \frac{1}{2} N'_{v_h} \sin 2\beta + N'_{v_a} \sin \beta \right], \quad (8)$$

and division by  $(\rho/2) A l U^2$  gives the dimensionless equation

$$N' = - \left[ \frac{1}{2} N'_{v_h} \sin 2\beta + N'_{v_a} \sin \beta \right]. \quad (9)$$

The yaw moment coefficients tabulated in Ref. 3 are based upon  $x^3$ . These are converted here to the  $A l$  basis by multiplying the Ref. 3 coefficients by  $l^2/A$  where  $A = \pi h^2/4$  and  $h$  is the maximum height of the main hull. The  $N'$  data for the full appendage case (3) are curve-fitted to Eq. (9) by a least square analysis and the coefficients are found to be  $N'_{v_h} = -0.72$  and  $N'_{v_a} = +0.36$ . Figure 1 shows the comparison of the curve-fitted theoretical representation

$$N' = 0.36 \sin 2\beta - 0.36 \sin \beta \quad (10)$$

with the experimental data. The standard deviation of the test data from the theoretical curve is 0.01, which is statistically equal to the standard deviation of the test data among themselves. These results indicate that the theoretical representation of the pitch moment in the equations of motion is accurate over the whole range of angle of attack.

The hydrodynamic vertical force arising from  $w$  is not separated because symmetry conditions dictate that both the hull and appendage forces vanish when  $\alpha = \pm k\pi$ . The representation

$$Z = -B_8 w U = \frac{\rho}{2} A U^2 Z'_w \sin \alpha \quad (11)$$

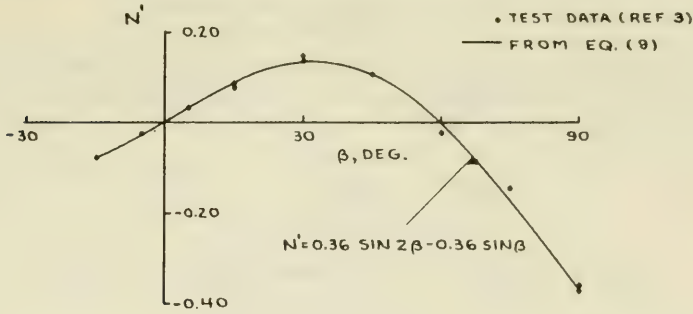


Fig. 1 - Comparison with test results to determine validity of theoretical representation of pitch moment in motion equations

given in Eq. (1b) has this property. The turning test results (3) are used for comparison with the representation given by Eq. (11). The expression for the side force which is analogous to Eq. (11) is

$$Y = -\frac{\rho}{2} AU^2 Y'_v \sin \beta,$$

or in dimensionless form

$$Y' = -Y'_v \sin \beta. \quad (12)$$

As before, the coefficient data (3) are converted to the A basis and least square fitted to Eq. (12), giving the result

$$Y' = 4.53 \sin \beta. \quad (13)$$

Figure 2 shows the comparison of Eq. (13) with the experimental data. Although the standard deviation of the test data from the theoretical curve is greater than the standard deviation of the data among themselves (i.e., 0.25 vs 0.091), part of this discrepancy arises because the tests were made at subcritical Reynolds numbers ( $Re$ ) at the high sideslip angles (i.e.,  $45^\circ \leq \beta \leq 90^\circ$ ). This causes the cross flow drag coefficient (which is equal to  $Y'$  at  $\beta = 90^\circ$ ) to be larger than would be the case if  $Re$  were higher than the critical value. If the data at  $\beta = 90^\circ$  had been obtained at  $Re \approx 1.0 \times 10^6$  rather than at  $Re \approx 4.4 \times 10^5$ , it is expected that much better correlation would have resulted. This conjecture plus the reasonably good representation by Eq. (12) of the  $Y$ -force data in the range  $-15^\circ \leq \beta \leq 90^\circ$  (even in the case of large variations in Reynolds number) accounts for the retention of the simple expression for the  $Z$ -force in the motion equations. Additional test data over the whole range of attack angle and a series of Reynolds numbers are needed for clarification.



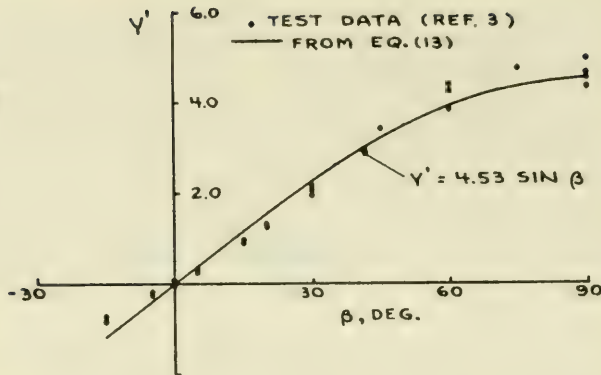


Fig. 2 - Comparison with test results to determine validity of theoretical representation of normal force in motion equations

#### IV. CONDITIONS FOR CRUISING, HOVERING, AND DYNAMIC STABILITY

##### (A) Cruising

When acceptable cruising conditions are possible, the vessel has inherent dynamic stability relative to a level, straight course equilibrium state  $(u_e, \theta_e, \delta_e)$ , where  $0 < u_e < u_m$  and  $|\delta_e| < \delta_m$ ,  $u_m$  and  $\delta_m$  being the maximum attainable speed and elevator angle, respectively. An inherently stable vessel can recover from very high degrees of control saturation, whereas the inherently unstable vessel cannot (4). In a proportional control, the degree of control saturation may be defined as the ratio of the elevator angle  $\delta$  (or elevator angular velocity  $\dot{\delta}$ ) that would be attained if there were no limit, to its limiting value  $\delta_m$  (or  $\dot{\delta}_m$ ).

Using small perturbation theory and Eqs. (1), it can be shown (5) that under constant  $X_c$  and zero  $Z_c$  and  $M_c$  conditions, a submersible is dynamically stable in cruising flight if

$$Z'_w M'_q - (m' + Z'_q) M'_w > 0 \quad (14a)$$

and

$$X'_0 \left[ Z'_w W' (z'_G - x'_G \theta_e) - M'_w (W' - B') \theta_e \right] + (W' - B') (Z'_\delta M'_w - Z'_w M'_\delta) \delta_e > 0, \quad (14b)$$

where

$$W' = \frac{W}{\rho/2} A u_e^2 \quad \text{and} \quad B' = \frac{B}{\rho/2} A u_e^2.$$

Inequality (14a) is the hydrodynamic stability criterion, and (14b) reduces to

$$z_G - x_G \theta_e > 0 \quad (15)$$

in the case where  $W = B$  (neutral buoyancy), the main operating condition of the submersible. The expressions for  $\theta_e$  and  $\delta_e$  are obtained from Eqs. (1) under the conditions that  $w_f = 0$ ,  $\theta_e$  is small,  $W = B$ , and  $M'_\delta = (1/2)Z'_\delta$ . This gives

$$\theta_e = \frac{Wx_G}{\left(M'_w - \frac{1}{2}Z'_w\right) \frac{\rho}{2} A l_{U_e}^2 - Wz_G} \quad (16)$$

and

$$\delta_e = \frac{-Z'_w Wx_G}{\left[\left(M'_w - \frac{1}{2}Z'_w\right) \frac{\rho}{2} A l_{U_e}^2 - Wz_G\right] Z'_\delta} \quad (17)$$

Use of Eq. (16) in Eq. (15) results in

$$z_G - \frac{Wx_G^2}{\left(M'_w - \frac{1}{2}Z'_w\right) \frac{\rho}{2} A l_{U_e}^2 - Wz_G} > 0 \quad (18)$$

as a necessary condition for dynamic stability. For a body with tail fins, the coefficients  $Z'_w$ ,  $Z'_q$ ,  $M'_w$ , and  $M'_q$  can be represented by the relations (Ref. 6)

$$Z'_w = Z'_{w_h} + Z'_{w_a}; \quad M'_w = M'_{w_h} + \frac{1}{2}Z'_{w_a} \quad (19a)$$

$$Z'_q = Z'_{q_h} + \frac{1}{2}Z'_{q_a}; \quad M'_q = M'_{q_h} + \frac{1}{4}Z'_{q_a} \quad (19b)$$

Since the quantity

$$M'_w - \frac{1}{2}Z'_w = M'_{w_h} - \frac{1}{2}Z'_{w_h} > 0, \quad (20)$$

Eq. (18) shows that

$$z_G > 0 \quad (21)$$

is a necessary condition for dynamic stability of the neutrally buoyant submersible.

The other basis for defining acceptable cruising conditions is that  $|\delta_e| < \delta_m$  or, from Eq. (17)

$$\left| \frac{Z'_w Wx_G}{\left[\left(M'_w - \frac{1}{2}Z'_w\right) \frac{\rho}{2} A l_{U_e}^2 - Wz_G\right] Z'_\delta} \right| < \delta_m \quad (22)$$

The criterion given by Eq. (14a) now is applied to the case of a tapered body of revolution with

$$\ell = 43.25 \text{ ft}; \quad A = 49.3 \text{ sq ft}; \quad B = 95,500 \text{ lb}; \quad I_{yy} = 2.94 \times 10^5 \text{ slug-ft}^2. \quad (23a)$$

The bare hull coefficients are

$$Z'_{w_h} = -0.71; \quad M'_{w_h} = 0.98; \quad Z'_{q_h} = -0.14; \quad \text{and} \quad M'_{q_h} = -0.07; \quad (23b)$$

and the mass coefficient  $m'$  for the neutrally buoyant case is 1.40. By substituting Eqs. (19) into (14a) and using the given numerical constants in the resulting expression, it is found that the submersible has neutral hydrodynamic stability when the tail appendage has a normal force rate coefficient  $Z'_{w_a} = -0.86$ . Thus,  $-Z'_{w_a} > 0.86$  defines one of the limit curves for acceptable cruising performance.

Similarly, the criteria given by Eqs. (18) and (22) are applied to the hydrodynamically stable conditions defined by

$$Z'_{w_a} = Z'_\delta = -1.5 \quad \text{and} \quad \delta_m = 0.35 \text{ rad} \quad (24)$$

and Eqs. (23). The cross-hatched areas  $u_e$  vs  $x_G$  with  $z_G$  as parameter shown in Fig. 3 define the region where acceptable cruising performance is possible. The acceptable cruising regions in the  $u_e, x_G$  plane are characterized by two roughly triangular areas with the same vertex on the  $x_G = 0$  axis and a  $u_e$  value there which is always greater than the critical speed\* (but approximately equal to it when  $z_G \geq 0.03$  ft). The upper area is always much larger than the lower, but both angles at the vertex increase with  $z_G$ . The base of the upper area is the maximum speed (assumed here to be  $u_m = 8.0$  ft/sec), and the base of the lower area is  $u_e = 0$ . For the hydrodynamically stable submersible, acceptable cruising conditions are limited by dynamic instability when  $z_G$  is small and by the excessively large elevator angles required for equilibrium when  $z_G$  is large. In either case, it is necessary to keep  $x_G$  small in order to permit cruising at low speeds.

The existence of the cruising regions near  $u_e = x_G = 0$  should *not* be interpreted as being acceptable hovering conditions. The reason for this is that cruising and hovering conditions refer to two completely different equilibrium states. In the cruising condition no attempt is made to keep the submersible motionless relative to the  $x_o, y_o, z_o$  frame (i.e.,  $\dot{x}_{o_e} = \dot{z}_{o_e} = 0$ ), whereas this is precisely the situation in hovering. For example, the case where the submersible drifts with a current  $u_f = 1.69$  ft/sec (1 knot), so that its longitudinal speed  $u_e = 0$  relative to the fluid, is a permissible and acceptable equilibrium cruising condition if  $z_G = 0.05$  ft and  $x_G = 0.005$  ft (see Fig. 3); however, it is not an admissible hovering condition because  $\dot{x}_{o_e} \neq 0$ .

\*The critical speed for a given  $z_G$  is that for which the denominator of Eq. (16) vanishes.

# Cruising and Hovering Response of a Tail-Stabilized Submersible

| A     | $\ell$ | W      | B      | $Z_{w_e}$ | $Z_w$ | $M_w$ | $Z_b$ | $M_b$ | $w_f$  | $\delta_m$ |
|-------|--------|--------|--------|-----------|-------|-------|-------|-------|--------|------------|
| sq FT | FT.    | LB.    | LB.    |           |       |       |       |       | FT/SEC | RAD.       |
| 149.3 | 43.25  | 95,500 | 95,500 | -1.5      | -2.21 | 0.23  | -1.5  | -0.75 | 0      | 0.35       |

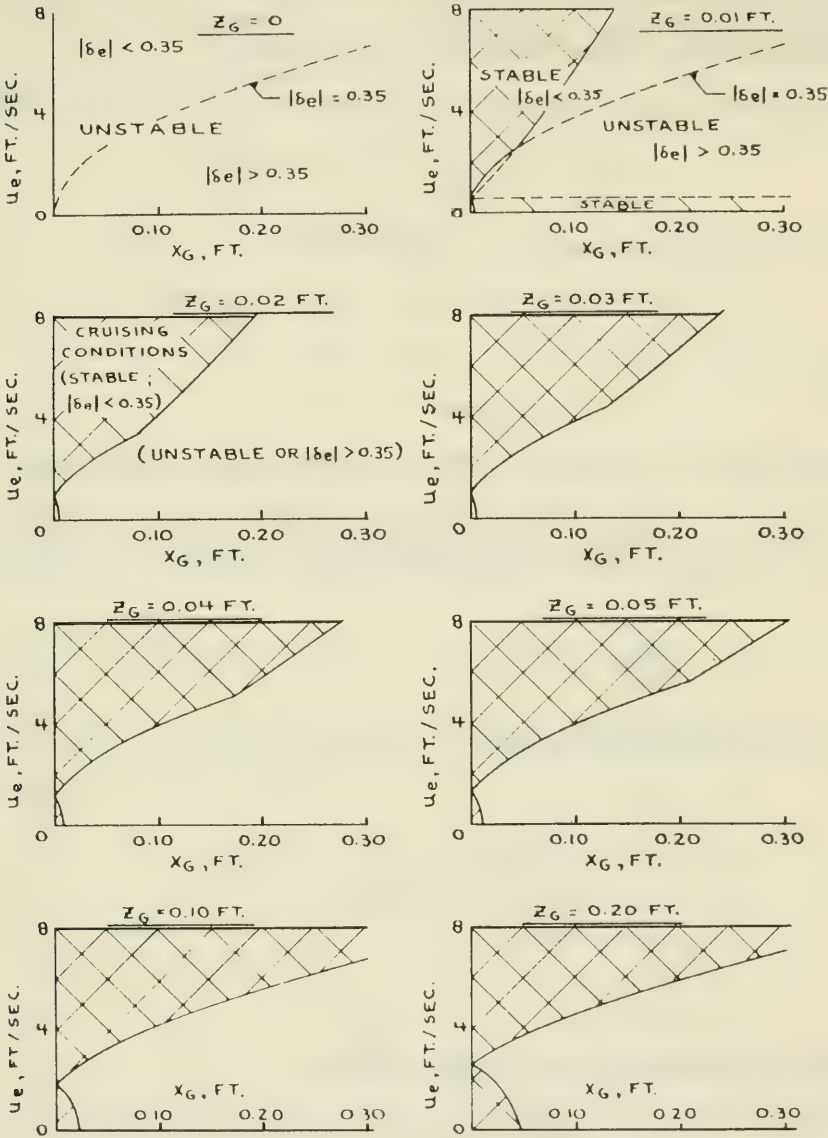


Fig. 3 - Regions of speed and CG position for stable cruising operations



## (B) Hovering

When acceptable hovering conditions are possible, the vessel has inherent dynamic stability relative to an equilibrium state with  $\dot{x}_{0e} = \dot{z}_{0e} = 0$ ,  $\theta = \theta_e$ , and where  $|u_e| < u_m$ ,  $|Z_{se}| < Z_m$  and  $|Z_{be}| < Z_m$ ,  $Z_m$  being the maximum vertical force obtainable with either the bow or stern thruster.

The inherent dynamic stability of the submersible in the hovering equilibrium state

$$\dot{x}_{0e} = \dot{z}_{0e} = \theta_e = \dot{\theta}_e = w_e = 0 \quad \text{and} \quad u_e = -u_f \quad (25)$$

has been investigated (1) under the conditions that

$$\delta = 0, \quad Z_s = Z_{se}, \quad Z_b = Z_{be}, \quad n_x = n_{xe}, \quad \text{and} \quad W - B = 0. \quad (26)$$

Letting

$$u = u_e + \bar{u}, \quad w = \bar{w}, \quad \theta = \bar{\theta}, \quad \text{etc.}, \quad (27)$$

and substituting Eqs. (25), (26), and (27) into the equations of motion give

$$n_{xe} = \left\{ \frac{-b_x |u_e| + [(b_x^2 + 4a_x A_7) u_e^2]^{1/2}}{2a_x} \right\} (\text{Sign } u_e) \quad (a) \quad (28)$$

$$Z_{se} = \frac{C_{13}}{2x_b} = -Z_{be} \quad (b)$$

for the equilibrium equations, and

$$A_1 \ddot{\bar{u}} + A_3 \ddot{\bar{\theta}} + A_{16} \bar{u} = 0 \quad (a)$$

$$B_2 \ddot{\bar{w}} + B_3 \ddot{\bar{\theta}} + B_{15} \dot{\bar{\theta}} + B_{17} \bar{w} = 0 \quad (b)$$

$$C_1 \ddot{\bar{u}} + C_2 \ddot{\bar{w}} + C_3 \ddot{\bar{\theta}} + C_{15} \dot{\bar{\theta}} + C_{12} \bar{\theta} + C_{17} \bar{w} = 0 \quad (c) \quad (29)$$

$$\dot{\bar{x}}_0 = \bar{u} \quad (d)$$

$$\dot{\bar{z}}_0 = -u_e \bar{\theta} + \bar{w} \quad (e)$$

for the linearized perturbation equations, where

$$\begin{aligned} A_{16} &= 2|u_e| - b_x |n_{xe}| \\ B_{15} &= B_4 u_e + B_5 |u_e|; \quad B_{17} = B_8 |u_e| \\ C_{15} &= C_4 u_e + C_5 |u_e|; \quad C_{17} = C_8 |u_e| + C_9 u_e, \end{aligned} \quad (30)$$

and the other constants are defined in Eq. (4).

The solution of Eq. (29) is of the form

$$\bar{u} = \sum u_i e^{\sigma_i t}, \quad \bar{w} = \sum w_i e^{\sigma_i t}, \quad \theta = \sum \theta_i e^{\sigma_i t} \text{ with } \sigma_i \neq \sigma_j, \quad (31)$$

and the  $u_i, w_i, \theta_i$  are constants depending upon the initial conditions. This system yields an indicial equation of the form

$$d_0 \sigma_i^4 + d_1 \sigma_i^3 + d_2 \sigma_i^2 + d_3 \sigma_i + d_4 = 0, \quad (32)$$

where

$$\begin{aligned} d_0 &= A_1(B_2C_3 - B_3C_2) - C_1A_3B_2 \\ d_1 &= A_1(B_2C_{15} - B_{15}C_2 - B_3C_{17} + B_{17}C_3) - C_1A_3B_{17} - A_{16}(B_3C_2 - B_2C_3) \\ d_2 &= A_{16}(B_{17}C_3 - B_3C_{17} - B_{15}C_2 + B_2C_{15}) - A_1(B_{15}C_{17} - B_{17}C_{15} - B_2C_{12}) \\ d_3 &= A_{16}(B_2C_{12} - B_{15}C_{17} + B_{17}C_{15}) + B_{17}A_1C_{12} \\ d_4 &= A_{16}B_{17}C_{12} \end{aligned} \quad (33)$$

The submersible is dynamically stable under the prescribed hovering conditions if the real parts of all four  $\sigma_i$  are negative. The Routh stability criteria give the necessary and sufficient conditions for this to be true. These are that

$$(a) \quad d_1 > 0 \quad \text{and} \quad (b) \quad d_4 < \frac{d_3}{d_1} \left( d_2 - \frac{d_3 d_0}{d_1} \right). \quad (34)$$

The other bases for defining acceptable hovering conditions are that  $|u_e| < u_m$  (or  $|n_{x_e}| < n_{x_m}$ ), and that each equilibrium vertical thrust force be less in magnitude than  $Z_m$ , or from Eq. (28)

$$\left\{ \frac{-b_x |u_e| + [(b_x^2 + 4a_x A_7) u_e^2]^{1/2}}{2a_x} \right\} < n_{x_m} \quad (a)$$

and (35)

$$\frac{W |x_G|}{2x_b} < Z_m. \quad (b)$$

These hovering criteria now are applied to the submersible with the constants given by Eq. (23) and

$$Z_m = 300 \text{ lb}; \quad x_b = 16.0 \text{ ft}; \quad a_x = 2.46 \text{ slug-ft}; \quad b_x = -6.40 \text{ slugs}. \quad (36)$$

The hydrodynamic data include

$$\begin{aligned}
 Z'_{w_a} &= -1.5; & Z'_w &= -2.21; & Z'_{q_a} &= M'_{w_a} = -0.75; & M'_q &= -0.38, & (a) \\
 X'_u &= -0.07; & X'_{wq} &= Z'_w = -1.273; & M'_q &= -0.054, & X'_o &= -0.15. & (b)
 \end{aligned}
 \tag{37}$$

The values of  $Z_m$  and  $x_b$  were given by SP,  $a_x$  and  $b_x$  were determined using propeller design charts (Ref. 7) for a three-bladed propeller with diameter of 4.0 ft and pitch-to-diameter ratio of 0.50, the coefficients of Eq. (37) were obtained using Eq. (19) and the previously given data (23b), the "added mass and inertia" coefficients were estimated on the basis of Lamb's "accession to inertia" coefficients for prolate spheroids, and the estimate of  $X'_o$  is an average value based on the fact that the submersible will operate over a range of speed from 0 to approximately 5 knots. The Schoenherr friction drag coefficients as well as estimates of form drag were made for the main hull, tail appendages, rescue skirt, thruster ducts and gear, and other exposed equipment to obtain the given value of  $X'_o$ . However, a rough estimate of  $X'_o$  is justified because it has been found (1) that reasonably large changes in its value have little influence on the measures of performance treated herein.

The cross-hatched area of Fig. 4 shows the region in the  $u_e, z_G$  plane of acceptable hovering performance defined by the criteria of stability, maximum propeller speed, and maximum vertical thrust forces. The region is closed at the top by the criterion  $u_e < 6.95$  ft/sec which was obtained by arbitrarily assuming a maximum propeller speed  $n_{x_m}$  of 20 rad/sec in Eq. (35a). The notation in Fig. 4 that  $|x_G| < 0.10$  ft for  $|z_{s_e}|$  and  $|z_{b_e}| < 300$  lb is the result of using the given data in Eq. (35b).

The other hovering limit curves in Fig. 4 are the result of applying the stability criteria. It is found that  $d_o$  and  $d_1$  are positive for all  $u_e$ , but  $d_4 > 0$  only if  $z_G > 0$ . This latter condition accounts for the  $z_G = 0$  limit line in Fig. 4. When  $u_e > 0$ ,  $z_G > 0$ , and  $|x_G| < 0.10$  ft,  $d_2$  and  $d_3$  are positive and condition (34b) is satisfied if

$$\bar{Z}'_w (M'_q - m'x'_G) - (m' + Z'_q) M'_w > 0, \quad \text{for } u_e > 0 \tag{38a}$$

or equivalently, if

$$\bar{Z}'_w (M'_q - m'x'_G) - [m' + (Z'_{q_h} + Z'_{q_a})] (M'_{w_h} + M'_{w_a}) > 0, \quad \text{for } u_e > 0, \tag{38b}$$

where the sign of each quantity is shown above its symbol. For the hydrodynamically stable submersible considered, the first product of Eq. (38) is positive (Stabilizing) and, although the second product gives a negative contribution (destabilizing), its effect is diminished by the stabilizing influence of the tail appendage contributions  $Z'_{q_a}$  and  $M'_{w_a}$ . Hence, Eq. (38) is satisfied, and the neutrally buoyant, tail stabilized submersible with a small margin of hydrodynamic stability can hover acceptably in large negative (bow-to-stern) ocean currents if  $z_G > 0$  and  $|x_G|$  is smaller than 0.10 ft.

# Cruising and Hovering Response of a Tail-Stabilized Submersible

| W<br>LB. | B<br>LB. | Z <sub>m</sub><br>LB. | x <sub>b</sub><br>FT. | Z <sub>w<sub>a</sub></sub><br>FT. | Z <sub>w</sub><br>FT. | Z <sub>h</sub><br>FT. | Z <sub>a</sub><br>FT. | M <sub>w<sub>h</sub></sub><br>FT/SEC | M <sub>w<sub>a</sub></sub><br>FT/SEC | M <sub>a</sub><br>FT/SEC | X <sub>oe</sub><br>FT/SEC | Z <sub>oe</sub><br>FT/SEC | Θ <sub>e</sub><br>DEG | n <sub>xm</sub><br>RAD/SEC |
|----------|----------|-----------------------|-----------------------|-----------------------------------|-----------------------|-----------------------|-----------------------|--------------------------------------|--------------------------------------|--------------------------|---------------------------|---------------------------|-----------------------|----------------------------|
| 95,500   | 95,500   | 300                   | 16.0                  | -1.5                              | -2.21                 | -0.14                 | -0.15                 | 0.98                                 | -0.75                                | -0.38                    | 0                         | 0                         | 0                     | 2.0                        |

$$|X_G| < 0.10 \text{ FT. FOR } |Z_{se}| \text{ AND } |Z_{be}| < 300^*$$

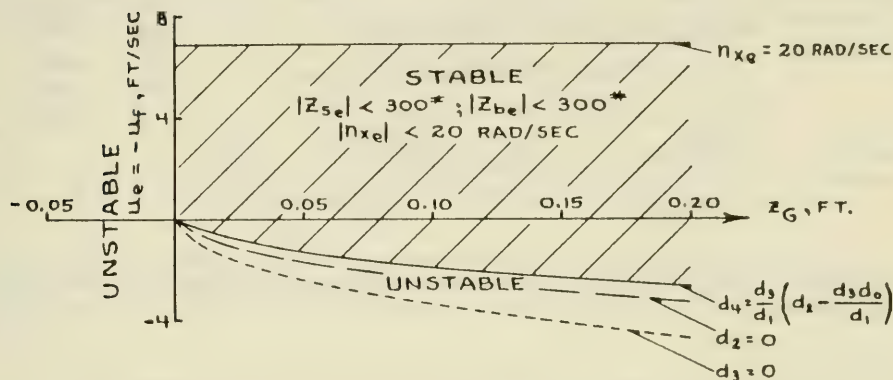


Fig. 4 - Regions of current velocity and CG position for stable hovering operations of a tail-fin-stabilized submersible

The sufficient condition for stability analogous to Eq. (38) for positive currents (or  $u_e < 0$ ) is that

$$Z'_w(M'_q + m'x'_G) - [m' + (Z'_{qh} - Z'_{qa})] \cdot (M'_{wh} - M'_{wa}) > 0 \quad \text{for } u_e < 0. \quad (39)$$

This inequality cannot be satisfied by the tail-stabilized submersible because the tail appendage contribution ( $Z'_{qa}$  and  $M'_{wa}$ ) to the second product in Eq. (39) causes a destabilizing effect which is greater than that introduced by the body of revolution itself. This places a severe restriction on the hovering capabilities of the tail stabilized vessel in positive (stern-to-bow) ocean currents and accounts for the lower limit of stability curve (marked  $d_4$ ) in Fig. 4. This lower limit curve is, to a first-order approximation, defined by

$$u_e > -6 \sqrt{z_G} \quad (z_G > 0), \quad (40)$$

where  $z_G$  is measured in feet and  $u_e$  in feet-per-second. Hence, it is the action of the metacentric pitch moment  $-Wz_G(\sin \theta)$  in the vertical plane motions of the vessel which accounts for presence of a small stable hovering region when  $u_e < 0$ . In pure horizontal plane motions where there is no stabilizing metacentric yaw moment present, the tail stabilized submersible is inherently unstable in hovering equilibrium conditions with  $u_e < 0$ . It also is noted that in the numerical applications of Eq. (34), the terms containing  $x_G$  are found to have



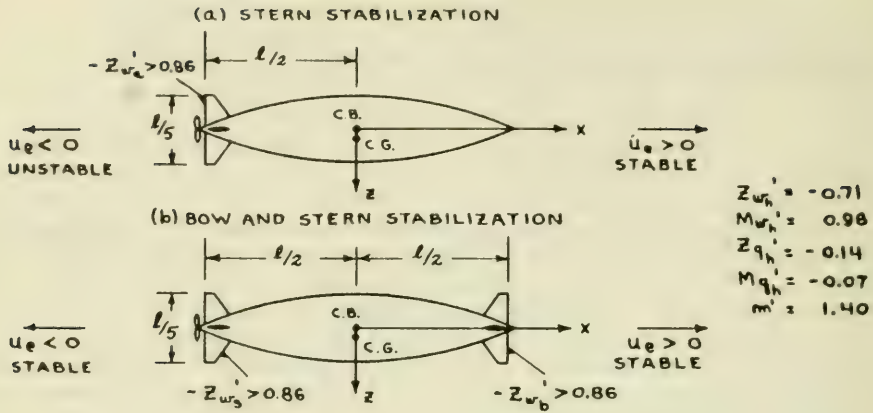


Fig. 5 - Comparison of hydrodynamic stability of vessel with (a) stern fins only and (b) bow and stern fins

a very small contribution when  $|x_G| < 0.10$  ft. Hence, the only restriction on  $x_G$  shown in Fig. 4 is that given by Eq. (35b).

One way in which the hovering stability problem can be eliminated is depicted in Fig. 5. Bow fins are added to the stern-finned submersible to obtain a vessel with fore-and-aft symmetry which is hydrodynamically stable in both directions of motion. To demonstrate this, representations of the hydrodynamic rate coefficients analogous to Eq. (19) first are written for the bow and stern fin case, i.e.,

$$\begin{aligned} Z'_w &= Z'_{w_h} + Z'_{w_b} + Z'_{w_s} & M'_w &= M'_{w_h} - \frac{1}{2} (Z'_{w_b} - Z'_{w_s}) \\ Z'_q &= Z'_{q_h} - \frac{1}{2} (Z'_{w_b} - Z'_{w_s}) & M'_q &= M'_{q_h} + \frac{1}{4} (Z'_{w_b} + Z'_{w_s}), \end{aligned} \quad (41)$$

when the subscripts b and s represent bow and stern fin contributions, respectively. Now assume that both fins are equivalent hydrodynamically, i.e., that

$$Z'_{w_b} = Z'_{w_s} = Z'_{w_1} \quad (42)$$

and use this in Eq. (41) to get

$$\begin{aligned} Z'_w &= Z'_{w_h} + 2Z'_{w_1} & M'_w &= M'_{w_h} \\ Z'_q &= Z'_{q_h} & M'_q &= M'_{q_h} + \frac{1}{2} Z'_{w_1}, \end{aligned} \quad (43)$$

which means that the appendages have the following net effects on the rate coefficients:

$$Z'_{w_a} = 2Z'_{w_1}; \quad Z'_{q_a} = M'_{w_a} = 0; \quad \text{and} \quad M'_{q_a} = \frac{1}{2} Z'_{w_1} \quad (44)$$

Use of these relations in Eqs. (38) and (39) with  $x_G = 0$  shows that the stability criteria for  $u_e > 0$  and  $u_e < 0$  reduce identically to the same one, namely

$$Z'_w M'_q - (m' + Z'_{q_h}) M'_{w_h} > 0, \quad (45)$$

or

$$Z'^2_{w_1} + \left( \frac{1}{2} Z'_{w_h} + 2M'_{q_h} \right) Z'_{w_1} + [Z'_w M'_q - (m' + Z'_{q_h})] M'_{w_h} > 0. \quad (46)$$

Substitution of the bare hull hydrodynamic coefficients given by Eq. (23b) and the mass coefficient  $m' = 1.40$  into Eq. (46) and solving for  $Z'_{w_1}$  shows that if

$$-Z'_{w_1} > 0.86$$

the bow- and stern-finned vessel will be hydrodynamically stable in both positive and negative ocean currents. It is noted that the value of  $Z'_{w_a}$  for neutral hydrodynamic stability of the submersible with only a stern appendage also is -0.86. This type of result occurs generally because a stern appendage alone, regardless of size, cannot stabilize the vessel when  $u_e$  is negative, and the bow appendage alone, regardless of size, cannot stabilize it when  $u_e$  is positive (6).

## V. LIMIT MANEUVER RESPONSE CHARACTERISTICS

The adequacy of stern stabilization alone in a submersible which must both cruise and hover in ocean currents is investigated further by resort to limit maneuver response calculations for the tail-stabilized submersible. Although Eqs. (1) through (4) were used in digital machine computations of one cruising maneuver and three hovering maneuvers (1), for brevity only one type of the latter class is considered herein. Similarly, the parametric studies covered in Ref. 1 were extensive, but the present discussions deal only with those aspects relating to the use of tail stabilization.

### (A) Cruising Limit Maneuvers

Figure 6 is an example of computed response in a cruising pitch overshoot maneuver (Run 102). The vessel is initially in straight, level, equilibrium flight with  $u_e = 5.07$  ft/sec. The water current velocity components  $u_f = w_f = 0$ ,  $Z'_\delta = 2M'_\delta = -0.50$ ,  $Z'_{w_a} = -1.5$ , and  $x_G = z_G = 0.01$  ft. The other data (where applicable) are the same as given previously in Eqs. (23), (36), and (37). The elevator then is deflected at the maximum rate  $\dot{\delta}_m = 0.20$  rad/sec until it reaches the maximum angle  $\delta_m = 0.35$  rad and held there until the time  $t_1$  when the

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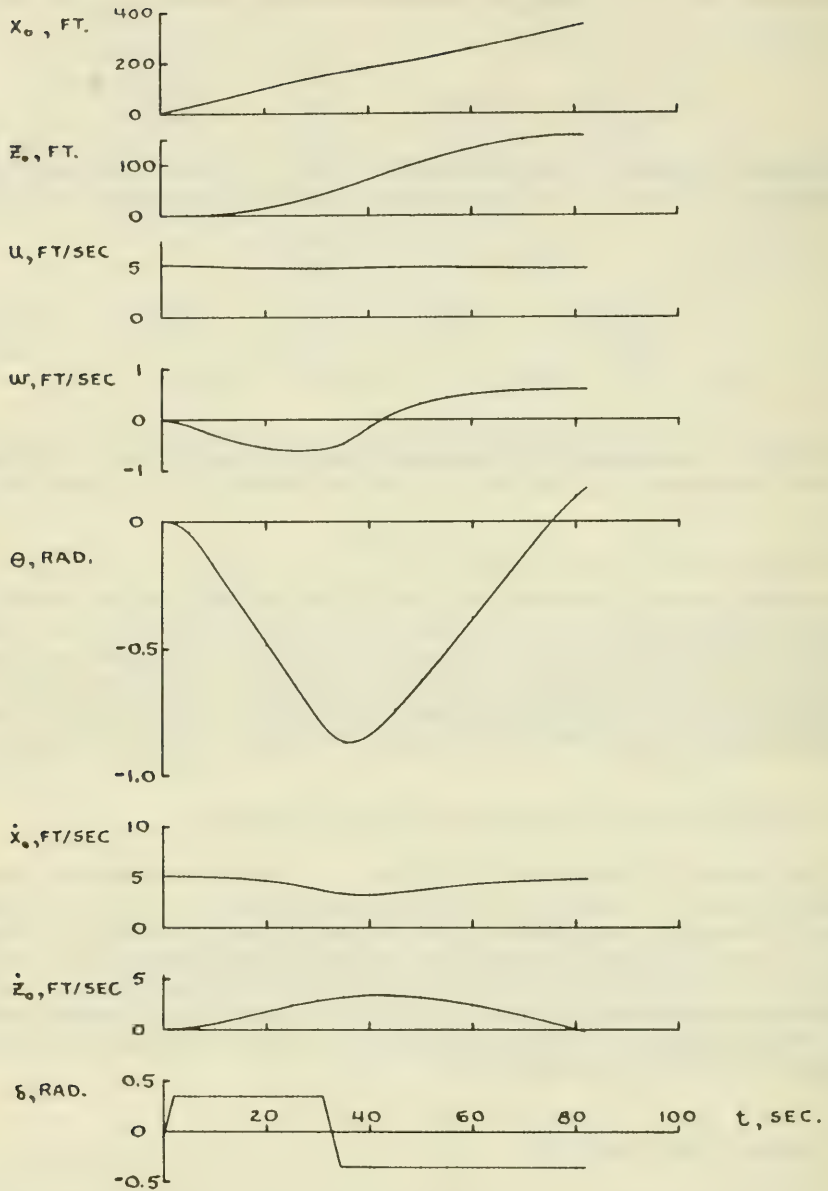


Fig. 6 - Example of computed response in cruising pitch overshoot maneuver, Run No. 102

vessel reaches the prescribed execute pitch angle  $\theta_1 = -0.80$  radian. At time  $t_1$ , the elevator is deflected at the rate  $-\dot{\delta}_m$  until it reaches the angle  $-\delta_m$  and held there until the time  $t_2$  when the depth velocity  $\dot{z}_o$  is zero. The run is stopped when  $t = t_2$ . The run also is stopped at any time when  $|\theta| > 1.57$  rad or  $t > 100$  sec. Some of the measures of performance are  $\theta_e$ ,  $\delta_e$ ,  $t_1$ ,  $z_{o1}$ ,  $z_{om}$ ,  $\theta_m$ , and  $t_2$ .

Figure 7 shows how the maximum depth  $z_{om}$  and the time  $t_2$  vary with  $Z'_{wa}$  for the previously given values of the other parameters. There exists a value  $Z'_{wa} \approx -1.3$  for minimum  $z_{om}$  and nearly minimum  $t_2$ . The submersible with  $0.86 < |Z'_{wa}| < 1.3$  may also respond rapidly but because of its marginal stability, requires larger depths than in the optimum case to complete the maneuver. A submersible with  $|Z'_{wa}| < 0.86$  cannot complete the maneuver because the pitch angle  $\theta$  exceeds the limit of 1.57 radians. For example, when  $Z'_{wa} = -0.70$ ,  $\theta$  exceeds 1.57 radians after only 26.0 seconds and while the depth velocity is still increasing. This confirms the result of the previous stability analysis. Lastly, when  $|Z'_{wa}| > 1.3$ , the vessel has an excessive amount of hydrodynamic stability and requires greater depth and longer time to complete the maneuver than in the optimum case. Since an automatic control with elevator deflected in proportion to an error in  $\theta$  would (in effect) add hydrodynamic stability to the system, it would be preferable to choose a tail fin with  $0.86 < |Z'_{wa}| < 1.3$  than one with  $|Z'_{wa}| > 1.3$ .

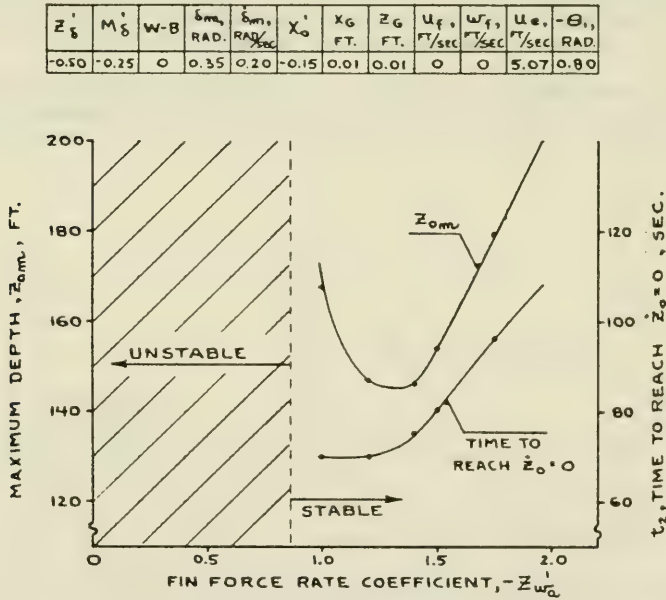


Fig. 7 - Effect of  $Z'_{wa}$  on maximum depth and time to complete cruising speed limit maneuver



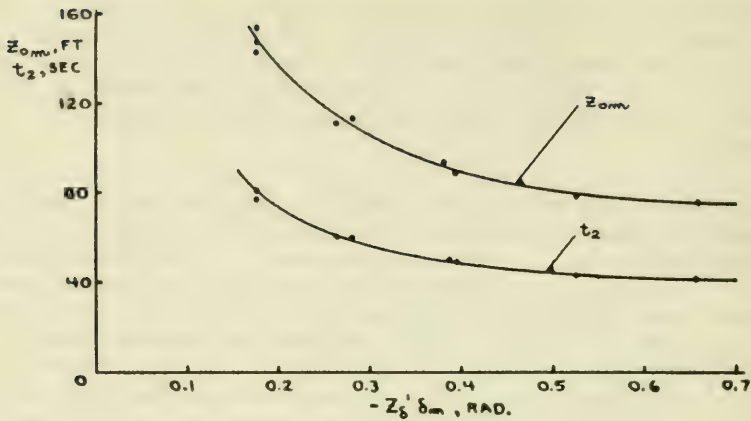


Fig. 8 - Effect of the product  $Z'_\delta \delta_m$  on cruising pitch overshoot maneuver for otherwise standard run conditions

Figure 8 shows the effect of maximum elevator force coefficient  $|Z'_\delta| \cdot \delta_m$  on cruising maneuver response. The standard condition with  $Z'_{w_a} = -1.5$  was used to obtain the fairly good collapse of the  $z_{om}$  and  $t_2$  data. These results show that both the depth and time required to complete the maneuver decrease as the maximum elevator force coefficient increases.

Figure 9 shows how  $t_1$ ,  $t_2$ , and  $z_{om}$  vary with metacentric height  $z_G$  for conditions with  $\delta_m = 0.175$  rad, and  $Z'_\delta = Z'_{w_a} = -1.5$ . The total depth  $z_{om}$  and

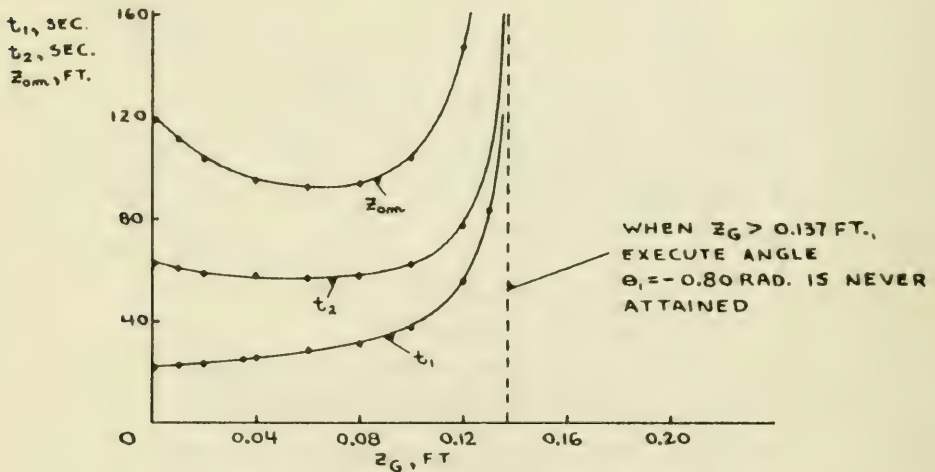


Fig. 9 - Effect of vertical CG position on cruising pitch maneuver response for otherwise standard run conditions

time  $t_2$  are minimized when  $z_G \approx 0.065$  ft. The submersible with  $0 < z_G < 0.065$  ft also responds rapidly but its marginal stability causes larger depth excursions than in the optimum case. All three measures of performance increase from the values at  $z_G = 0.065$  ft to infinite values at  $z_G = 0.137$  ft. It can be shown analytically (1) that the submersible with  $Z'_\delta = -1.50$  and  $\delta$  fixed at  $\delta_m = 0.175$  radians attains a steady, downward, straight course condition with  $|\theta| < 0.80$  radians for  $z_G \geq 0.137$  ft. It is for this reason that the execute angle of  $\theta_1 = -0.80$  radian is never attained in the cruising maneuver computations when  $z_G > 0.137$  ft. Hence  $z_G = 0.137$  ft is a limiting value for the standard maneuver conditions.

However, values of  $z_G$  which are excessive for cruising operations might be desirable for hovering maneuvers. Therefore, the variation of the limiting values of  $z_G$  with both  $\theta_1$  and  $Z'_\delta \delta$  was investigated (1) using the otherwise standard conditions given previously. Figure 10 shows this variation with  $\theta_1$ , and indicates that the limiting value of  $z_G$  decreases with increasing  $|\theta_1|$ , approximately like  $-0.1/\sin \theta_1$ . If a value of  $z_G < 0.09$  ft is chosen for cruising operations, all pitch angles will be attainable for the submersible with standard conditions.

Figure 11 shows how this limiting value of  $z_G$  can be increased by increasing the maximum elevator force coefficient  $|Z'_\delta| \cdot \delta_m$ . It is a plot of the limiting value of  $z_G$  vs  $|Z'_\delta| \cdot \delta_m$  for  $\theta_1 = -0.80$  radian and otherwise standard conditions. The limit value of  $z_G$  increases linearly with  $|Z'_\delta| \cdot \delta_m$ , and has a value of about 0.30 ft when  $|Z'_\delta \delta_m| = 0.60$ . If  $|Z'_\delta \delta_m| = 0.60$ , it can be shown (1) that all values of  $\theta_1$  are attainable when  $z_G < 0.209$  ft.

Therefore, it has been shown that for a given value of  $Z'_\delta \cdot \delta_m$ , there exists a tail appendage force rate coefficient  $Z'_{wa}$  and a metacentric height  $z_G$  which yields optimum cruising overshoot maneuver response. However, the optimum

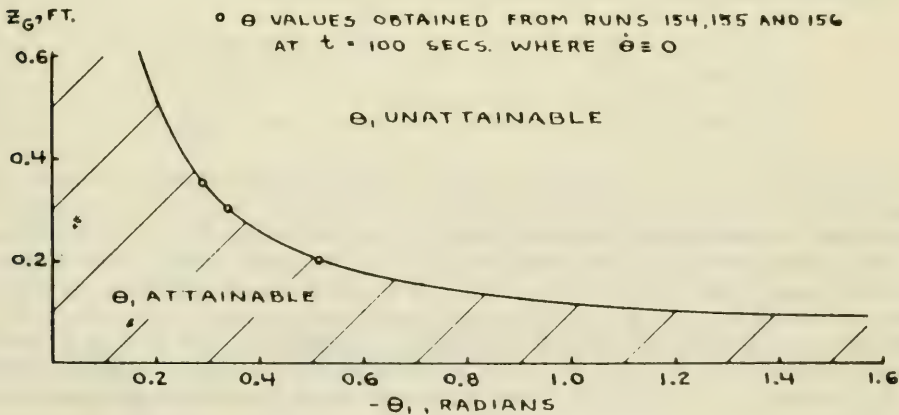


Fig. 10 - Limit values of vertical CG position above which execute pitch angle cannot be attained for otherwise standard cruising pitch maneuver conditions

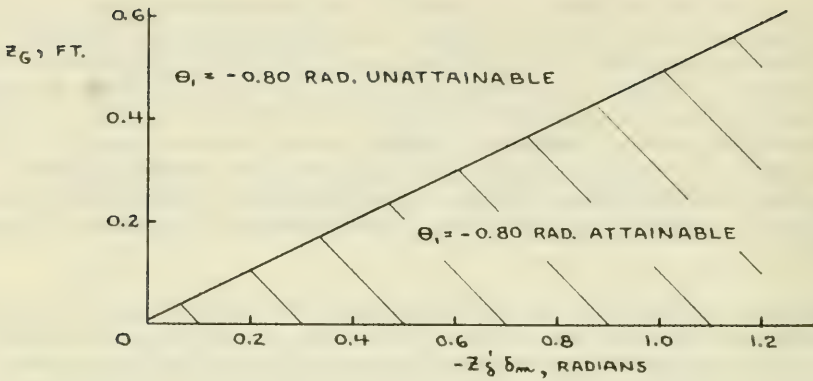


Fig. 11 - Limit values of  $z_G$  for  $\theta_1 = -0.8$  radian as a function of maximum elevator force coefficient for otherwise standard cruising pitch maneuver conditions

value of  $z_G$  for cruising is small, and it may be desirable to increase  $z_G$  for hovering operations. This can be done, without restricting the attainable pitch angles of the submersible in cruising, by increasing the value of the maximum elevator force coefficient  $|Z'_8| \cdot \delta_m$ .

### (B) Hovering Limit Maneuvers

Figure 12 is an example of computed response in a hovering heave overshoot maneuver (Run 237). The tail-stabilized submersible is well behaved in this example because it is operating in a negative (bow-to-stern) current  $u_f = -1.69$  ft/sec under otherwise standard run conditions of

$$Z'_{w_a} = -1.5, \quad \delta = 0, \quad M'_{qq} = -0.42,^* \quad Z_m = 300, \quad \dot{Z}_m = 100 \text{ lb/sec},$$

$$W = B = 95,500 \text{ lb}, \quad x_G = z_G = 0.01 \text{ ft}, \quad w_f = 0, \quad \text{and } z_{o_1} = 20 \text{ ft.} \quad (47)$$

With the vessel initially motionless with zero pitch angle relative to the  $x_o, y_o, z_o$  inertial frame, the bow and stern vertical thrust forces  $Z_b$  and  $Z_s$  are increased from their respective equilibrium values  $Z_{b_e} = -29.84$  lb and  $Z_{s_e} = 29.84$  lb at the maximum rate  $\dot{Z}_m = 100$  lb/sec until the stern vertical thrust force reaches the maximum value  $Z_m = 300$  lb and the bow vertical thrust force reaches the value of 240.32 lb. These forces are maintained until time  $t_1$  when the prescribed execute depth  $z_{o_1} = 20$  ft is reached. At time  $t_1$ , the  $Z_b$  and  $Z_s$  forces begin to reverse at the rate  $-\dot{Z}_m$  until  $Z_b$  reaches the maximum negative value of -300 lb and  $Z_s = -240.32$  lb. These vertical thrust forces are maintained until

\* $M'_{qq}$  was determined analytically (1) using strip theory. The resulting expression is  $M'_{qq} = (1/16) Z'_{w_h} + (1/4) Z'_{w_a}$ .

Cruising and Hovering Response of a Tail-Stabilized Submersible

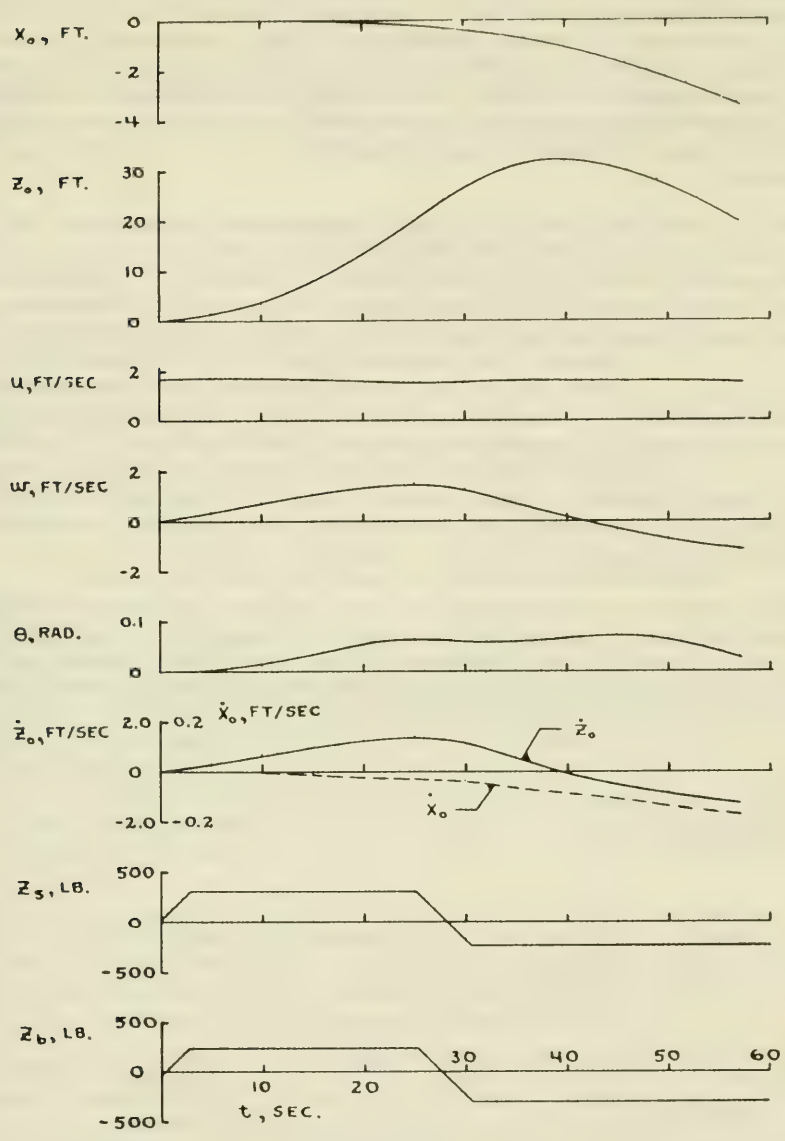


Fig. 12 - Example of computed response in hovering heave overshoot maneuver, Run No. 237



the time  $t_2$  when the depth  $z_{o1}$  of 20 ft again is attained, and the run is stopped. A run also is stopped if  $|z_o| > 100$  ft,  $|\theta| > 1.57$  rad, or  $t > 200$  sec. In a completed hovering heave maneuver, there is a time in the interval  $(t_1, t_2)$  where  $\dot{z}_o = 0$ . It is noted that the control moment  $M_c$  and the longitudinal propeller speed  $n_x$  have the equilibrium values  $M_{c_e} = x_b(Z_{s_e} - Z_{b_e})$  and  $n_{x_e}$  for all time  $(0, t_2)$  in a hovering heave maneuver. Hence, any net unbalanced pitch moments or longitudinal forces are due entirely to transient inertial and hydrodynamic effects. Some of the measures of performance are  $Z_{b_e}$ ,  $Z_{s_e}$ ,  $t_1$ ,  $\theta_1$ ,  $x_{o1}$ ,  $\theta_m$ ,  $x_{om}$ ,  $z_{om}$ , and  $t_2$ .

Figure 13 shows the effect of  $Z'_{wa}$  on six measures of performance in the hovering maneuver for otherwise standard conditions with  $u_e = 0$  (i.e.,  $u_f = 0$ ). Although the vessel is marginally stable for all  $Z'_{wa}$  when  $u_e = 0$ , the asymmetry of the submersible due to the presence of the tail appendage is manifested in the increasingly large heave-to-surge and heave-to-pitch coupling effects that are obtained as  $|Z'_{wa}|$  is increased. For example, when  $Z'_{wa} = -1.5$ , the vessel attains a pitch angle of almost -33 deg and moves forward a distance of 80 ft before the heave maneuver is completed. This occurs without applying any unbalanced pitch moments  $M_c$  or longitudinal forces  $X_c$  to the system. The source of this behavior is the nosedown hydrodynamic pitch moment due to the tail appendage  $(-C_{8w}U)$  which acts when a pure downward force  $Z_c$  is applied to the vessel. The tail-stabilized submersible also is characterized by its slow response and large depth overshoot in hovering heave maneuvers with  $u_e = 0$ .

Figure 14 demonstrates the strongly adverse effects that positive longitudinal current velocities  $u_f$  have on the hovering heave maneuver response of the tail-stabilized submersible for otherwise standard conditions. These results confirm the trends predicted by the analysis of dynamic stability. For a positive

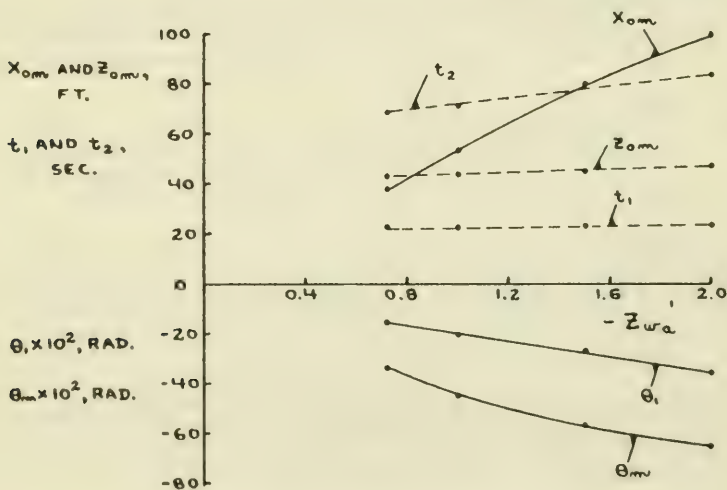


Fig. 13 - Effect of tail fin force rate coefficient on hovering heave overshoot maneuver for otherwise standard run conditions

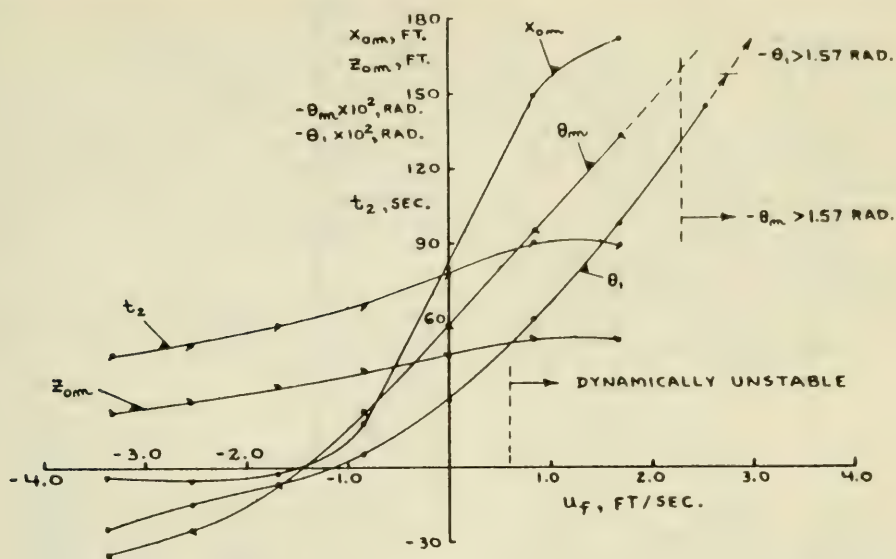


Fig. 14 - Effect of longitudinal current velocity on hovering heave maneuver for otherwise standard run conditions

current velocity of 1 knot, the vessel attains a pitch angle  $\theta$  and longitudinal displacement of almost 80 degrees and 4 hull lengths, respectively, before the maneuver is completed. At slightly higher positive currents than 1 knot the maneuver cannot be completed at all because the pitch angle exceeds the limit value. On the other hand, for a negative current velocity of 1 knot, both the response time  $t_2$  and the heave overshoot are significantly reduced and the coupling effects are negligible in comparison with those obtained for the positive 1 knot current.

It is noted that attempts to improve hovering response in positive longitudinal currents by large reductions in tail appendage size were unsuccessful. These reductions did not improve heave response significantly for  $u_f > 0$ , and also had large detrimental effects on performance for  $u_f < 0$ .

Because the pitch and surge motions of the tail-stabilized submersible are so sensitive to the action of a pure vertical force  $Z_c$ , some amelioration of these coupled motions can be obtained by reducing the magnitude of  $Z_m$ . Figure 15 shows this effect for otherwise standard conditions with  $u_e = 0$ . Decreasing  $Z_m$  also reduces the heave overshoot and causes large increases in response time. Therefore, if the slower response can be accepted, small vertical control forces (50 or 100 lb) could be used to move the vessel vertically. This will permit the hydrodynamic pitch moments arising from the vertical motion to be counteracted by a pure control moment  $M_c$ .

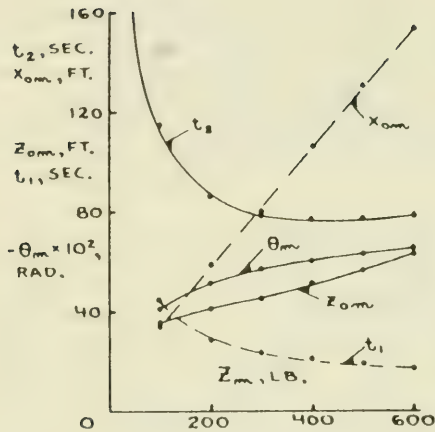


Fig. 15 - Effect of maximum vertical MPD force on hovering heave maneuver for otherwise standard run conditions

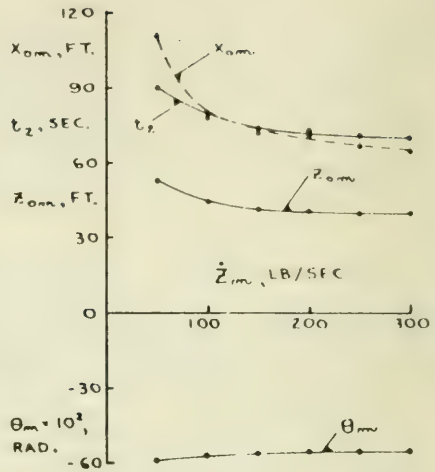


Fig. 16 - Effect of maximum MPD vertical force rate on hovering heave maneuver for otherwise standard run conditions

It is noteworthy that reducing the rate of application  $\dot{z}_m$  of the vertical thrust force does not have the same mitigating effects on hovering heave maneuver response as does reducing  $z_m$ . Figure 16 shows that reducing  $\dot{z}_m$  increases the coupling effects as well as the time of response and depth overshoot. However, any improvement in performance to be gained by increasing  $\dot{z}_m$  will be small for  $\dot{z}_m > 100$  lb/sec.

The only other method found to improve the vertical plane, hovering limit maneuver response of the tail-stabilized submersible is to increase the metacentric height  $z_G$ . Figure 17 shows this for otherwise standard conditions with  $u_e = 0$ . Increasing  $z_G$  from 0.01 to 0.10 ft decreases  $x_{om}$  from 79.5 to 21.5 ft,  $\theta_m$  from -0.57 to -0.33 rad,  $t_2$  from 78 to 61 sec, and  $z_{om}$  from 44.9 to 38.2 ft. Further increase in  $z_G$  has little effect on response time and depth overshoot, but continues to reduce the pitch and surge coupling effects.

## VI. CONCLUDING REMARKS

Analyses of inherent dynamic stability and limit maneuver response have shown that the tail-stabilized submersible can perform adequately in cruising flight, but is subject to highly coupled, unstable hovering motions, especially in stern-to-bow ocean currents. Two methods have been suggested for improving its performance, but these are restricted in applicability. The first is to reduce pitch and surge coupling effects by limiting vertical velocity through application of only small unbalanced vertical control forces and the use of corrective pitch control moments. This method will be difficult to apply in the vertical plane



because of the extreme sensitivity of the hovering control and motions of the vehicle to small positive or negative buoyancy forces and vertical currents. It also will be difficult to apply in the horizontal plane where large lateral currents can act. In these cases, control saturation can cause loss of hovering control of the inherently unstable submersible.

The second method is to increase the inherent vertical plane hovering stability of the vessel by increasing its metacentric height. The larger metacentric pitch moment acts as an inherent pitch angle control, and a reduction in cruising capability can be avoided by increasing the maximum elevator force coefficient  $|Z'_\delta| \cdot \delta_m$ . However, in horizontal plane motions where metacentric yaw moments are absent, large sway-to-yaw coupling effects cannot be avoided with the tail-stabilized vessel.

Another approach to the design of a submersible which must cruise and hover also has been considered. If all appendages are confined to lie within the maximum hull radius, this is to use bow and stern stabilizers which are hydrodynamically equivalent. A design of this type is inherently stable with respect to all equilibrium hovering conditions, and its fore-aft symmetry eliminates one source of hydrodynamic coupling that is present in the tail-stabilized submersible. The elevators can be part of the stern stabilizers for better turning and diving effectiveness in cruising. The bow stabilizers are not expected to have any detrimental influence on cruising performance. In particular, they do not affect cruising stability, and may have a beneficial effect on turning ability.

## VII. ACKNOWLEDGMENTS

The author wishes to thank Mrs. Kae Wilkens, Mr. Conrad Sedlak, and Mr. Herant Deroian for their assistance in preparing this manuscript.

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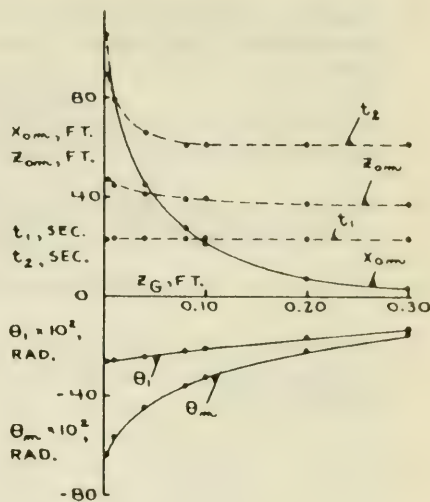


Fig. 17 - Effect of metacentric height on hovering heave maneuver response for otherwise standard run conditions



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# ADEPT AT DEPTH

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*Washington, D.C.*

There is only one recorded instance when an individual maintained complete control of all his faculties while in the depths of the sea. Today we are gingerly exploring this capability in our Man-in-the-Sea experiments, but it is not likely that man will venture much beyond 1000 feet while exposed to the environment. The alternative is to encase himself in a pressure vessel and, thus, permit visits to the greatest depths. This very act of encasement, however, presently denies to him the ability to perform all but the most primitive functions. How frustrating it is for man, who can thread a needle or write the Ten Commandments on the head of a pin in his normal environment, not to be able to perform but the most elemental, dexterous task in the deep oceans. Need this be so?

It is the aim of this paper to bring into focus those technologies which may be brought to bear in order to permit man to, indeed, be adept at depth. Of course, for one to be adept, he must be adept at something. The less complex the task assigned, the easier the design of a tool with which a man is adept. It is not economically or technologically feasible to permit man to extend his full dexterity through the pressure vessel. On the other hand, he will not be content to merely observe. We must then decide which general tasks are worth performing and the most effective means toward those ends. In order that the scope of this paper be bounded, let us postulate such a task to be performed and present the means by which successful accomplishment could be attained.

The task is to perform a rescue mission from a disabled submarine. It has been determined that this mission requires a manned submersible vehicle faced with the following constraints: (a) air transportability, (b) operation in variable water currents, (c) operation in various visibility states, (d) distressed submarine at angles to 45°, (e) under-ice operation, (f) not sea state limited, (g) transportable by mother submarine, (h) endurance to complete rescue mission, (i) adequate depth capability, and (j) maximum secondary mission capability.

The vehicle possessing these attributes must be nimble, indeed, to perform hatchmating and carry at least one prosthetic device for accomplishing such tasks as hatch clearance of debris.

What are the necessary elements which will permit success? The answer is obviously a highly responsive vehicle and arm, but consider those elements which contribute to responsiveness. For man to impose his will effectively, there are three areas of concern. These may be classed as sensory information, command translators, and the vehicle and arm dynamics.

The closest attention to these elements is mandatory, for the entire system performance is limited by the weakest link. Sensory information is not limited to vision alone, but embraces acoustic as well as position and rate data and may include feel in the form of feedback controls. The sensed information must be displayed to the operator in meaningful formats and utilized effectively as inputs to control systems. If this task is done well, the operator will know what is happening and what needs to happen. The next concern is to permit the man to impose his will upon the machine through manipulation of controls. These must be meaningful controls embodying the best practices of good human factors experience. The allocation of all dynamic control tasks to a single operator is mandatory. Last and most important, at least to hydrodynamicists, is the method of generating and controlling reaction forces in the water medium. It is to this element that the bulk of this paper is devoted.

The generally accepted criteria of excellence for propulsive devices is propulsion efficiency. The history of submersibles leads one ever upward in the quest for higher speed. Maneuvering capability has been solely for the high-speed region with little or no willingness to sacrifice higher speeds for low-speed maneuverability. When one considers the military requirements of submarines, it is clear that this emphasis is correct. There has been little interest in performing such tasks as rendezvousing submerged or bottom contour following at slow speeds for oceanographic or search missions except the recent Palomares effort.

Recent years have seen a growing interest in the general field of ocean engineering, generating a need for maneuvering capability in the zero and near zero speed range. Costeau's Saucer Vehicle is, of course, the best known example of this new breed of craft. Other later examples are the Alvin, a craft for oceanographic research, constructed for the U.S. Navy, and several commercial vehicles which may be typified by the Deepstar, Deepquest, and Aluminant. In each of these vehicles, the best-known proven state-of-the-art propulsion and control schemes were used.

For the task now at hand, the U.S. Navy carried out an exhaustive search of maneuvering and control devices conducting many tests on those thought to be the most promising, which could perform within the constraints enumerated earlier. The two surviving schemes (Figs. 1 and 2) are generally referred to as the ducted thruster with ring tail and the tandem propeller. The first needs little or no explanation to this audience, being a combination of the well-known bow thruster principle and steerable shroud. The tandem propeller is, perhaps, a new concept to many of you, but is an extension of principles presented four years ago at this symposium.

The departures from this design are the result of redirection of the performance emphasis from high speed to low speed. The two principal changes are (a) the tilting of the blade axis, resulting in swept conical surfaces, and (b) increase in blade area. The first change permits the direction of the large lift forces in the cross body axis without greatly reducing axial capability. The large blade areas provide better force-to-power ratios by moving greater mass through smaller accelerations and at lower rotational speeds. It is apparent that very large force-to-power ratios are possible if blade size or thruster duct



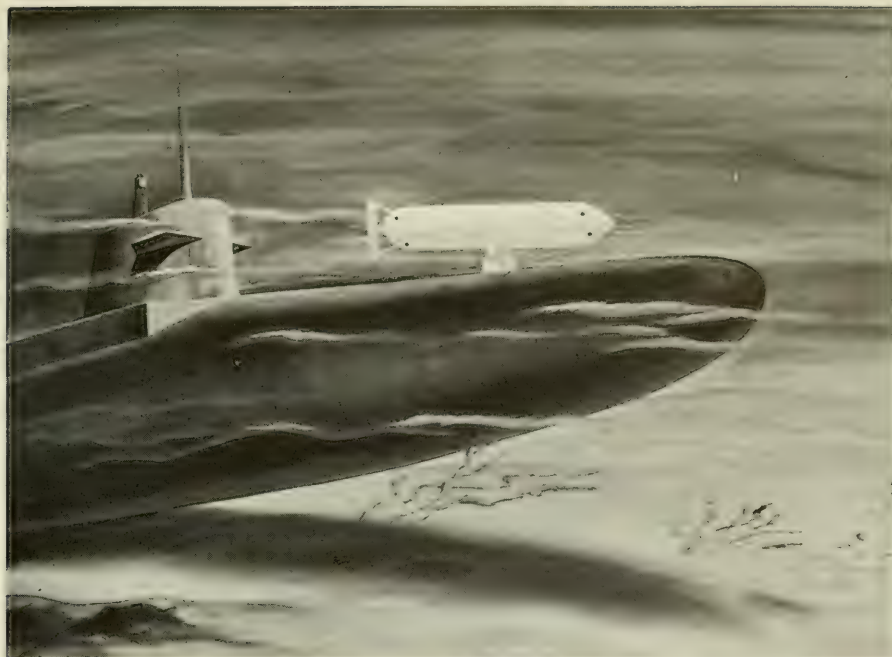


Fig. 1 - Ducted thruster with ring tail

diameter are unlimited. The two designs shown represent reasonable optimizations within the constraints imposed.

We will show some of the hydrodynamic test results which, when considered along with such other criteria as mission power consumption, permitted a clear choice to be made. Figure 3 compares large-scale-model test data, extrapolated to full scale, in the sway or heave mode. The measure of excellence, or performance factor, is chosen as developed pounds of thrust per horsepower plotted against velocity in the applied direction. Performance in pitch or yaw is easily derived from these data. Surge performance is derived in quite a different way and posed, perhaps, the most difficult decision criteria. On the one hand, an independent system is provided for surge, and, therefore, it and the body shape may be optimized for this mode without undue penalty to the other degrees of freedom. On the other hand, in the tandem propeller, surge is provided by the same system which provides the other degrees of freedom. While it enjoys a redundancy factor in that either propeller can provide surge motions, it suffers from not providing the ultimate in cruising efficiency. A comparison of the two systems' energy consumption per mile over a range of 0 to 5 knots is shown in Fig. 4. This imposes no penalty for power which may be required for thruster control below 1-1/2 knots, where shroud control force reversals occur. The sixth degree of freedom, roll, is hydrodynamically possible only with the tandem propeller system.





Fig. 2 - Tandem propeller

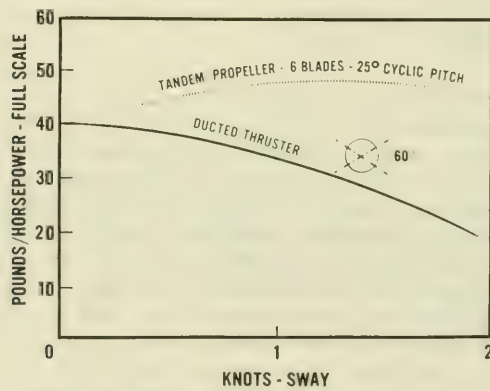


Fig. 3 - Comparison of the performance factor of the tandem propeller and of the ducted thruster (pounds of thrust per horsepower vs velocity in the applied direction)

## Adept at Depth

Special modes of operation are possible such as the antimud mode shown in Fig. 5 wherein the cyclic pitch is augmented with a counter-thrusting collective pitch to provide a relatively undisturbed field below the rescue vehicle. The final criteria for choice might well be the necessary number of main propulsion motors required for the two systems. It is obvious that a given horsepower will be obtainable for less weight in two motors than in five.

Being agile or adept at depth is not merely a result of providing directionality to force. One must consider force magnitude and response times as well as damping and stability factors. It is clear, for instance, that if a designer chooses to provide stability in pitch and roll through the use of meta-centric height rather than through the control system, he imposes burdens upon the control systems engineer. Likewise, if he attains arrow stability in the

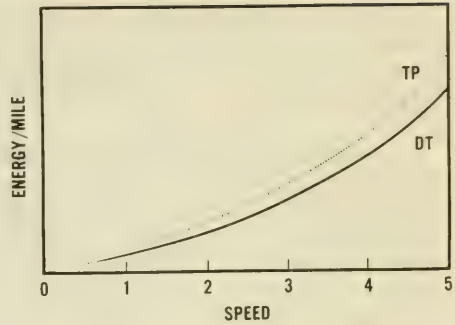


Fig. 4 - Comparison of the energy consumption of the tandem propeller and of the ducted thruster

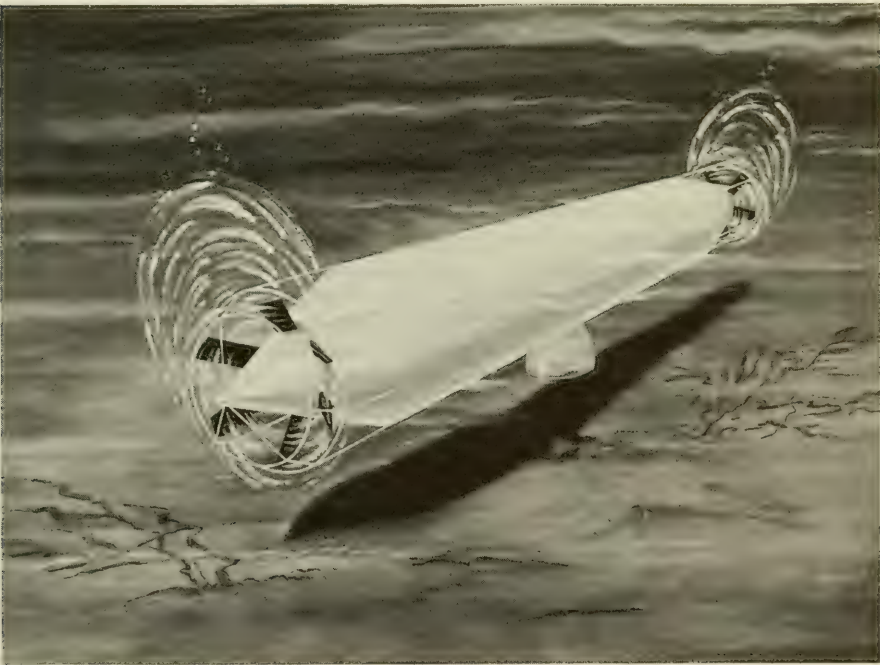


Fig. 5 - Antimud mode of operation

surge mode through large fixed control surfaces rather than with the control system, he reduces not only the maximum control rates attainable, because of the added drag, and the time to attain these rates, but introduces also motion crosscouplings.

The most important factor contributing to vehicle agility is the ability to rapidly develop large control forces in the desired orientation. Not only does it permit coping with fluctuating forces imposed by currents flowing about bottom obstructions, but it greatly simplifies the control system designers' task and that of the operator. From a control system consideration, it is desirable that vehicle orientation changes be under the command of the human operator and that vehicle stability be under the command of the control system. A good control system, for example, will make the operator oblivious to changing current conditions and will depend on high-grade sensory inputs. This last requirement places severe demands on the sensor designer, particularly those to sense sway, heave, and surge components. Roll, pitch, and yaw data impose trivial demands on the state-of-the-art. The operator's input device must be so orientated that its motions are directly relatable in the most straightforward way to those of the vehicle. That is to say, a pitch command, for instance, must be a pitch motion or force on the control input device. A good input device has zero reference detents with sufficient breakout force to prevent undesirable control crosscoupling inputs by the operator, and damping. An example of such a device is shown in Fig. 6.

We have concentrated on adept vehicles. What if controlling the vehicle in a precision manner is not the task but that accomplishing useful work is. This implies that a prosthetic device of some description is to be operated from the vehicle and will be used to extend man's desires to perform useful work functions. There are two basic approaches to this problem; the first is to bottom or otherwise anchor the vehicle physically with respect to the work area and then proceed to concentrate on the operation of the prosthetic arm to accomplish the task at hand. The second is to assume an untethered vehicle as the reference platform. The first has the advantage of placing the operator in a situation wherein he may leisurely go about his task, wherein the second places much tighter requirements upon vehicle control and upon the prosthetic arm response characteristics. There are, of course, situations wherein each can perform where the other approach will not. For the vehicle at hand, it would appear that the free flight approach would be most appropriate because the vehicle already has imposed upon it requirements of control and stability nearly equivalent to anchoring itself to the bottom. Once this decision has been made, it is possible to design a control system which is prosthetic-arm-or-hand oriented. A first requirement is that the dynamic response characteristics of the hand be significantly faster than the vehicle. It has been demonstrated that the operator of an arm is able to ignore vehicle stability in all six degrees of freedom and arm response. In consonance with the principle of assigning all dynamic tasks to a single operator, wherein possible, this arm control task presents an excellent example. In the fulfilling of a specific task with the arm, such as picking up an object resting on the ocean floor while performing a search mission, the operator should not have to make motion assignments between the vehicle and the arm. His concern should be that of grasping the object with the hand. His orders should be in terms of hand positioning control with no conscious concern

## Adept at Depth

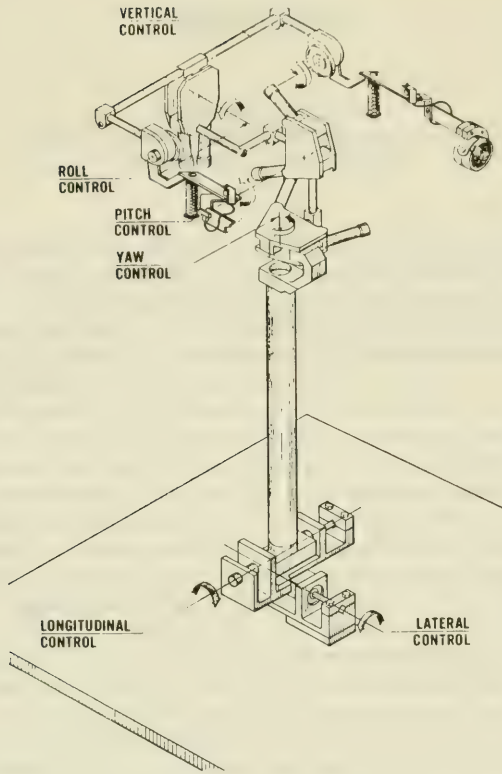


Fig. 6 - Control input device used for a model tandem propeller

as to whether supporting motions are provided by vehicle motions or arm motions. This mode of operation could well be another special mode like the anti-mud mode described previously and in which the apportionment of motions is assigned in a logical sequence by the control computer.

In summary, the features which go to make up a high-performance vehicle are: (a) six degrees of uncoupled freedom, (b) as much installed power as possible within imposed constraints, (c) neutral stability, (d) a precision sensor suit with meaningful displays to the operator and inputs to the control computer, (e) an automatic orientation holding control system, (f) a vehicle-oriented control input device, (g) a high-response prosthetic arm, and (h) a hand-positioning control mode.

There have been discussed the many functions which, when properly integrated, will enable the production of a vehicle which is indeed adept at depth.

\* \* \*



## DISCUSSION

LCdr A. Skolnick

*Deep Submergence Systems Project Office  
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The author has stated privately that the historical event to which his first sentence refers is Moses in the Red Sea. One can think of at least one other biblical instance, i.e., Jonah, where the doubly aggravating circumstance of being in the depths of a whale in the depths of the sea was the case. The prospect of mating a deep-diving manned submersible on the hatch of a distressed submarine is in a sense an aggravation of the third power. The rendezvous and docking of two vehicles in outer space is less difficult; the visibility is good there, but this is hardly the case for the submersible; the space docking occurs unbuffeted by winds, whereas underseas chaotic flow conditions can take place as a result of water currents and the effect of local protuberances; finally, space vehicles are smartly responsive, whereas submersibles are well noted for their sluggishness. The requirement of mating upon an underway or hovering mother submarine (present in support of the operation) portends no *lesser* task than the requirement of mating upon the distressed ship.

The author is correct in underscoring the necessity for developing large control forces rapidly; no amount of control system sophistication or human operator unburdening can compensate for a lack of performance upon demand. Without proper power available, control augmentation devices merely become additional "cargo" and their purpose is severely compromised. Members of the audience familiar with the effects of transportation lag (pure time delay) upon automatic control systems can well imagine the situation when a human pilot is introduced into the loop. Any action taken to alleviate long response times is sure to contribute to success in operation. Finally, one wonders about the whole business of analytical description and electronic analog simulations vis-a-vis the accuracy of the hydrodynamics model for near-zero velocity operation. The controls engineer has made great strides in the last 15 years and stands at the ready — it is past time for the hydrodynamicist to develop valid zero-velocity-regime-describing functions so steady team progress can be made in this rapidly developing field.

\* \* \*

## DISCUSSION

Roy S. Rice  
*Cornell Aeronautical Laboratory, Inc.*  
*Buffalo, New York*

The author has described several of the important considerations involved in the development of vehicles able to perform the many functions required for true agility in the depths of the sea. Each of these considerations is worthy of extended discussion, but two of them are especially significant to this symposium because of the need for an understanding of low-speed hydrodynamics in order to realize suitable vehicle configurations.

The author has pointed out that emphasis on improving maneuverability in submersible vehicles has been mainly concerned with the high-speed or cruise regime. Indeed, until the formation of the Deep Submergence Systems Project, very little work, either experimental or theoretical, had been done toward determining the hydrodynamic coefficients and stability derivatives of underwater vehicles operating at low speed. Now, with the extreme importance of establishing fundamental understanding of the stability and control characteristics of the DSSP craft in order to assure ability to meet rescue, search, and recovery mission requirements, the absence of an adequate body of knowledge of vehicle dynamics in the near-zero speed range is creating substantial design problems.

Most of you are familiar with the techniques developed at the Davidson Laboratory and the David Taylor Model Basin for evaluating the hydrodynamic characteristics of submarines when a dominant forward velocity exists. These methods permit analysis of the nondimensionalized stability derivatives of the vehicle in a convenient form and computation of the dynamic response characteristics. Extrapolation of data to reasonably similar hulls may be made with confidence, and preliminary design studies can result in obtaining substantial insight into vehicle performance.

The absence of a dominant speed in hovering operation means, of course, that no established flow field exists from which perturbations for maneuvering may be examined, and therefore the hydrodynamic characteristics must be treated with respect to the motion of interest. This introduces many nonlinearities which, coupled with the fact that the flow conditions range from laminar through turbulent (with much of the operation in the transition region), pose severe problems in theoretical studies.

A second point of special interest brought out by the author has to do with the generation of control forces. For the precision of location and attitude needed for certain missions to be achieved, decoupled control of motion is indeed a necessity. Yet this capability must be attained within the constraints of a design which is both energy and power limited on a vehicle having characteristic response times severely mismatched with dynamic responses that the pilot is well qualified to handle. Here, again, an understanding of the hydrodynamics — the distortion of the flow field due to motions of the hull resulting in disorientation

Haselton

of the control forces; degradation in force-generating capability due to motion in other degrees of freedom; the increased importance of unsteady flow effects in the short term application of control forces — is generally lacking.

In effect, the whole range of problems that have concerned the aerodynamicist in helicopter and VTOL designs is now facing the hydrodynamicist in the development of the high-maneuverability deep-submergence vehicle.

\* \* \*

## REPLY TO DISCUSSION

F. R. Haselton

I wish to thank Dr. Skolnick and Mr. Rice for their comments, with which I am fully in accord.

The time has arrived to open our eyes, roll up our sleeves, and go about the business of producing a work boat capable of effective use in ocean exploitation.

\* \* \*

Friday, September 30, 1966

Afternoon Session

## MANEUVERABILITY AND OCEAN PLATFORMS

Chairman: W. P. A. van Lammeren

Netherlands Ship Model Basin  
Wageningen, The Netherlands

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# A MODEL TESTING TECHNIQUE AND METHOD OF ANALYSIS FOR THE PREDICTION OF STEERING AND MANOEUVRING QUALITIES OF SURFACE SHIPS

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## ABSTRACT

A semitheoretical method of investigating steering and manoeuvring qualities of surface vessels is described. A general mathematical model, consisting of nonlinear equations of motion for a body free to move in the horizontal plane, is used as a basis. The hydrodynamic coefficients of the equations are obtained by carrying out captive-model experiments for each particular vessel, after which any desired manoeuvre can be predicted with the aid of a computer.

The hydrodynamic coefficients are determined using a planar-motion mechanism system to test 6-m (20-ft) wax models in a conventional towing tank. Details of the design, construction, and use of this mechanism are given, together with a description of the methods used to analyse the measured forces. The convenient manner in which nonlinear and crosscoupling coefficients, necessary for the accurate prediction of radical manoeuvres, can be obtained with a planar-motion mechanism technique is described.

Short mention is made of the influence of frequency and tank-interference effects on the measured hydrodynamic forces, and it is shown that troublesome resonant standing-wave conditions can be avoided.

## INTRODUCTION

There is no completely analytical method available at this time for predicting the steering and manoeuvring characteristics of ships. Due to this lack of theoretical methods, experimental techniques using free-running models have been employed by model basins for many years. Of more recent development is

a semitheoretical technique which utilizes the experimental results of captive-model tests in conjunction with the equations of motion expanded to include significant nonlinear terms. The captive model tests may be experiments using either a rotating arm or a planar-motion mechanism, and the predictions of manoeuvring characteristics are obtained from a solution of the equations of motion by means of either a digital or an analog computer.

A semitheoretical technique of this kind has been adopted at the Hydro- and Aerodynamics Laboratory (HyA) for investigating steering and manoeuvring qualities of ships and is dealt with in this paper. The HyA-method utilizes a planar-motion mechanism system for experimental measurement of the hydrodynamic coefficients in the equations of motion, and uses the HyA-GIER digital computer for the numerical solution of the equations. Before launching into technical details, however, it is relevant to consider briefly the reasons which make such an approach desirable.

Free-running model tests constitute the simplest and most direct means of assessing the behaviour of a given ship design during the execution of a particular manoeuvre. The tests are made by simulating full-scale conditions as closely as possible. The disadvantages of free-running model tests are firstly that they can only be conducted with difficulty in a conventional long narrow towing tank and preferably require a large manoeuvring basin. Secondly, whereas free-running tests provide information enabling engineering decisions to be made as to the suitability or otherwise of a given design, the reasons for the observed performance are not apparent from the test results. The tests are unsatisfactory in that little insight is gained into the hydrodynamic phenomena involved, and a rational basis for improvements in design is difficult to establish. Finally, even when large facilities are available for making free-running model tests, certain scaling problems are difficult to avoid. It is, for instance, difficult to obtain the correct conditions of propeller loading corresponding to the ship propulsion point. When making captive-model tests this is very simply achieved by applying a towing force via the rigid connection to the carriage.

The semitheoretical technique employing captive-model tests in combination with a mathematical model is a more analytical and potentially more powerful approach which overcomes these disadvantages. If a planar-motion mechanism is employed for the execution of the captive model tests, the disadvantages of the free-running tests mentioned above are eliminated and scale effects in general are reduced because of the large models which can conveniently be used. If a rotating arm is used for the captive model experiments, the disadvantage that a special manoeuvring basin is necessary is not eliminated. Furthermore, because of the small size of most rotating arm facilities, scale effects are likely to be troublesome.

The equations of motion, which constitute the mathematical model of the situation under investigation, must closely represent the real physical occurrences and be capable of yielding results with an accuracy at least as good as those obtainable from free-running model tests. The ease with which this can be done depends largely on the degree of complexity of the situation considered. Course stability characteristics involving motions only deviating infinitesimally from zero are thus more amenable to simulation by these means than are for

example the interaction effects between two ships passing in a shallow canal. The difficulty in handling the more complex situations arises not, however, as a result of the mathematical model, which even in a very general case is almost ideally suited to computer solution, but rather in the determination of the hydrodynamic coefficients of the equations of motion.

Although some of the hydrodynamic coefficients can be approximated theoretically, the only reliable way at this time of obtaining values with the accuracy needed for quantitative simulation is to conduct captive model tests. This state of affairs has the merit of great flexibility. As theory progresses, more and more coefficients will be obtainable by calculation and fewer experiments will be necessary. In the meantime, the orderly experimental approach necessitated by the structure of the mathematical model is conducive to systematic and progressive collation of data, itself tending to make testing redundant and to stimulate the creation of useful theoretical concepts.

## MATHEMATICAL MODEL

The mathematical model at present in use at HyA for simulation of steering and manoeuvring characteristics of surface ships is based on the general equations of motion of a rigid body moving in the horizontal plane. The detailed derivation of these equations in terms of the now commonly accepted notation (1,2) employed in steering and manoeuvrability work is given by Abkowitz (3). For a body having freedom in surge, sway, and yaw but restricted in heave, pitch, and roll\* the equations, developed for a coordinate system fixed in the symmetry plane of the body, are

$$\begin{aligned} X &= m(\dot{u} - rv - x_G r^2) \\ Y &= m(\dot{v} + ru + x_G \dot{r}) \\ N &= I_z \dot{r} + mx_G(\dot{v} + ru), \end{aligned} \tag{1}$$

where terms on the right side describe mass and inertial responses, and the left side expresses the external hydrodynamic forces and moments acting on the body (see Fig. 1).

The hydrodynamic forces and moments are functions of body geometry, motions, and orientation. For a given body with a single control surface, moving in unrestricted water, they may be expressed as the general functions

$$\left. \begin{matrix} X \\ Y \\ N \end{matrix} \right\} = f(\dot{u}, \dot{v}, \dot{r}, u, v, r, \delta), \tag{2}$$

---

\*Rolling and heel have been neglected in the present model, since they are felt to have little influence on the prediction of steering and manoeuvring characteristics, with the possible exception of fast warships, which heel appreciably in turns.



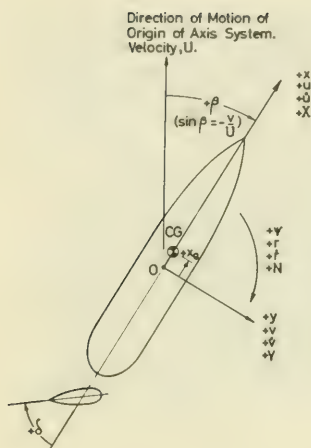


Fig. 1 - Diagrammatic definition of motion, orientation, and force parameters in terms of body-axis coordinates

which must be defined before simulation studies can be carried out.

The functions describing the hydrodynamic forces and moments have been developed into a useful form for analysis purposes with the aid of a Taylor expansion of the functions. If the Taylor expansion is limited to the first-order terms, the linearized equations are obtained (3,4). The present stage of development, which enables realistic simulations of ship manoeuvres to be made, is based largely on a third-order Taylor expansion of the functions. Introducing the assumptions that (a) forces and moments have appropriate port and starboard symmetry except for a constant force and moment caused by the propeller, and (b) there are no second- or higher-order acceleration terms, and that crosscoupling between acceleration and velocity parameters is negligible, the validity of which assumptions has been verified, for instance, by the measurements reported in Ref. 5, the third-order Taylor expansion reduces to the expressions

$$\begin{aligned}
 X = & X_u \dot{u} + X_* + X_u \Delta u + \frac{1}{2} X_{uu} \Delta u^2 + \frac{1}{6} X_{uuu} \Delta u^3 \\
 & + \frac{1}{2} X_{vv} v^2 + \frac{1}{2} X_{rr} r^2 + \frac{1}{2} X_{\delta\delta} \delta^2 + \frac{1}{2} X_{vuv} v^2 \Delta u + \frac{1}{2} X_{rru} r^2 \Delta u + \frac{1}{2} X_{\delta\delta u} \delta^2 \Delta u \\
 & + X_{vr} vr + X_{v\delta} v\delta + X_{r\delta} r\delta + X_{vru} vr \Delta u + X_{v\delta u} v\delta \Delta u + X_{r\delta u} r\delta \Delta u
 \end{aligned} \quad (3)$$

$$\begin{aligned}
 Y = & Y_v v + Y_r r + Y_* + Y_u \Delta u + Y_{uu} \Delta u^2 \\
 & + Y_v v + \frac{1}{6} Y_{vvv} v^3 + \frac{1}{2} Y_{vrr} vr^2 + \frac{1}{2} Y_{v\delta\delta} v\delta^2 + Y_{vu} v \Delta u + \frac{1}{2} Y_{vu u} v \Delta u^2 \\
 & + Y_r r + \frac{1}{6} Y_{rrr} r^3 + \frac{1}{2} Y_{rvv} rv^2 + \frac{1}{2} Y_{r\delta\delta} r\delta^2 + Y_{ru} r \Delta u + \frac{1}{2} Y_{ru u} r \Delta u^2 \\
 & + Y_{\delta\delta} \delta + \frac{1}{6} Y_{\delta\delta\delta} \delta^3 + \frac{1}{2} Y_{\delta vv} \delta v^2 + \frac{1}{2} Y_{\delta rr} \delta r^2 + Y_{\delta u} \delta \Delta u + \frac{1}{2} Y_{\delta u u} \delta \Delta u^2 \\
 & + Y_{vr\delta} vr\delta
 \end{aligned} \quad (4)$$

The corresponding expression for N is obtained by replacing Y by N in Eq. (4).

Equating the hydrodynamic forces and moments based on Eqs. (3) and (4) with mass and inertial responses, Eqs. (1), then the nonlinear mathematical model finally becomes (4)

## Prediction of Steering and Manoeuvring of Ships

$$X \text{ equation:} \quad (\dot{m} - X_u) \dot{u} = f_1(u, v, r, \delta)$$

$$Y \text{ equation:} \quad (\dot{m} - Y_v) \dot{v} + (m x_G - Y_r) \dot{r} = f_2(u, v, r, \delta) \quad (5)$$

$$N \text{ equation:} \quad (m x_G - N_v) \dot{v} + (I_z - N_r) \dot{r} = f_3(u, v, r, \delta),$$

where

$$\begin{aligned} f_1(u, v, r, \delta) = & X_* + X_u \Delta u + X_{uu} \Delta u^2 + X_{uuu} \Delta u^3 + X_{vv} v^2 \\ & + (X_{rr} + m x_G) r^2 + X_{\delta\delta} \delta^2 + X_{vvu} v^2 \Delta u + X_{rru} r^2 \Delta u + X_{\delta\delta u} \delta^2 \Delta u \\ & + (X_{vv} + m) v r + X_{v\delta} v \delta + X_{r\delta} r \delta + X_{vru} v r \Delta u + X_{v\delta u} v \delta \Delta u + X_{r\delta u} r \delta \Delta u \\ f_2(u, v, r, \delta) = & Y_* + Y_u \Delta u + Y_{uu} \Delta u^2 + Y_{vr} v r \delta \\ & + Y_v v + Y_{vvv} v^3 + Y_{vrr} v r^2 + Y_{v\delta\delta} v \delta^2 + Y_{vu} v \Delta u + Y_{vuu} v \Delta u^2 \\ & + (Y_r - m u) r + Y_{rrr} r^3 + Y_{rvv} r v^2 + Y_{r\delta\delta} r \delta^2 + Y_{ru} r \Delta u + Y_{ruu} r \Delta u^2 \\ & + Y_{\delta\delta} \delta + Y_{\delta\delta\delta} \delta^3 + Y_{\delta vv} \delta v^2 + Y_{\delta rr} \delta r^2 + Y_{\delta u} \delta \Delta u + Y_{\delta uu} \delta \Delta u^2 + Y_{\delta\delta\delta u} \delta^3 \Delta u \\ f_3(u, v, r, \delta) = & N_* + N_u \Delta u + N_{uu} \Delta u^2 + N_{vr} v r \delta \\ & + N_v v + N_{vvv} v^3 + N_{vrr} v r^2 + N_{v\delta\delta} v \delta^2 + N_{vu} v \Delta u + N_{vuu} v \Delta u^2 \\ & + (N_r - m x_G u) r + N_{rrr} r^3 + N_{rvv} r v^2 + N_{r\delta\delta} r \delta^2 + N_{ru} r \Delta u + N_{ruu} r \Delta u^2 \\ & + N_{\delta\delta} \delta + N_{\delta\delta\delta} \delta^3 + N_{\delta vv} \delta v^2 + N_{\delta rr} \delta r^2 + N_{\delta u} \delta \Delta u + N_{\delta uu} \delta \Delta u^2 + N_{\delta\delta\delta u} \delta^3 \Delta u \end{aligned}$$

The factorials,  $1/2$  and  $1/6$ , have been dropped as they unnecessarily complicate the bookkeeping, and the fourth-order terms  $Y_{\delta\delta\delta u}$  and  $N_{\delta\delta\delta u}$  have been introduced to obtain sufficient flexibility of expression for the influence of speed on rudder action.

Experience has shown that several of the hydrodynamic coefficients or derivatives contained in the above mathematical model are negligibly small. In particular the coefficients which give the changes of the nondimensional coefficients  $Y_*$ ,  $Y_v$ ,  $Y_r$ ,  $N_*$ ,  $N_v$ ,  $N_r$ ,  $X_{vv}$ ,  $X_{rr}$ ,  $X_{vr}$ ,  $X_{v\delta}$ , and  $X_{r\delta}$  with speed are small, as will be discussed later.

It is not suggested that the above mathematical model represents the ultimate in sophistication. It is felt, however, that further significant elaboration will not be justified before advances are made in theoretical or experimental methods of defining the pertinent hydrodynamic coefficients. The next logical step is perhaps to include freedom in roll, which might be necessary for simulation of radical manoeuvres of fast warships, which heel appreciably in high-speed turns. Additional terms would of course also be necessary, if it was desired to simulate more complex situations.

The solution of the mathematical model using a digital computer is described in detail in Ref. 4.

## GENERAL CONSIDERATIONS ON TESTING PROCEDURES

It is seen that the mathematical model, Eqs. (5), is comprised of numerous coefficients such as  $Y_{vvv}$ ,  $N_r$ , and  $X_{\delta\delta}$ . These coefficients in general depend on the particular geometry and design of a ship, and they must be known with reasonable accuracy before manoeuvres can be simulated by solving the mathematical model with the aid of a computer.

Ideally, numerical values for the coefficients would be evaluated by theoretical means, but although some of the hydrodynamic coefficients can be calculated approximately, the only reliable way at this time of obtaining values with the accuracy needed for simulations is to conduct captive model tests.

In a captive model test, the model is forced to perform precisely controlled movements, one or two of the different motion and rudder parameters being assigned values simultaneously. The resulting hydrodynamic forces and moments acting on the model are measured as functions of the parameters, and the coefficients are subsequently obtained from these measurements. The expressions for the hydrodynamic forces and moments, Eqs. (5), may be considered having been developed on the basis of a superposition process, the accuracy of which is progressively improved by successive inclusion of "crosscoupling" terms expressing deviations from simple superposition. Similarly, in the execution of the captive model tests, parameters are first explored one at a time, all other parameters being zero. The resulting forces and moments can then be expressed by coefficients which are functions of the single parameter. The next step in the captive model tests is to vary two parameters simultaneously, and if the resulting forces and moments differ from the superimposed results of the individually measured values, then the difference is expressed as a two-variable function of the parameters, and the coefficient representing the "cross-coupling" effect can be determined. Whereas, it is possible to generate three or more parameters simultaneously and so obtain crosscoupling terms in more than two variables, such terms have been found to be less than the accuracy of measurement. It is perhaps interesting in this connection to consider a free-sailing ship as a special case of captive model testing in which the ultimate stage of superposition has been reached. The totals of forces and moments acting can be inferred from the accelerations, and compared with model data expressed as one- and two-variable functions. This is done in effect when full-scale trajectories are compared with simulations based on model results.

The range of motion and rudder parameters explored during testing should, in principle, cover the range of subsequent simulation. Surge, sway- and yaw-accelerations, speed loss, drift angle, yaw velocity, and rudder angle should therefore be varied systematically up to the values corresponding to maximum-rudder manoeuvres for the free-sailing ship.

Captive model tests in which measurements are made of hydrodynamic forces and moments resulting from drift angle, from rudder angle, and from combinations of drift and rudder angles, can be conducted with relatively simple equipment in a conventional towing tank.

Many methods have been used to measure forces and moments due to angular velocity, among them being such devices as curved models, curve-flow

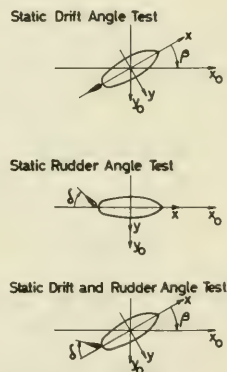


channels, and freely decaying oscillators. All of these systems have major disadvantages, being either inaccurate or unwieldy or both. The use of a rotating arm in a circular tank is at present by far the most widely used approach and is furthermore a most satisfactory means of obtaining crosscoupling terms in angular velocity and drift angle, and angular velocity and rudder angle. The disadvantages of a rotating arm, apart from its high capital costs, are that it is not practicably possible to measure acceleration derivatives and that it is not well suited to the generation of small angular velocities.

The planar-motion mechanism system conceived and developed by Gertler (6) and Goodman (7) provides a means of conducting captive model tests in which angular and straight-line motion can be imposed on a model in a conventional towing tank. Developed as a technique for submerged body research, the original mechanism generated motions of the body in the vertical plane. For application to surface ships, a planar-motion mechanism must operate in the horizontal plane, and the following description of the HyA planar-motion mechanism system is consequently given in terms of sway and yaw motions.

A planar-motion mechanism can be used in two different modes of operation, designated static and dynamic. In the static mode the model is constrained to travel along a straight path at constant velocity, and the mechanism is used to set discrete drift angles. Figure 2 schematically represents the three main types of tests made in the static mode. Forces and moments resulting from drift angle, rudder angle, and combinations of drift and rudder angles are measured in these tests.

Fig. 2 - Various examples of tests executed in the static mode of planar-motion mechanism operation



The unique feature of the planar-motion mechanism is its ability to generate oscillatory motions which are produced in the dynamic mode of its operation. Sinusoidal motions are imposed on the model with sway and yaw phased in such a way as to produce conditions of "pure sway" and "pure yaw." In the "pure-sway test," the bow and stern are oscillated in phase, and pure side-velocity and acceleration result, as shown schematically in Fig. 3.

In the "pure-yaw test" bow and stern are oscillated with phase-angle chosen such that pure angular velocity and acceleration result. Various examples of



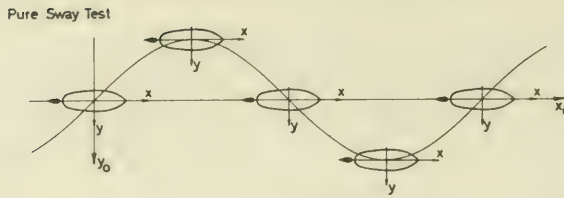


Fig. 3 - Pure sway test executed in the dynamic mode of planar-motion mechanism operation

this type of test are shown in Fig. 4, the two developments of the "pure-yaw test," i.e., the "yaw and drift angle test," and the "yaw and rudder angle test," constitute a means of measuring crosscoupling terms in the equations of motion.

As a means of experimentally measuring acceleration as well as angular velocity, drift angle, and rudder angle derivatives, the planar-motion mechanism system constitutes an almost ideal method of obtaining all the linear terms needed for course stability studies (6,7). The same basic approach can be extended to measurement of nonlinear and crosscoupling terms in angular velocity,

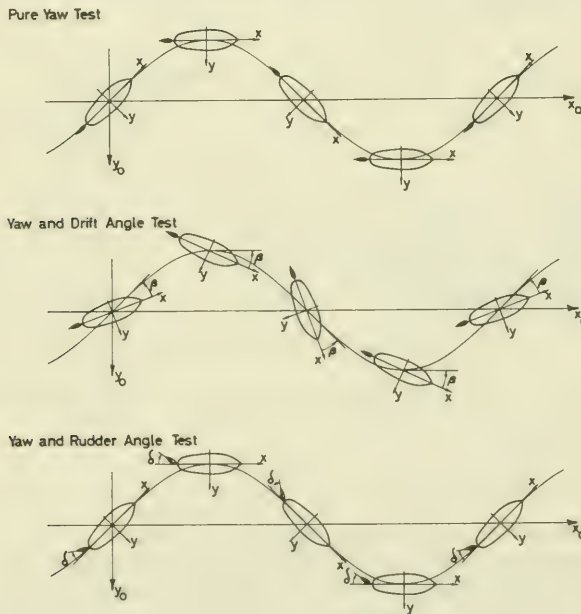


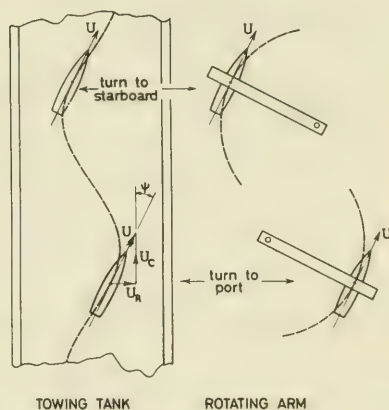
Fig. 4 - Various examples of yaw tests executed in the dynamic mode of planar-motion mechanism operation

thus enabling all of the terms in the above nonlinear mathematical model, Eqs. (5), to be measured.

# PLANAR-MOTION MECHANISM — PRINCIPLES OF MOTION GENERATION

The difficulty of generating a yaw velocity in a conventional tank is due to the fact that if a constant angular velocity is maintained for more than a short length of time, while traveling at the necessary speed, a collision with one of the side walls will result. Use of a planar-motion mechanism overcomes this problem by forcing the model to travel with alternately port and starboard yaw velocities. This results in an S-shaped path which in rough terms may be considered as being built up of segments of the circular path traveled by a model under a rotating arm (Fig. 5). To be more precise, a planar-motion mechanism generates a sinusoidal trajectory and the angular velocity is thus constantly varying, but the rotating arm comparison is nonetheless valid, because the variation is so slow that quasi-steady conditions are obtained.

Fig. 5 - Comparison of pure yawing motion generated with a planar-motion mechanism and a rotating arm



A continuously varying yaw velocity is necessarily accompanied by a continuously varying yaw acceleration to which the same quasi-steady conditions apply.

Reference to Fig. 5 shows that if the model is to move with a "pure yaw" motion, i.e., with zero drift angle, then it must at all times move along the tangent to its path. In terms of the body-axis system fixed in the model (Fig. 1), this means that the side velocity  $v$  must be zero, or the total velocity vector  $U$  of the origin  $O$  of the axis system must lie along the model centre line. It can be seen that for a model moving down the tank with carriage speed  $U_C$  this is achieved by moving the model perpendicularly to the carriage centre line with a relative velocity  $U_R$ . The magnitude of this velocity is given by

$$\tan \psi = \frac{U_R}{V_C}, \quad (6)$$

where  $\psi$  is the angular displacement from the tank centre line. Thus the generation of an angular motion and linear motion *relative to the carriage* results in a pure angular motion of the model *relative to the water*.

A practical realization of this requirement as adopted in the HyA planar-motion mechanism is diagrammatically illustrated in Fig. 6. The arrangement consists of two scotch-yokes connected by a rigid shaft. Rotation of the shaft results in periodic movement perpendicular to the carriage centre line of two points F and A fixed in the model. By variation of the phase angle  $\phi$  between the forward and aft scotch-yokes, various combinations of angular velocity and movement perpendicular to the carriage centre line are obtained. The phase angle, resulting in the fulfilment of the conditions for pure angular motion of the model relative to the water, is found as follows. Let

$\eta_F, \eta_A$  = lateral displacements relative to the carriage centre line of points F and A fixed in the model,

$d$  = distance of points F and A from origin O of the body-axis system fixed in the model,

$a$  = amplitude of the scotch-yokes and of the lateral displacement of points F and A,

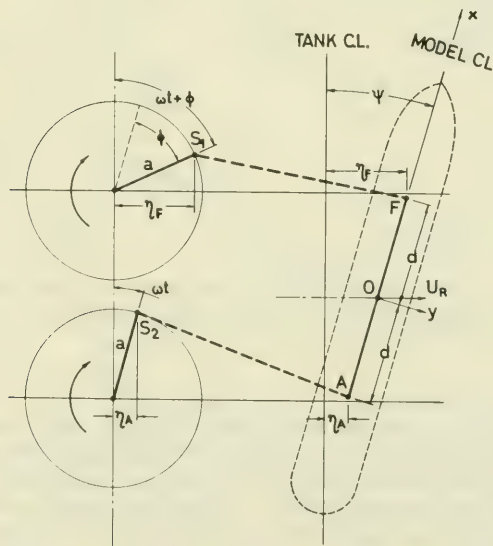


Fig. 6 - Motions generated by a planar-motion mechanism

## Prediction of Steering and Manoeuvring of Ships

$\phi$  = phase angle by which the forward scotch-yoke leads the aft,

$\omega$  = rate of rotation of the shaft connecting the scotch-yokes,

$\psi$  = angular displacement of the model centre line relative to the tank centre line, and

$t$  = time.

Then

$$\eta_F = a \sin (\omega t + \phi)$$

$$\eta_A = a \sin \omega t$$

$$U_R = \frac{d}{dt} \left( \frac{\eta_F + \eta_A}{2} \right) = a\omega \cos \frac{\phi}{2} \cos \left( \omega t + \frac{\phi}{2} \right).$$

Also

$$\sin \psi = \frac{\eta_F - \eta_A}{2d} = \frac{a}{d} \sin \frac{\phi}{2} \cos \left( \omega t + \frac{\phi}{2} \right). \quad (7)$$

Substituting for  $\sin \psi$  and  $U_R$  in Eq. (6),

$$\begin{aligned} \frac{1}{\cos \psi} \frac{a}{d} \sin \frac{\phi}{2} \cos \left( \omega t + \frac{\phi}{2} \right) &= \frac{a\omega}{U_C} \cos \frac{\phi}{2} \cos \left( \omega t + \frac{\phi}{2} \right) \\ \tan \frac{\phi}{2} &= \frac{\omega d}{U_C} \cos \psi. \end{aligned} \quad (8)$$

Since  $\psi$  is small, we make the approximation  $\cos \psi = 1$ , giving

$$\tan \frac{\phi}{2} = \frac{\omega d}{U_C}. \quad (9)$$

Thus the phase-angle  $\phi$  is not a cyclic quantity, since it only depends on frequency of oscillation and carriage speed, the distance  $d$  being fixed in the model.

The present HyA planar-motion mechanism has maximum amplitude  $a$  at the scotch-yokes of 100 mm, and the distance  $d$  is 1050 mm, giving a maximum possible value of 5.46 degrees for  $\psi$  when the error in the approximation  $\cos \psi = 1$  is less than 0.5 percent. This value is obtained only at zero speed when the phase-angle between the scotch-yokes is 180 degrees. In a normal operating condition, the maximum value of  $\psi$  is reduced by the factor  $\sin (\phi/2)$  to approximately half this value when the error is 0.1 percent. While these considerations give a good indication that the error involved is negligible, the only rigorous way of checking this is to analyze the "contaminating" cyclic motions and corresponding impure forces by carrying through the exact calculation. This has been done in another connection and has shown that the error is indeed negligible.



Accepting that a phase angle between the forward and aft scotch-yokes given by Eq. (9) results in pure angular motion of the model relative to the water, the expressions for the model's angular velocity and angular acceleration are obtained by differentiating Eq. (7) with respect to time:

$$\text{angular displacement, } \psi = \frac{a}{d} \sin \frac{\phi}{2} \cos \left( \omega t + \frac{\phi}{2} \right) \quad (7)$$

$$\text{angular velocity, } \dot{\psi} = -\frac{a\omega}{d} \sin \frac{\phi}{2} \sin \left( \omega t + \frac{\phi}{2} \right) \quad (10)$$

$$\text{angular acceleration, } \ddot{\psi} = -\frac{a\omega^2}{d} \sin \frac{\phi}{2} \cos \left( \omega t + \frac{\phi}{2} \right) \quad (11)$$

It is seen that angular velocity and angular acceleration are out of phase with each other.

Similarly, when the scotch-yokes are in-phase ( $\phi = 0$ ), the model centre line is always parallel to that of the tank and carriage, and the model experiences pure swaying motions. In terms of body-axis parameters, lateral displacement, velocity, and acceleration are then given by:

$$\text{sway displacement, } y = a \sin \omega t \quad (12)$$

$$\text{sway velocity, } \dot{y} = a\omega \cos \omega t \quad (13)$$

$$\text{sway acceleration, } \ddot{y} = -a\omega^2 \sin \omega t. \quad (14)$$

To summarize, hydrodynamically pure yawing or swaying motions can be generated by setting the phase angle between forward and aft scotch-yokes to an appropriate value. In both cases, the velocity and acceleration are 90 degrees out of phase with each other.

A constant drift angle or rudder angle can easily be superimposed on the pure yaw motion as shown in Fig. 4. Nonzero drift angle is obtained by changing the lengths of the arms shown symbolically in Fig. 6 as  $S_1F$  and  $S_2A$ . This has the effect of adding a sizable constant component  $\beta$  to the small cyclic value of  $\psi$  resulting from yawing motion. The phase-angle is then found from Eq. (9) by substituting  $d \cos \beta$  for  $d$ , and similarly  $d$  is replaced by  $d \cos \beta$  in Eqs. (7), (10), and (11).

## DESCRIPTION OF THE HyA PLANAR-MOTION MECHANISM

### Mechanical Structure

The HyA planar-motion mechanism is in many ways similar to instruments previously reported (6,7,9,10). Hence the following description is limited to a general outline of the main structural arrangement and of some novel features.

The HyA towing tank facility measures 240 x 12 x 6 metres in length, width, and depth respectively (8), enabling resistance and propulsion experiments to be

performed with 6 or 7 metre models without encountering serious blockage effects. When designing the planar-motion mechanism it was considered important that the large, accurate, wax models used for resistance and propulsion work should also be used for planar-motion mechanism tests. This enabled the wide experience available regarding these models to be utilized and was also advantageous from an economic point of view. The conventional wax models have been found to be sufficiently robust when reinforced by two transverse wooden bulkheads and one or two laminates of glass-fibre on the inside surface (see Fig. 7).

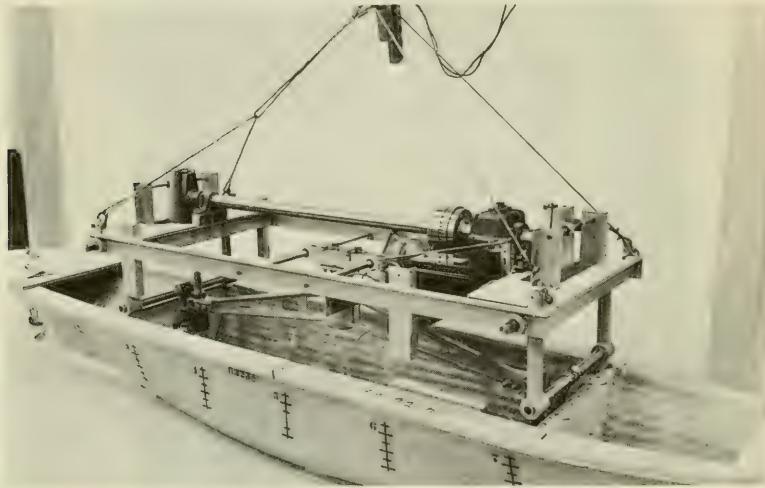


Fig. 7 - The HyA planar-motion mechanism shown suspended over a 6-m (20-ft) wax model for photographic purposes

In order to obtain acceleration and damping of the same order of magnitude, and to reduce the likelihood of frequency problems while still enabling sizeable yaw velocities to be generated, the HyA planar-motion mechanism was designed for lower frequencies and larger amplitudes of oscillation than the mechanism of Gertler (6) and Goodman (7). This proved to be a fortunate decision, as frequency problems caused by reflection of surface waves from the tank walls arose at lower frequencies than expected.

A general impression of the layout of the mechanism is given in Figs. 7 and 8. It is contained in a flat frame of welded angle sections which is in itself flexible in torsion but very stiff when in use, by virtue of being solidly clamped to the strong working-section of the carriage.

The solid main shaft is driven via a 1:50 Moss worm-and-wheel reduction gearbox and a 1:2 Power-Grip "timing" belt drive by a 2-1/2-hp Thrige synchronous electric motor.

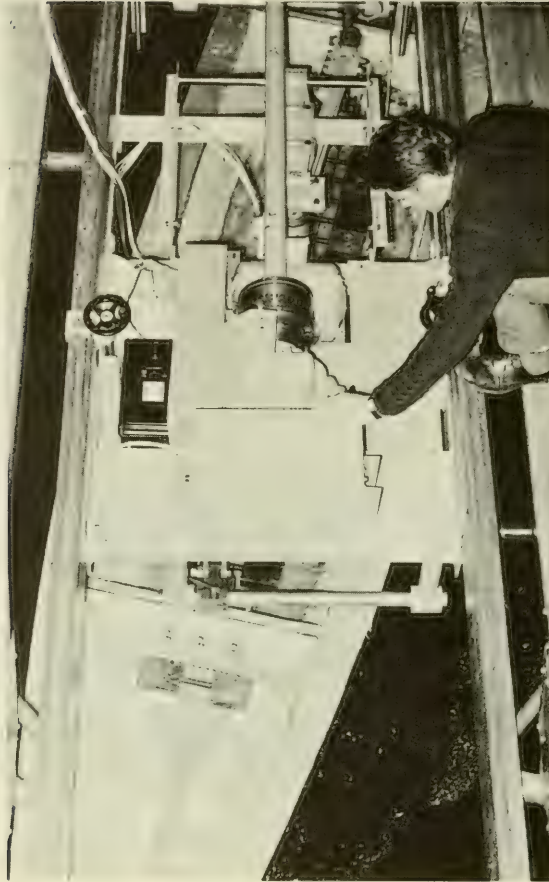


Fig. 8 - The HyA planar-motion mechanism mounted on the towing carriage during testing

Both worm-and-wheel reduction gearbox and belt drive are designed and manufactured to allow a chatter-free and vibration-free angular motion even under the most rapidly changing load conditions. Current is supplied to the motor by a Ward Leonard system installed on the carriage, enabling revolutions to be continuously varied from 2 to 40 rpm. By these means a predetermined revolution value can be maintained within fine limits, independent of loading.

The uniform rotary motion of the main shaft is converted to periodic translatory motion in the horizontal plane by a scotch yoke attached to each end (Fig. 9). The amplitudes of the scotch yokes can be continuously varied from 0 to 100 mm, and the settings are indicated with an accuracy of 0.1 mm by small built-in mechanical digital counters. No appreciable wear has occurred in the brass-to-steel block and slider systems after several hundred hours of operation.



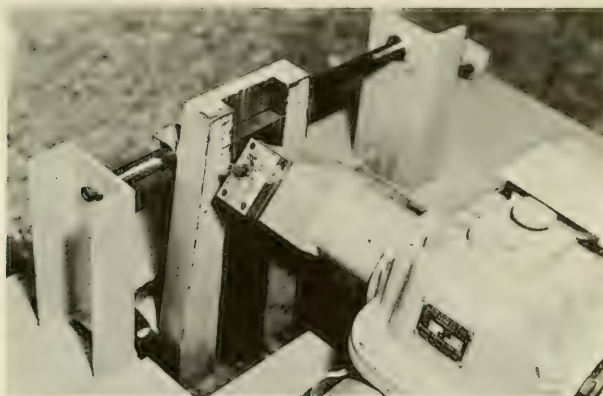


Fig. 9 - Variable amplitude scotch yoke

The translatory motion at each scotch yoke is transmitted to the model by a rectangular crossframe and two links. The upper arm of each crossframe is supported by four axial bearings which also permit the frames to swing forward and aft. Constant drift angles can be introduced by sliding the short link on each lower arm and clamping it (see Fig. 8), when it effectively becomes part of the crossframe. The links are connected to axial and rotary bearings at each end of a longitudinal beam on the model centre line. The beam is restrained at its centre point to prevent cyclic longitudinal movements of the model, and it maintains the links at a constant distance from the centre of the mechanism during all types of oscillatory motions. The beam together with the force gauges is mounted in the model in the workshop prior to being transported to the carriage and connected with the rest of the mechanism. This facilitates accurate positioning of the beam symmetrically around the origin of the model's body axes as well as permitting model and mechanism to be transported and attached to the carriage in a simple and convenient manner.

A Stromag electromagnetic tooth coupling in the main shaft permits the phase angle between forward and aft scotch yokes to be precisely set in discrete intervals of 1.5 degrees (Fig. 10). Also mounted on the coupling is the synchronous switch that controls the integrator circuits used in conjunction with the force-measurement system. The switch consists of a perspex ring having four black lines spaced at 90 degree intervals. As the shaft rotates, the black lines interrupt light beams shining on two photocells, giving rise to two trains of electrical impulses. The perspex ring can be rotated relative to the main shaft and locked so as to give impulses at positions of zero and maximum yaw or sway displacement. This is accomplished by rotating the ring through half of the phase angle between the scotch yokes.

A mechanical microswitch, giving one impulse per revolution of the main shaft, is used for identification purposes.



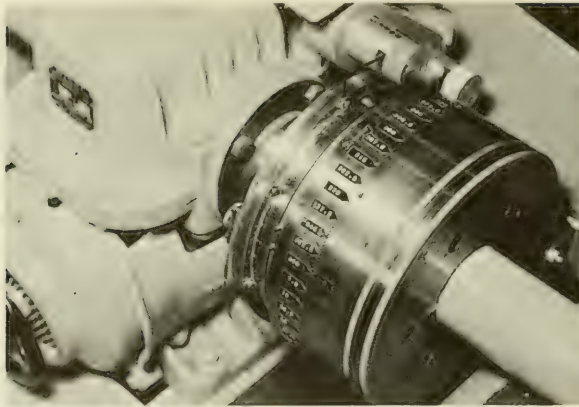


Fig. 10 - Electromagnetic phase-angle coupling and synchronous switch arrangement

#### Dynamometry and Recording Instrumentation

Two modular force-gauges are attached to each end of the centre beam (Fig. 11). The physical proportions of the gauges are such that they are flexible in only one direction and they are orientated to pick up either the side force  $Y$  or the longitudinal force  $X$ . Gimbals, also seen in Fig. 11, bolted to wooden bulkheads ensure that the model behaves as a "simply supported" beam. The model hull, force gauges, and centre beam form a closed loop, eliminating any errors due to friction. Total  $X$  and  $Y$  forces and  $N$  moment acting on the model are obtained from the gauge forces as follows:

$$\begin{aligned} X &= X_F + X_A \\ Y &= Y_F + Y_A \\ N &= (Y_F - Y_A)l, \end{aligned} \tag{15}$$

where the subscripts  $F$  and  $A$  denote forces measured at forward and aft gauges, and  $l$  is the distance from the origin to the gimbal axis (1150 mm).

The electrical signals from the two  $X$  gauges pass through Disa control units, are electrically summed, and then are recorded on a 10-inch-span Philips pen-recorder. The control units are used to adjust sensitivity and zero position and also contain accurate reference (span-check) signals, which can conveniently be regarded as equivalent to known forces at the gauges. When calibrated together with the gauges, these reference signals enable widely differing sensitivities to be used.

The signals from the  $Y$  gauges are processed in one of two alternative ways dependent on the nature of the test. If the planar-motion mechanism is being

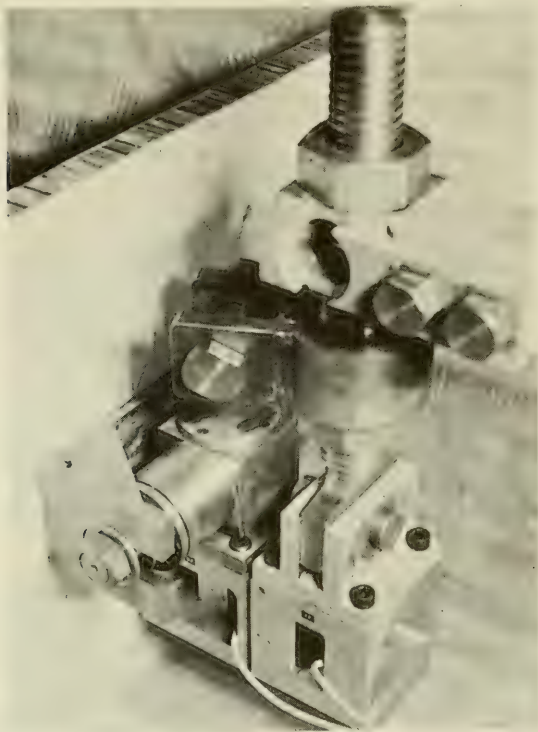


Fig. 11 - X and Y force gauges and a two-degree-of-freedom gimbal attached to the bulkhead in the model

used in the static mode, i.e., if only constant forces resulting from drift and/or rudder angles are to be measured, then the gauge signals are passed via control units to pen recorders in exactly the same manner as the X forces, except that the signals from forward and aft gauges are recorded individually and not summed. When the mechanism is used in its "dynamic" mode, and periodic forces are acting at the gauges, the signals are integrated before being recorded on the pen recorders.

## INTEGRATION OF PERIODIC FORCES

### Basic Principles

The periodic forces acting at the Y gauges in "pure sway" and "pure yaw" tests are composed of velocity-dependent and acceleration-dependent components. Assuming in the following, in order to simplify the discussion, that the periodic forces are directly proportional to the motions and that there is no

interaction between velocity-dependent and acceleration-dependent forces, then the cyclic forces, like the generated velocities and accelerations, will be of sine and cosine form. Each Y gauge thus simultaneously senses a sine-wave force and a cosine-wave force, almost certainly of different amplitude but necessarily of the same frequency, having the form

$$\text{gauge force} = a_1 \sin \omega t + a_2 \cos \omega t,$$

where  $a_1$  and  $a_2$  are constants in time.

If such a gauge force is recorded directly on a pen recorder, the record will be a sinusoidal curve of amplitude  $a_3$  given by

$$a_1 \sin \omega t + a_2 \cos \omega t = a_3 \sin (\omega t + e),$$

where  $a_1 = a_3 \cos e$  and  $a_2 = a_3 \sin e$ .

It is possible to measure  $a_3$  and  $e$  from a paper record and so obtain  $a_1$  and  $a_2$ , but the authors can say from experience that this is not a practicable solution, if only because of the difficulty of coping with the great quantities of paper records involved.

In the method chosen for the HyA planar-motion mechanism the force-signal is integrated electrically, with periodic reversals of polarity as illustrated in Fig. 12. It is seen that if polarity is reversed after half a period, then the sine component is measured and the cosine component eliminated. If polarity is reversed after one quarter and three quarters of a period, then the sine component is eliminated and the cosine component measured. Figure 12 also shows that if a constant force component is included to give the more general expression

$$\text{gauge force} = a_0 + a_1 \sin \omega t + a_2 \cos \omega t,$$

then either  $a_0$ ,  $a_1$ , or  $a_2$  may be obtained by suitable polarity reversal. A constant force signal will, for example, result from a constant rudder or drift angle superimposed on pure yaw motion or from imperfect gauge zero-adjustment.

These simple concepts form the basis used in the HyA planar-motion mechanism system for the measurement of periodic forces.

### Integration Programs

The integration of the force signals is, in practice, controlled by programming circuits contained in a processing unit (Fig. 13) designed for this particular purpose. The same programming circuits simultaneously control two parallel and identical measuring channels, used in conjunction with the forward and aft Y gauges.

The circuits are built up from flip-flops, pulse shapers, relays, etc., shown in the circuit diagram, Fig. 14. The operations controlled by the programs

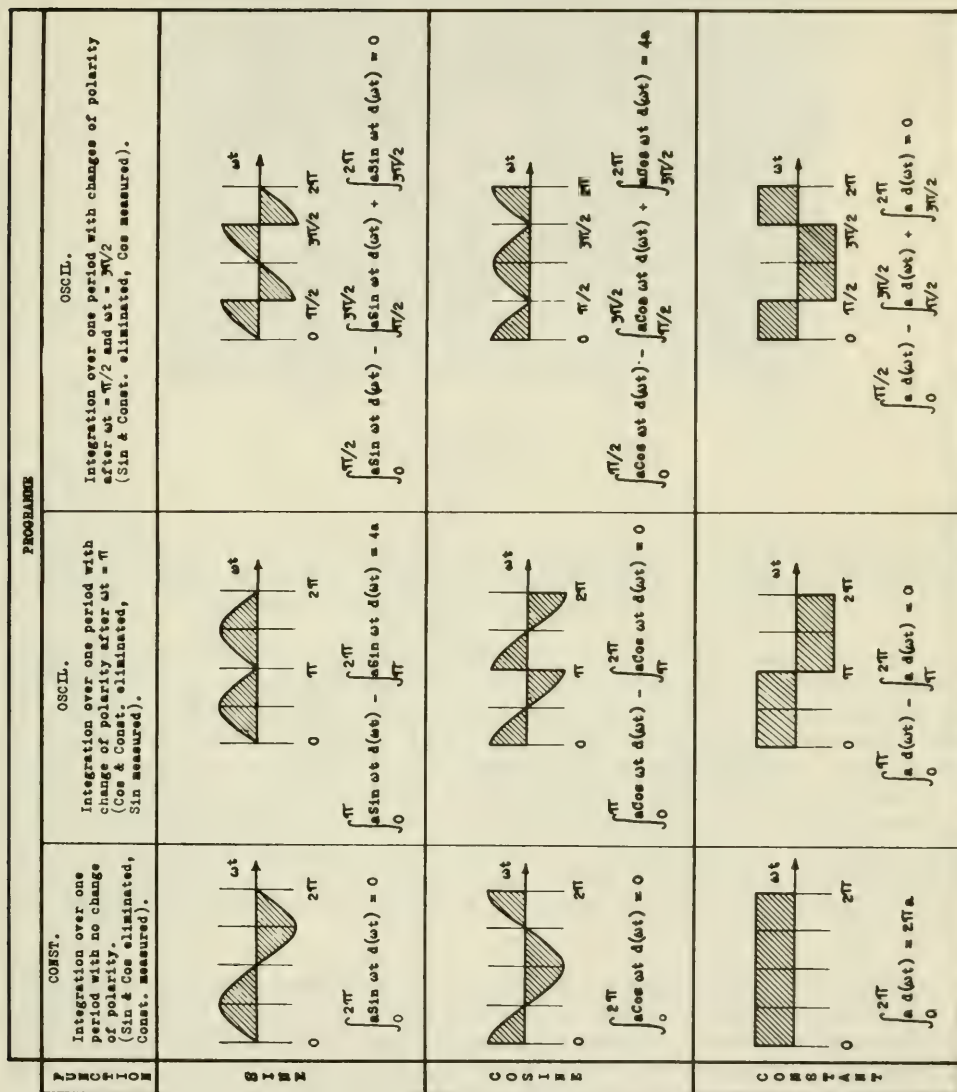


Fig. 12 - Integration with periodic polarity reversals



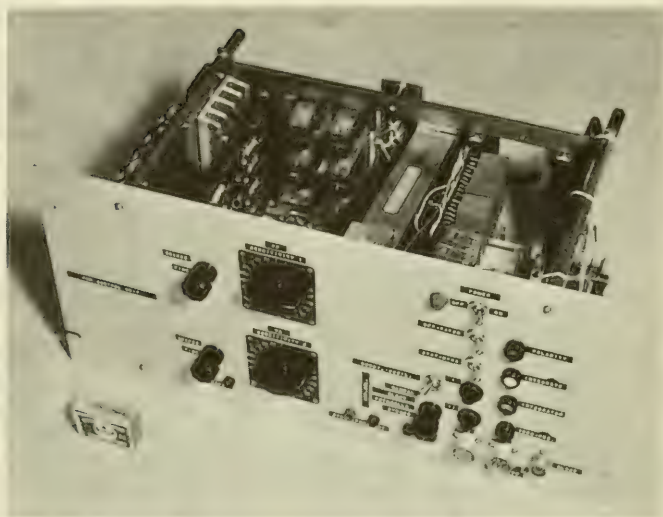


Fig. 13 - Processing unit

consist of reversing the polarity of the signals fed to integrator-condensers, controlling the limits of integration, activating the output and recording mechanisms, and resetting the condensers after each integration period.

The operations are activated by electrical impulses fed alternately to Input 1 and Input 2 of the processing unit. The trains of impulses are generated by the two photocells of the synchronous switch mounted on the shaft coupling. Four impulses are generated by each photocell during each revolution of the mechanism. Impulses can also be generated by a miniature synchronous motor in conjunction with an analogous photocell arrangement or by means of manually operated pushbuttons. The mini-synchronous motor is positioned inside the processing unit and is used for integration of span-check signals after each measuring run and for calibration. The pushbuttons are mainly used for checking and adjusting but are also useful for special tests needing long integration-times.

Integration can be made using either of two different programs, designated Oscil. and Const., the sequence of events of which is shown in Table 1 and Fig. 15. The only difference between the two programs is that the polarity of a signal integrated by means of the Oscil. program is negative for the first half and positive for the second half of an integration period, whereas the polarity of a signal integrated by means of the Const. program is the same during a whole integration period. In both cases, the integration period corresponds to one complete revolution of the mechanism and hence to one period of the force signal. The complete programs, however, use  $1\frac{1}{4}$  periods, resulting in the start of each integration period occurring  $\pi/2$  later than the one preceding it. This facilitates alternate elimination of sine and cosine force-components, when

# Prediction of Steering and Manoeuvring of Ships

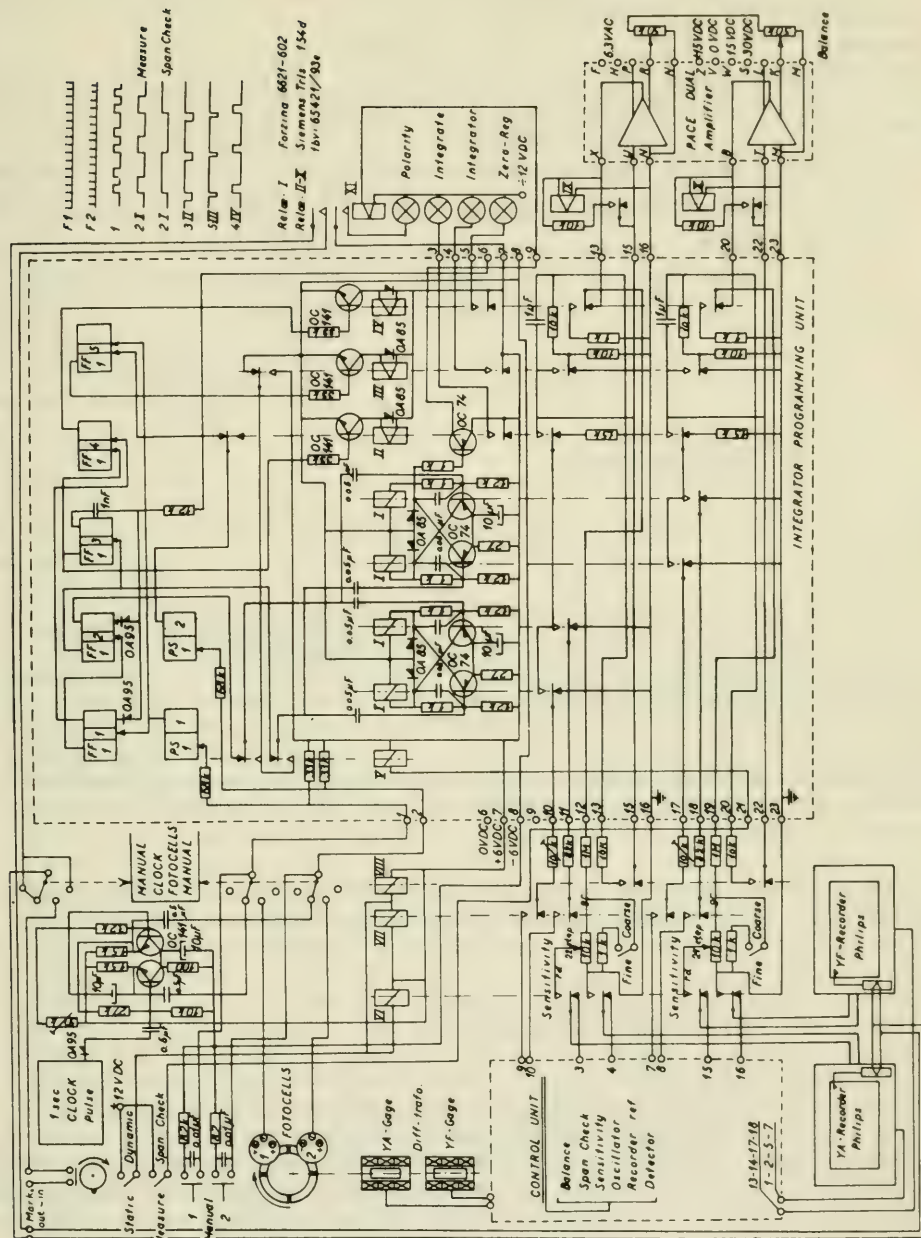


Fig. 14 - Programming circuits

Table 1  
Sequence of Events Controlled by Programming Circuits

| Number of Impulses Received |         | Events                                                   |                                        |
|-----------------------------|---------|----------------------------------------------------------|----------------------------------------|
| Input 1                     | Input 2 | Oscil. Program                                           | Const. Program                         |
| 0                           | 0       | Zero condition (condensers at zero)                      | Zero condition (condensers at zero)    |
| 1                           | 0       | Pens switched OFF. Integration begins. Polarity changed. | Pens switched OFF. Integration begins. |
| 1                           | 1       | —                                                        | —                                      |
| 2                           | 1       | Pens switched ON                                         | Pens switched ON                       |
| 2                           | 2       | —                                                        | —                                      |
| 3                           | 2       | Polarity changed                                         | —                                      |
| 3                           | 3       | —                                                        | —                                      |
| 4                           | 3       | —                                                        | —                                      |
| 4                           | 4       | —                                                        | —                                      |
| 5                           | 4       | Integration ends                                         | Integration ends. Polarity changed.    |
| 5                           | 5       | Condensers reset to zero                                 | Condensers reset to zero               |

using the Oscil. program. Subsequent integrations of, for instance, the sine component will furthermore have opposite polarity sequences enabling any differences in positive and negative integration rates to be averaged out.

The duration of the hold period, in which the desired final value is recorded, is governed by the time lag between the impulses fed to Input 1 and Input 2 of the unit.

#### Integration of Forces in Pure Yaw Tests

Periodic forces resulting from "pure yaw" motions are processed by the Oscil. program. Figure 16 shows that the angular velocity of the origin of the body axes is out of phase, and angular acceleration is in phase (but of opposite sign) with angular displacement. It is therefore convenient to use displacement as a reference, and the synchronous switch is consequently adjusted to give impulses to Input 1 at positions of zero, maximum, and minimum angular displacement, i.e., at  $\omega t + (\phi/2) = 0, \pi/2, \pi, 3\pi/2, 2\pi$ , etc. This is accomplished by rotating the synchronous switch to a position halfway between the angular displacement of the forward and aft scotch yokes; i.e., if the forward yoke leads the aft by a phase angle  $\phi$ , the synchronous switch has a phase angle of  $\phi/2$ .



## Prediction of Steering and Manoeuvring of Ships

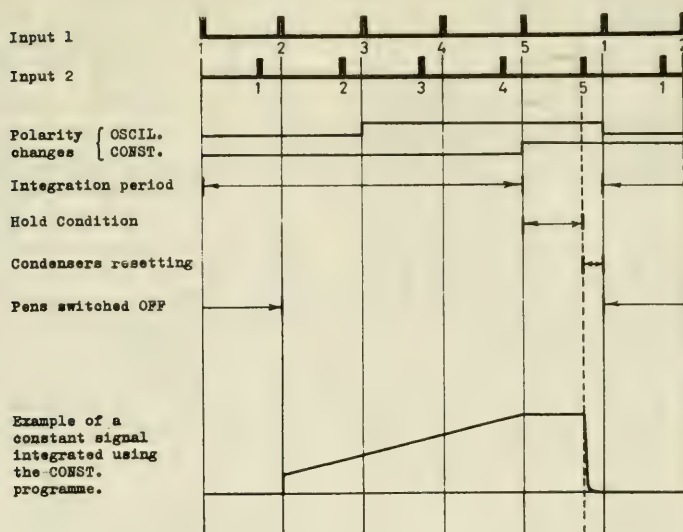


Fig. 15 - Sequence of events controlled by the programming circuits

In the first integration period shown in Fig. 16, it is seen that because the polarity is reversed at  $\omega t + (\phi/2) = \pi$ , the cosine and any constant component are eliminated and the sine component is measured. The integrated value is designated OUT  $\oplus$ . The designation OUT  $\oplus$  is used to indicate that the measured force is out of phase with angular displacement. The symbol  $\oplus$  indicates that the force corresponds to positive yaw velocity, as the effective yaw velocity has been positive throughout the whole of the integration period.

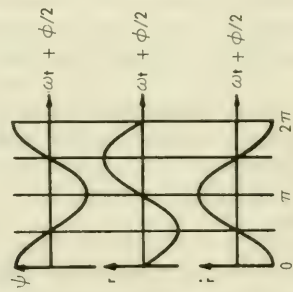
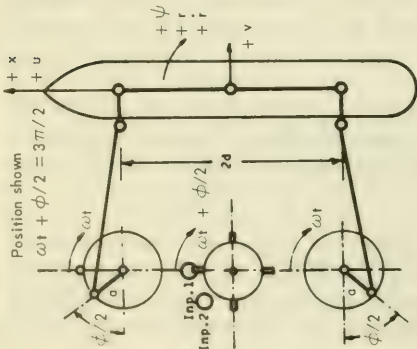
The next quarter revolution of the mechanism is used for the hold condition, activating the recording and output mechanism, and to discharge the condenser. As a result, the second integration period starts at  $\omega t + (\phi/2) = \pi/2$ , and not at  $\omega t + (\phi/2) = 0$  as did the first. The polarity is reversed halfway through the integrating period, now corresponding to  $\omega t + (\phi/2) = 3\pi/2$ , and in this case the sine and any constant components are eliminated, and the cosine component is measured. The final integrated value is designated IN  $\ominus$ , as the measured force is in phase with the angular displacement and corresponds to negative acceleration.

In a similar way the third and fourth integration periods measure forces corresponding to negative velocity and positive acceleration, and are designated OUT  $\ominus$  and IN  $\oplus$  respectively. The fifth integration period is the same as the first, after which the cycle is repeated.

Characteristic features of forces recorded during "pure yaw" tests are illustrated in the typical records at the bottom right of Fig. 16. A margin pen, activated by the mechanical microswitch once per revolution of the main shaft, is used to identify the different force components and is shown symbolically on



GENERATION OF YAWING MOTIONS



$$\begin{aligned}\psi &= \frac{a}{d} \sin \frac{\phi}{2} \cos(\omega t + \frac{\phi}{2}) \\ \dot{\psi} &= r = \frac{a}{d} \sin \frac{\phi}{2} \omega \sin(\omega t + \frac{\phi}{2}) \\ \ddot{\psi} &= t = -\frac{a}{d} \sin \frac{\phi}{2} \omega^2 \cos(\omega t + \frac{\phi}{2})\end{aligned}$$

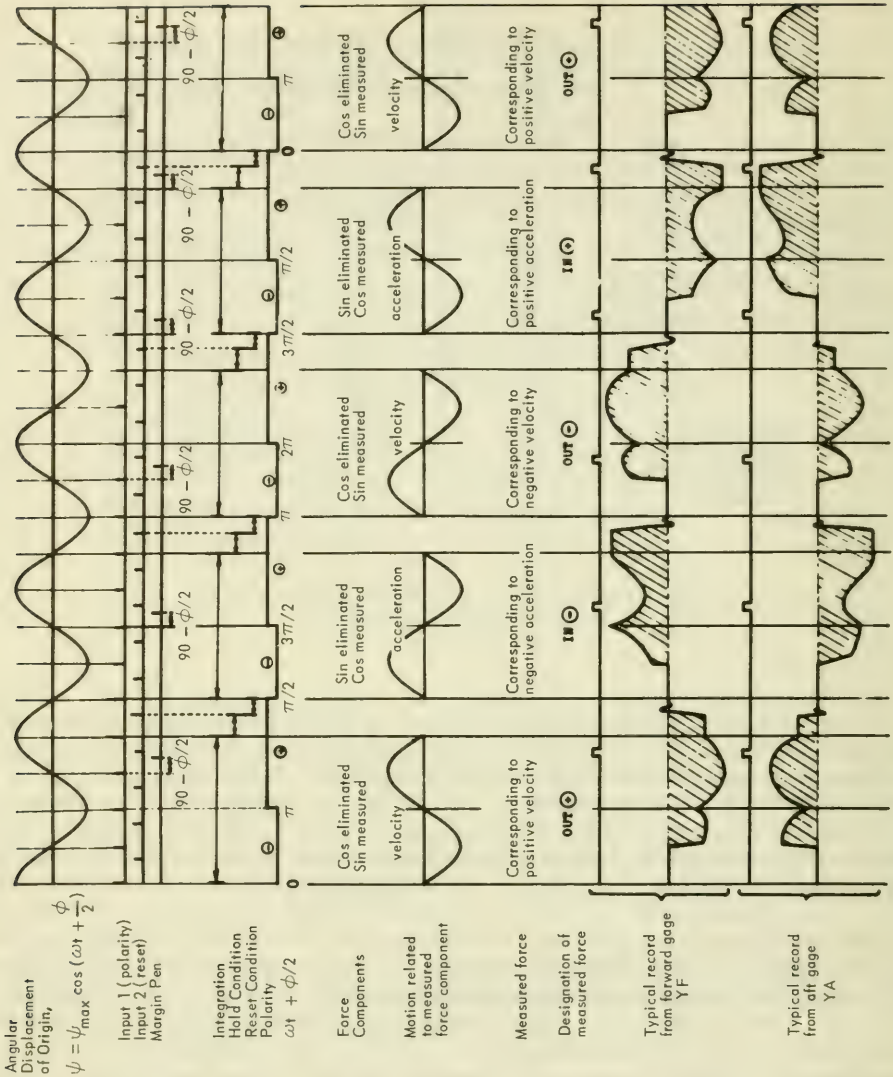


Fig. 16 - Yaw test analysis

the top of the forward scotch yoke. It is seen, for example, that an IN  $\oplus$  record is characterized by margin-pen marks 90 - ( $\phi/2$ ) after the start and stop of the integration period.

The forward and aft IN  $\oplus$  and IN  $\ominus$  forces are of the same order of magnitude but of opposite sign. This is natural, as mass and submerged lateral-area are distributed nearly symmetrical about the origin, and because yaw acceleration causes the forebody to swing the port while the afterbody swings to starboard, and vice versa.

The forward and aft OUT  $\oplus$  (and OUT  $\ominus$ ) forces are of the same order of magnitude but of opposite sign, the aft force being positive. The centrifugal effect of the ship's mass acts in the same direction at both forward and aft gauges, adding to the hydrodynamic force at the forward gauge and reducing the effect of the hydrodynamic force at the aft gauge. The forward gauge-force is therefore usually of greater absolute magnitude than the aft. For ships with pronounced rake of keel, however, the hydrodynamic force aft is greater than that forward and the measured forces become of almost equal absolute magnitude.

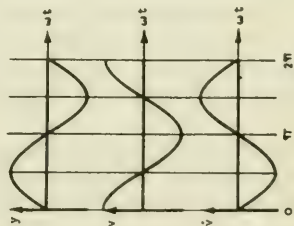
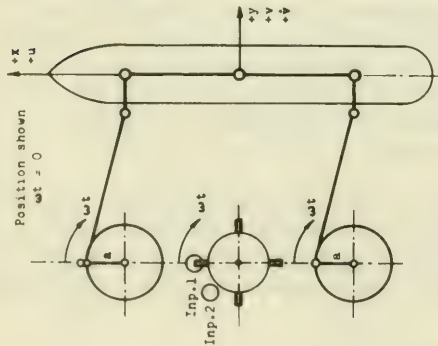
Forces dependent on acceleration are usually of greater absolute magnitude than those dependent on velocity. The relative proportions of the two change with frequency of oscillation and model speed, however, and, for low frequencies or high speeds, the velocity-dependent forces can be the largest.

By adding and subtracting the forward and aft IN and OUT components according to Eqs. (15), side forces and turning moments are obtained as functions of yaw acceleration and yaw velocity. When these values are faired, the slopes of the fairing lines at the origin give the terms  $Y_r - m\dot{x}_G$ ,  $N_r - I_z$ ,  $Y_r - m\dot{u}$ , and  $N_r - m\dot{x}_G\dot{u}$ . These terms contain the effects of known model mass and inertia, which may be eliminated, leaving the hydrodynamic terms  $Y_r$ ,  $N_r$ ,  $Y_r$ , and  $N_r$ . Ship mass and inertia values can then be reintroduced. This ability to account for differences in mass and inertia between model and ship is convenient, as it permits the model to be constructed without paying any regard to its inertia. Model displacement is usually kept to the scaled ship value, and ballast is adjusted until the correct trim is obtained.

### Integration of Forces in Pure-Sway Tests

The forces measured in "pure-sway" tests are treated in a manner exactly analogous to the "pure yaw" tests described above. As seen in Fig. 17, the forces are integrated with sign reversals related to lateral instead of angular displacement. The synchronous switch is thus adjusted to give impulses to Input 1 at positions of zero, maximum, and minimum sway displacement. The measured forces are again designated IN and OUT to indicate in phase or out-of-phase relationships with sway displacement.

Typical records of forces measured at the forward and aft gauges, shown in the bottom right of Fig. 17, are explained in similar manner to the results of "pure yaw" tests. Forward and aft IN  $\oplus$  forces are, in this case, of the same sign because forebody and afterbody accelerate simultaneously to port or



$$y = a \sin \omega t$$

$$\dot{y} = v = a\omega \cos \omega t$$

$$\ddot{y} = \dot{v} = -a\omega^2 \sin \omega t$$

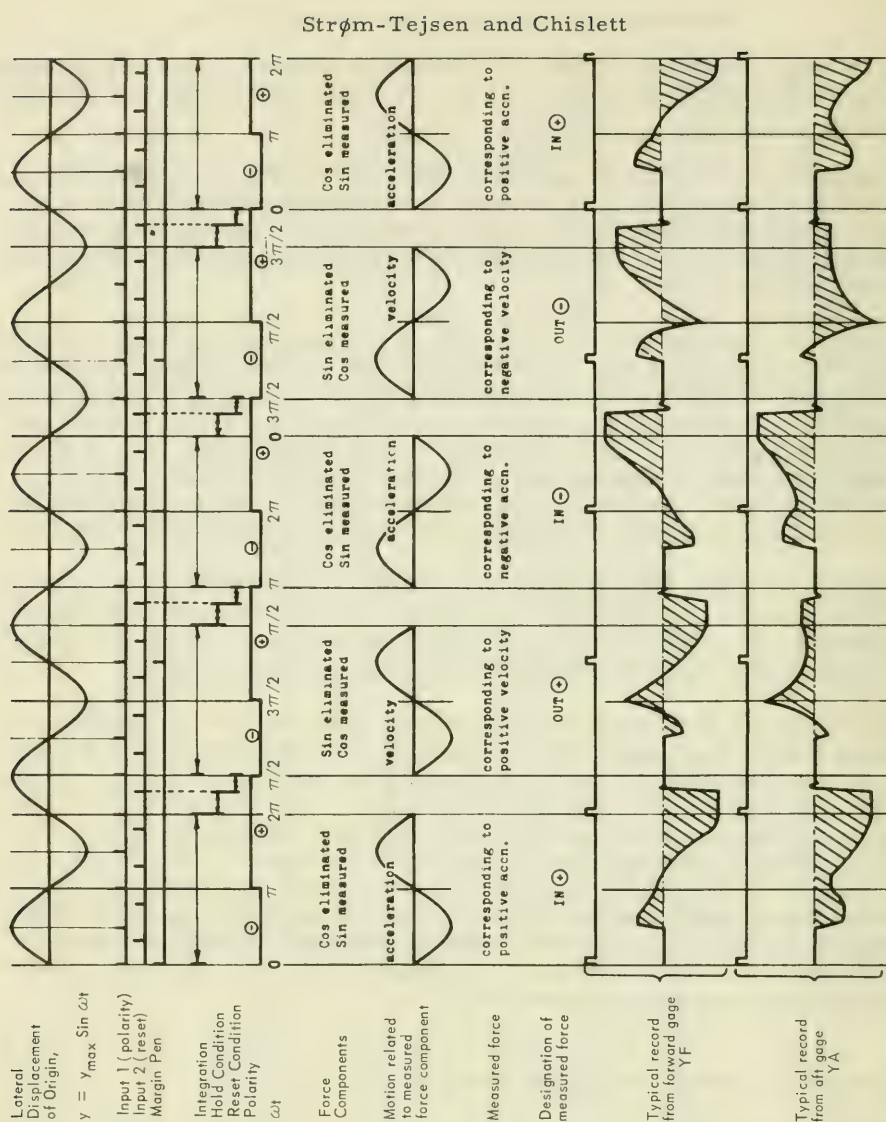


Fig. 17 - Sway test analysis



to starboard. Forward and aft OUT  $\oplus$  forces, resulting from positive sway-velocity (or negative drift-angle) are largely functions of lift, due to circulation built up around the hull, which acts at approximately the quarter-chord point, i.e., close to the forward Y gauge for a 6-m model.

Addition and subtraction of the in-phase force components according to Eqs. (15) give the side force and turning moment due to sway-acceleration. The slopes of straight lines used to fair these data give the terms  $Y_{\dot{v}} - m$  and  $N_{\dot{v}} - m \times G$ , again including the effects of the model mass, the elimination of which results in the hydrodynamic derivatives  $Y_{\dot{v}}$  and  $N_{\dot{v}}$ .

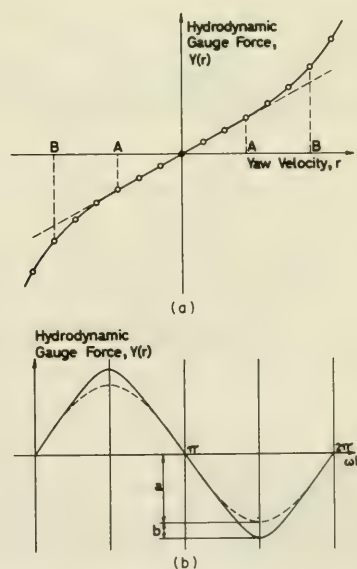
Out-of-phase force components resulting from sway velocity or drift angle are generally to some extent frequency and/or amplitude dependent and are discarded in favor of results obtained from tests made with steady drift angles in the static mode.

### Integration and Interpretation of Nonlinear Forces

It has been assumed in the foregoing that the periodic forces measured in "pure yaw" and "pure sway" tests are proportional to the motions. This is, however, not a necessary condition for the use of the integration technique described but was introduced to simplify the discussion. When the forces are related to the motions in a nonlinear fashion, the same principles can be used to obtain the nonlinear hydrodynamic coefficients.

Suppose, for instance, that for a particular hull the hydrodynamic force or moment resulting from yaw velocity is of the character shown in the top of Fig. 18.

Fig. 18 - Nonlinear force response to steady-state yawing motion (top) and to sinusoidal yawing motion (bottom)





When an oscillator technique is used to generate the yaw velocity, the force-response at a particular instant is basically the same as in the steady state provided frequency of oscillation is sufficiently low. If the sinusoidal angular velocity is of small amplitude, the forces are proportional to the motion and are also sinusoidal. This is the case within the range of linear force-response A-A. Tests made within this area provide the linear terms  $Y_r$  and  $N_r$  as discussed above. It is, however, desirable to explore the full range of yaw velocities that a vessel can experience, and this will probably extend into the nonlinear area indicated by B-B. If an oscillatory test is made in which a sinusoidal angular velocity of this amplitude is impressed on the model, the force-response will be of sinusoidal character within the region of lesser yaw velocities, A-A, but progressively deviate from sinusoidal in the regions A-B resulting in a gauge force of the character illustrated in the bottom of Fig. 18.

A curve of the character shown at the top in Fig. 18 can be expressed with good accuracy by a linear and a cubic term:

$$Y(r) = Y_r r + Y_{rrr} r^3.$$

Since the yaw velocity impressed on the model is sinusoidal, described by

$$r = r_{\max} \sin \omega t,$$

it follows that the cyclic forces experienced by the force gauges (Fig. 18, bottom) are expressed by

$$Y(r) = Y_r r_{\max} \sin \omega t + Y_{rrr} r_{\max}^3 \sin^3 \omega t$$

or

$$Y(r) = a \sin \omega t + b \sin^3 \omega t,$$

where  $a$  and  $b$  are constants in time. Integration over one period with a change of polarity after half a period then gives

$$\int_0^\pi Y(r) - \int_\pi^{2\pi} (Yr) = 4a + \frac{8}{3} b,$$

i.e.,

$$\begin{aligned} \frac{\text{integrated value}}{4} &= a + \frac{2}{3} b \\ &= Y_r r_{\max} + \frac{2}{3} Y_{rrr} r_{\max}^3. \end{aligned}$$

The true cubic term  $Y_{rrr}$  corresponding to the steady-state condition is thus 1-1/2 times greater than the cubic term obtained by fairing force-values which have been integrated over one period of sinusoidal yaw-velocity.

Figure 19 shows an example of nonlinear forces obtained in this manner by means of the HyA planar-motion mechanism. The integrated forces have been obtained as a result of pure yawing motion forced on a 20-ft model of the Mariner hull-form. The model, the condition tested, and the nomenclature are identical to that described in Ref. 5, except that the tests were made at a ship speed of 10 knots instead of 15 knots in order to facilitate the generation of large yaw velocities. The measured forces have been faired with linear and cubic terms, the coefficients of the fairing polynomials being included in the figure. Steady-state

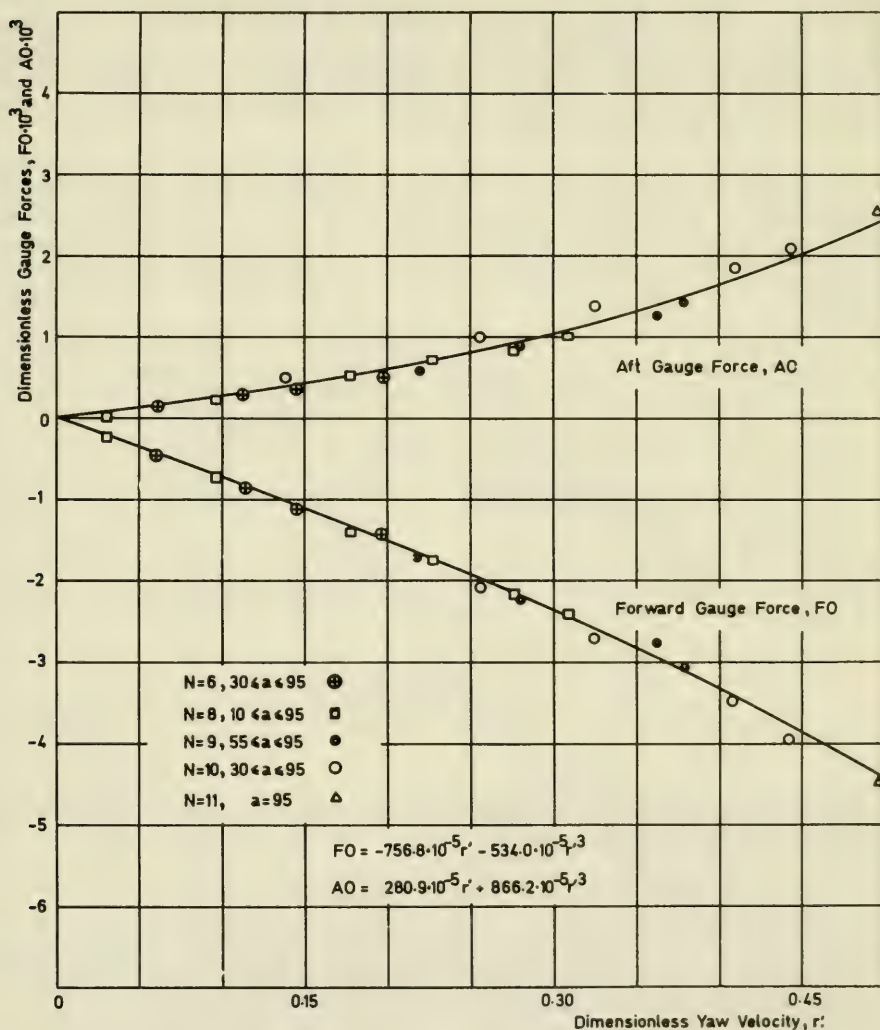


Fig. 19 - Pure yaw results measured with the HyA planar-motion mechanism: transverse (Y) gauge forces as functions of yaw velocity

side forces and turning moments can be obtained from the gauge-forces according to Eqs. (15), the cubic terms having been multiplied by 3/2 as described above.

### Integration and Interpretation of Crosscoupling Terms in Yaw and Drift-Angle Tests

Crosscoupling effects due to simultaneous yaw velocity and drift angle are expressed in the mathematical model by the terms  $Y_{vrr}$ ,  $N_{vrr}$ ,  $Y_{rvv}$ , and  $N_{rvv}$ . Similarly the crosscoupling due to yaw velocity and rudder angle is expressed by  $Y_{\delta rr}$ ,  $N_{\delta rr}$ ,  $Y_{r\delta\delta}$ , and  $N_{r\delta\delta}$ . These two sets of coefficients can be obtained from the "yaw and drift-angle" and the "yaw and rudder-angle" tests respectively, outlined in Fig. 4. As the same principles are used for measurement of both sets of terms, only the derivation of the crosscoupling between yaw velocity and drift angle is described in the following.

The terms  $vrr$  and  $rvv$ , shown diagrammatically in Fig. 20, constitute a flexible means of expressing the experimental data, while conforming to the port and starboard symmetry condition  $f(r, v) = -f(-r, -v)$ . The two terms have essentially the same character, as can be seen by replotting on a base of  $v$  instead of  $r$ .

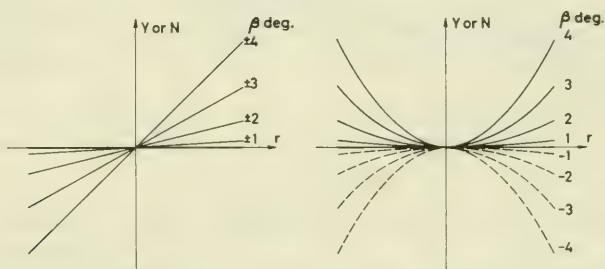


Fig. 20 - Crosscoupling terms  $Y_{rvv}$ ,  $N_{rvv}$ ,  $Y_{vrr}$ , and  $N_{vrr}$

The side force acting on a model in a "yaw and drift-angle" test can be expressed on a time basis by

$$Y(t) = Y_{\dot{r}}\dot{r}(t) + Y_{rr}r(t) + Y_v v + Y_{rvv}r(t)v^2 + Y_{vrr}vr(t)^2, \quad (16)$$

where  $v$  corresponds to drift angle, held constant during each measuring run, and  $r(t)$  and  $\dot{r}(t)$  are the sinusoidally varying yaw velocity and acceleration. Again, the corresponding expression for  $N(t)$  is exactly analogous.

The first three terms in Eq. (16) are known from the "pure yaw" and "static drift angle" tests. The manner in which the last two terms can be derived from the "yaw and drift-angle" test is shown schematically in Fig. 21.

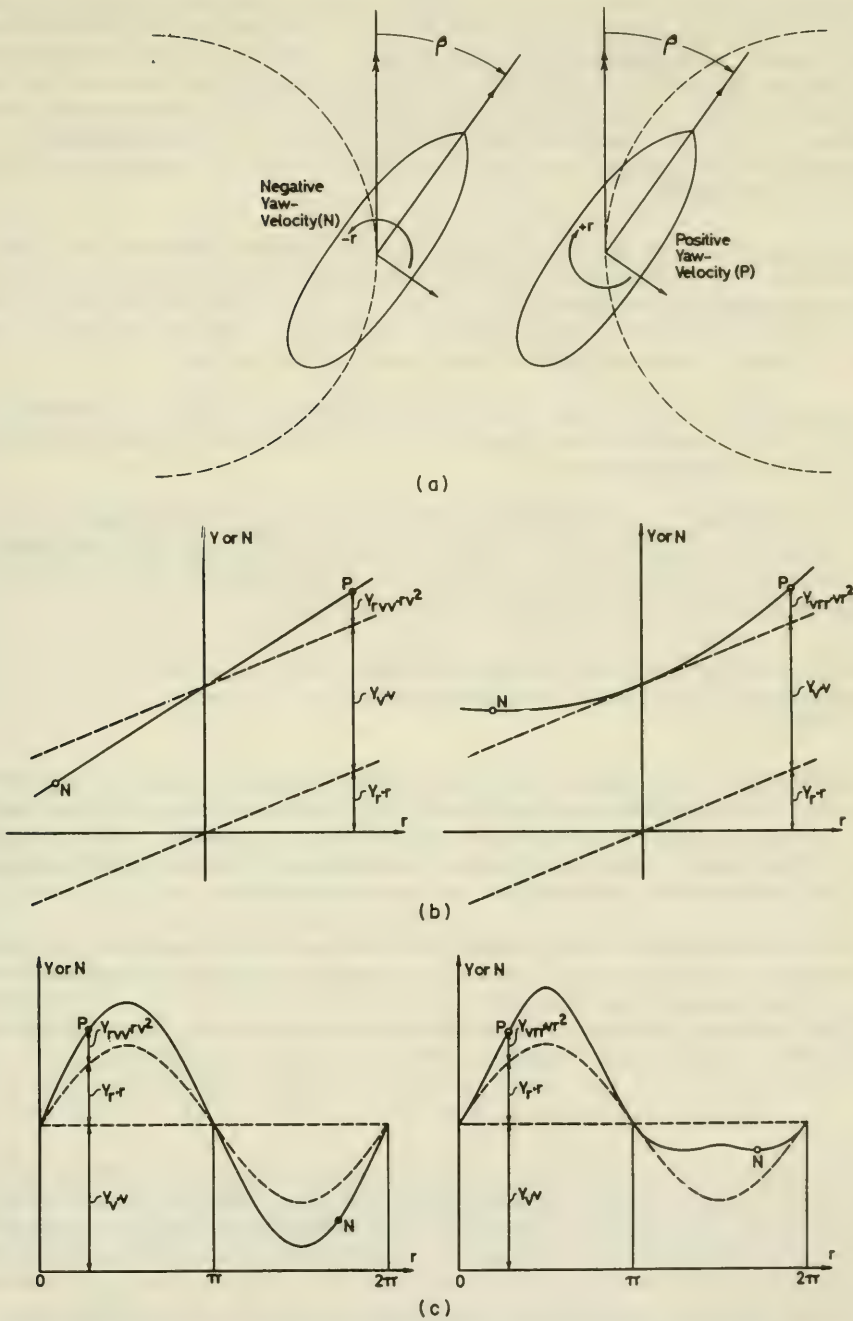


Fig. 21 - Generation, interpretation, and measurement of crosscoupling effects in yaw and drift-angle tests



The top of Fig. 21 illustrates two characteristic situations, one for positive (P) and the other for negative (N) yaw velocity. It is again helpful to relate the periodic forces, resulting from periodic motions generated by the planar-motion mechanism, to the corresponding steady-state measurements shown at the center in Fig. 21. Each of the two center diagrams in Fig. 21 illustrates one of the two terms  $r_{vv}$  and  $v_{rr}$ , assuming the other to be zero. The corresponding forces acting at the Y gauges during one revolution of the mechanism are shown at the bottom in Fig. 21.

It is seen that using the Oscil. program, the in-phase measurements obtained from a "yaw and drift-angle" test correspond to  $Y_r \dot{r}(t)$ , and the out-of-phase measurements correspond to  $Y_r r(t) + Y_{rvv} r(t)v^2$ , whereas the forces corresponding to the remaining two terms are eliminated. The component  $Y_{rvv} r(t)v^2$  is then obtained by subtracting the known value corresponding to  $Y_r r(t)$ .

Similarly, use of the Const. program gives measurements corresponding to  $Y_{vv} + Y_{vrr} vr(t)^2$ , and the component  $Y_{vrr} vr(t)^2$  is obtained by subtracting the known value corresponding to  $Y_{vv}$ .

In practice, the  $r_{vv}$  term has been found to be of significant magnitude and the  $v_{rr}$  term to be small. The  $r_{vv}$  term is linear with  $r$  and thus independent of the range of yaw velocity, whereas the  $v_{rr}$  term is second order in  $r$  and can only be measured if an adequate range is covered.

#### DESIGN OF THE EXPERIMENTAL PROGRAM AND DETERMINATION OF THE HYDRODYNAMIC COEFFICIENTS

The complete experimental program, which should be carried out in order to determine the hydrodynamic coefficients, would, in addition to the various planar-motion mechanism tests illustrated in Figs. 2, 3, and 4, consist of conventional open-water, resistance, and self-propulsion tests and an experiment for the determination of the model polar moment of inertia.

Two of the coefficients are furthermore obtained from numerical calculations instead of experimentally. The added mass of the ship in surge acceleration  $X_u$  is, for instance, normally assigned a value of  $-0.05 m$  based on theoretical considerations. Similarly, the ship moment of inertia  $I_z$  is computed on the basis of the longitudinal weight distribution of the full-scale ship.

#### Design of the Experimental Program

The range of motion and rudder parameters explored during testing should in principle cover the range of subsequent simulation. Sway and yaw acceleration, speed loss, drift angle, yaw velocity, and rudder angle should therefore be varied systematically up to the values corresponding to maximum-rudder manoeuvres for the free-sailing ship. Typical values for the range of motion parameters experienced by a cargo ship during the execution of a 35-degree rudder-angle turning circle and a 20-20-degree zig-zag manoeuvre are given in Table 2. Corresponding maximum values obtained with naval ships are often 1.5 to 2 times greater.

Table 2  
Typical Range of Dimensionless Motion Parameters for a Cargo Ship

| Motion Parameter | Typical Range of Values  |             |                      |
|------------------|--------------------------|-------------|----------------------|
|                  | 35-Degree Turning Circle |             | 20-20-Degree Zig-Zag |
|                  | Transition Period        | Steady Turn |                      |
| $u'$             | —                        | - 0.6       | - 0.3                |
| $v'$             | —                        | $\pm 0.16$  | $\pm 0.13$           |
| $r'$             | —                        | $\pm 0.42$  | $\pm 0.33$           |
| $\dot{u}'$       | 0.10                     | —           | 0.03                 |
| $\dot{v}'$       | $\pm 0.12$               | —           | $\pm 0.12$           |
| $\dot{r}'$       | $\pm 0.55$               | —           | $\pm 0.50$           |

When testing with the HyA planar-motion mechanism, the range of motion parameters a ship will encounter can be adequately covered with the exception of yaw velocity. In the static mode of operation, for example, drift-angle tests can be executed with drift angles up to  $\pm 20$  degrees corresponding to  $v' = \pm 0.34$ , which is comparable to the maximum sway velocity experienced by naval ships. In the dynamic mode of operation, the range of motion parameters can be adjusted by varying the amplitude at the scotch yokes and the shaft revolutions. Adequate coverage of the desired range of yaw and sway accelerations can easily be obtained in these tests.

For the crucial yaw velocity, Fig. 22 gives, for maximum amplitude, the relationship between nondimensional yaw velocity and shaft revolutions as a function of model speed and model length. For example, in the case of a cargo ship model with a maximum nondimensional yaw velocity  $r' = 0.4$ , model length 6 metres, and model speed 1.4 metres/sec, Fig. 22 shows that shaft revolutions of 12 per minute would be adequate. Although this is well within the structural capability of the mechanism, experience has shown the practical limit of operation to be around 8 to 10 revolutions per minute.

This limitation is due to wave reflections from the basin walls, which for higher shaft revolutions interfere with the force measurements. As the problem is associated with a system of standing waves built up in the basin at certain frequencies, it is in general independent of model speed and size.

With a maximum shaft rpm of approximately 10, it is seen from Fig. 22 that the HyA planar-motion mechanism gives an acceptable coverage for 6 to 8 metre cargo ship models, whereas for naval ships, when smaller models would often be used, the desired range of yaw velocities can be covered only at lower speeds.

It would normally be desirable to cover the complete range as given in Table 2. In some circumstances it is possible, however, to economize on the number of captive-model experiments necessary for accurate simulation of a

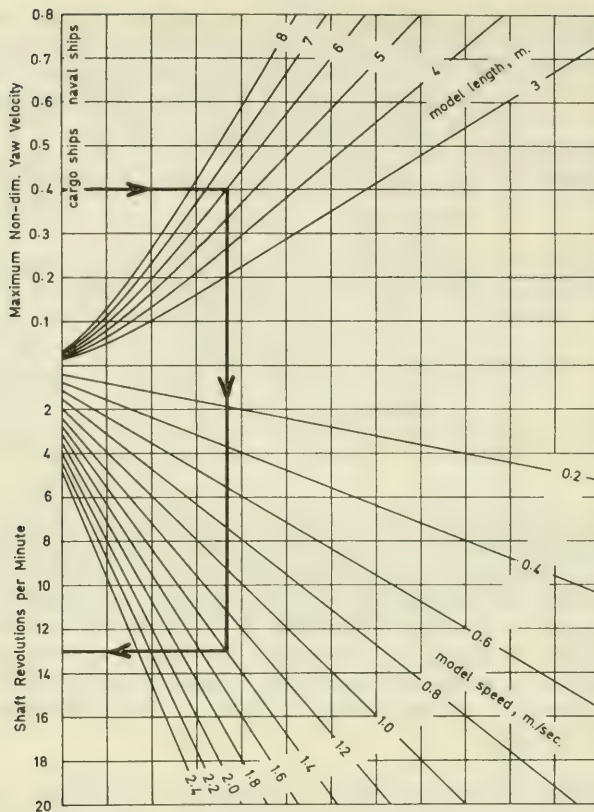


Fig. 22 - Relationship between the non-dimensional yaw velocity and the shaft revolutions as a function of model speed and model size (for 100 mm amplitude at the scotch yokes)

particular manoeuvre. For example, if interest is centred on the steady turning state corresponding to maximum rudder-angle, then acceleration terms are of minor importance and tests need only be made for one rudder angle, and for drift angles, yaw velocities, and speeds covering a range centred about the estimated equilibrium state.

On the other hand, to obtain coefficients which truly represent the symmetry inherent in ship forms and reflected in the design of the mathematical model, it is necessary to plan the experiments accordingly. In particular, two-variable crosscoupling terms should preferably be obtained from measurements corresponding to a range of parameters in which at least one of the parameters is assigned both positive and negative values. The numerical values of the coefficients will then uniquely describe the steering and manoeuvring properties of



the particular ship in question and can be used without reservation within the performance envelope corresponding to the ship's rudder action, instead of only being valid within a limited area.

In designing the experimental program, emphasis should be placed on the more important terms. In general, the six linear velocity-dependent terms,  $Y_v$ ,  $N_v$ ,  $Y_r$ ,  $Y_{\delta}$ , and  $N_{\delta}$ , are predominant, while acceleration-dependent, higher-order, and crosscoupling terms are of lesser influence (though many are far from negligible) in the accurate prediction of manoeuvres.

The larger hydrodynamic acceleration-dependent terms,  $Y_{\ddot{v}}$  and  $N_{\ddot{v}}$ , are of the same order of magnitude as their related mass  $m$  and inertia  $I_z$  values, which effectively doubles their accuracy. Acceleration terms are, furthermore, influential only in transitional manoeuvres, having no influence during steady turning states.

Sufficient experimental measurements should be made to justify the use of least-squares fairing procedures. If too few points are used, the fairing expression may give a good fit without truly representing the trend of the actual curves.

It is generally recognized that the propeller slipstream has a considerable influence on the characteristics of the rudder coefficients. In carrying out the model tests, it is consequently important to scale the slipstream as accurately as possible, for instance by executing the tests for a propeller rpm corresponding to the ship propulsion-point as opposed to that of the model. When a ship enters a manoeuvre, its speed will reduce and the propeller rpm will consequently vary somewhat, the variation being dependent on the type of engine and the engine control settings maintained during the manoeuvre. If it is simply assumed that a constant propeller rpm is maintained during the entire ship manoeuvre, the model tests carried out to measure the speed-dependent coefficients such as  $Y_{\delta_{uu}}$  and  $Y_{\delta_{uuu}}$  can simply be made with the model propeller rpm maintained at the value corresponding to the initial speed of the ship. It is also possible, however, as will be discussed later, to compute the variations of propeller rpm which occur as a ship reduces speed and to obtain speed-dependent coefficients which correspond to specified machinery types and control settings.

### Analysis of Force Measurements

In the last stage of the testing procedure, the forces and moments measured in the course of the various planar-motion mechanism tests, are analyzed to yield the different hydrodynamic coefficients. The analysis consists, in principle, of fairing the experimental data, using the mathematical model as the approximating function, and obtaining the hydrodynamic coefficients as those coefficients which give the best fit to the measured values.

The results from a "static drift angle" test would, for example, consist of measurements of  $X$  and  $Y$  forces and the  $N$  moment as functions of drift angle and rudder angle. According to the mathematical model, these forces and moments are described by the expressions



$$X = X_* + X_{vv}v^2 + X_{v\delta}v\delta + X_{\delta\delta}\delta^2$$

$$Y = Y_* + Y_vv + Y_{vvv}v^3 + Y_{v\delta\delta}v\delta^2 + Y_{\delta\delta}\delta + Y_{\delta\delta\delta}\delta^3 + Y_{\delta vv}\delta v^2$$

$$N = N_* + N_vv + N_{vvv}v^3 + N_{v\delta\delta}v\delta^2 + N_{\delta\delta}\delta + N_{\delta\delta\delta}\delta^3 + N_{\delta vv}\delta v^2$$

Hence, the hydrodynamic coefficients

$$X_*, X_{vv}, X_{v\delta}, X_{\delta\delta}$$

$$Y_*, Y_v, Y_{vvv}, Y_{v\delta\delta}, Y_{\delta\delta}, Y_{\delta\delta\delta}, Y_{\delta vv}$$

$$N_*, N_v, N_{vvv}, N_{v\delta\delta}, N_{\delta\delta}, N_{\delta\delta\delta}, N_{\delta vv}$$

are simply the coefficients of these expressions, when they are used as approximating functions in fairing the experimental measurements.

Figure 23 and Table 3 further illustrate the principle of the analysis. The figure shows results from a "static drift-angle" test carried out for the Mariner hull-form. The test was executed at 15 knots, and forces and moments were measured over the complete range of drift angles ( $\beta = -9.5, -7.5, -4.5, -2.0, -1.0, -0.5, 0.0, 0.5, 1.0, 2.0, 4.5, 7.0, 9.5$  degrees) and of rudder angles ( $\delta = 0$  to  $\pm 40$  degrees in steps of 10 degrees). The experimental points are plotted in the figure as circles, whereas the solid curves show the expressions obtained from fairing the data.

Table 3 presents the corresponding output from the computer program used in the analysis. The experimental and faired values are indicated by e and f, and the fairing polynomials are given at the bottom of the table.

Similar principles are applied in the analysis of results from tests made in the dynamic mode of operation. Figure 24 and Table 4 show, for example, the analysis of forces measured in the "pure yaw" test.

The out-of-phase forces are in this case faired by the expressions

$$X = (X_{rr} - m\dot{x}_G) \dot{r}^2$$

$$Y = (Y_r - m\dot{u}) \dot{r} + \frac{2}{3} Y_{rrr} \dot{r}^3$$

$$N = (N_r - m\dot{x}_G \dot{u}) \dot{r} + \frac{2}{3} N_{rrr} \dot{r}^3$$

and the in-phase forces by expressions of the form

$$Y = (Y_r - m\dot{x}_G) \dot{r}$$

$$N = (N_r - I_z \dot{r}) \dot{r}$$

The hydrodynamic coefficients are determined after corrections for differences in mass and moment of inertia for model and ship.

# Prediction of Steering and Manoeuvring of Ships

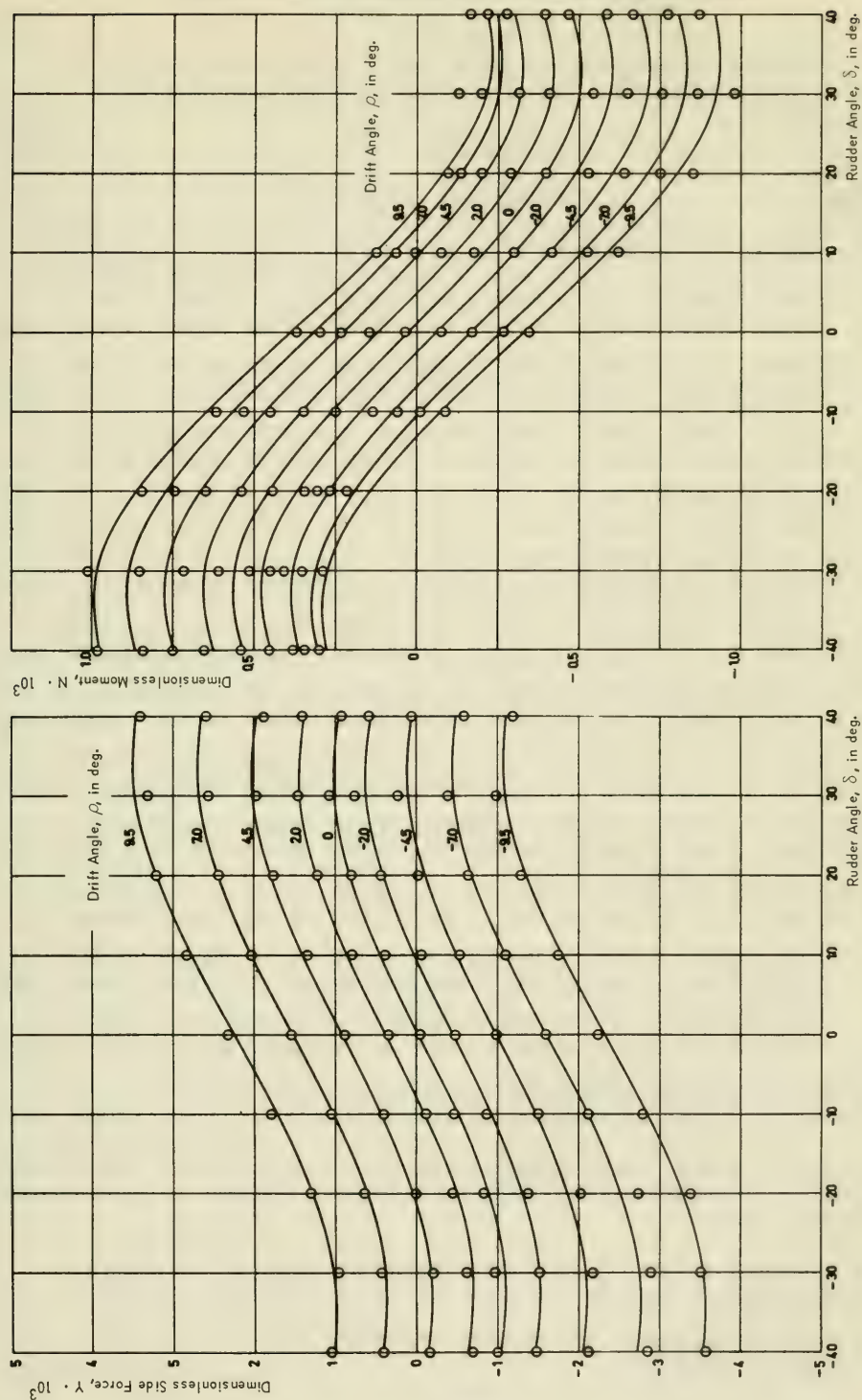


Fig. 23 - Example of results from a static drift-angle test: side force and turning moment as functions of drift angle and rudder angle

Table 3  
Example of Analysis of a Static Drift-Angle Test: Fairing  
of Measured Side Force and Turning Moment

| Drift<br>Angle*<br>(deg)                                                                                                                                        | Nondimensional Side Force, $Y \times 10^3$ |        |        |        |        |        |        |        |        |       |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------|--------|--------|--------|--------|--------|--------|--------|--------|-------|
|                                                                                                                                                                 | Rudder Angle in deg:                       |        |        |        |        |        |        |        |        | rms   |
|                                                                                                                                                                 | -40.0                                      | -30.0  | -20.0  | -10.0  | 0.0    | 10.0   | 20.0   | 30.0   | 40.0   |       |
| 9.5e                                                                                                                                                            | 1.050                                      | 0.962  | 1.304  | 1.794  | 2.326  | 2.833  | 3.213  | 3.311  | 3.405  | 0.081 |
| f                                                                                                                                                               | 1.020                                      | 1.016  | 1.277  | 1.715  | 2.242  | 2.769  | 3.208  | 3.472  | 3.471  |       |
| 7.0e                                                                                                                                                            | 0.408                                      | 0.429  | 0.639  | 1.048  | 1.537  | 2.030  | 2.430  | 2.559  | 2.588  | 0.052 |
| f                                                                                                                                                               | 0.404                                      | 0.375  | 0.611  | 1.023  | 1.524  | 2.025  | 2.439  | 2.676  | 2.649  |       |
| 4.5e                                                                                                                                                            | -0.151                                     | -0.214 | -0.003 | 0.396  | 0.878  | 1.340  | 1.759  | 1.970  | 1.876  | 0.049 |
| f                                                                                                                                                               | -0.136                                     | -0.183 | 0.036  | 0.430  | 0.913  | 1.396  | 1.792  | 2.011  | 1.965  |       |
| 2.0e                                                                                                                                                            | -0.693                                     | -0.618 | -0.445 | -0.117 | 0.343  | 0.791  | 1.216  | 1.457  | 1.402  | 0.044 |
| f                                                                                                                                                               | -0.637                                     | -0.693 | -0.485 | -0.101 | 0.372  | 0.845  | 1.230  | 1.438  | 1.382  |       |
| 1.0e                                                                                                                                                            | -0.877                                     | -0.847 | -0.630 | -0.251 | 0.131  | 0.540  | 0.981  | 1.214  | 1.176  | 0.053 |
| f                                                                                                                                                               | -0.835                                     | -0.893 | -0.687 | -0.304 | 0.167  | 0.638  | 1.021  | 1.227  | 1.169  |       |
| 0.5e                                                                                                                                                            | -0.932                                     | -0.922 | -0.770 | -0.391 | 0.040  | 0.470  | 0.873  | 1.156  | 1.046  | 0.042 |
| f                                                                                                                                                               | -0.935                                     | -0.994 | -0.788 | -0.405 | 0.065  | 0.536  | 0.918  | 1.124  | 1.065  |       |
| 0.0e                                                                                                                                                            | -1.006                                     | -0.972 | -0.825 | -0.457 | -0.047 | 0.376  | 0.796  | 1.073  | 0.921  | 0.062 |
| f                                                                                                                                                               | -1.036                                     | -1.094 | -0.889 | -0.507 | -0.036 | 0.434  | 0.816  | 1.022  | 0.963  |       |
| -0.5e                                                                                                                                                           | -1.098                                     | -1.086 | -0.947 | -0.531 | -0.132 | 0.307  | 0.749  | 1.067  | 0.902  | 0.076 |
| f                                                                                                                                                               | -1.138                                     | -1.196 | -0.990 | -0.608 | -0.138 | 0.333  | 0.715  | 0.921  | 0.862  |       |
| -1.0e                                                                                                                                                           | -1.234                                     | -1.196 | -1.034 | -0.651 | -0.219 | 0.229  | 0.625  | 0.900  | 0.794  | 0.057 |
| f                                                                                                                                                               | -1.241                                     | -1.300 | -1.093 | -0.710 | -0.239 | 0.232  | 0.614  | 0.821  | 0.762  |       |
| -2.0e                                                                                                                                                           | -1.484                                     | -1.510 | -1.369 | -0.871 | -0.476 | -0.060 | 0.442  | 0.758  | 0.584  | 0.067 |
| f                                                                                                                                                               | -1.455                                     | -1.511 | -1.302 | -0.918 | -0.445 | 0.028  | 0.413  | 0.621  | 0.564  |       |
| -4.5e                                                                                                                                                           | -2.123                                     | -2.174 | -2.016 | -1.496 | -0.988 | -0.541 | -0.024 | 0.236  | 0.057  | 0.089 |
| f                                                                                                                                                               | -2.038                                     | -2.083 | -1.864 | -1.469 | -0.986 | -0.503 | -0.108 | 0.110  | 0.063  |       |
| -7.0e                                                                                                                                                           | -2.852                                     | -2.885 | -2.727 | -2.123 | -1.585 | -1.096 | -0.635 | -0.386 | -0.578 | 0.111 |
| f                                                                                                                                                               | -2.722                                     | -2.749 | -2.511 | -2.098 | -1.597 | -1.096 | -0.683 | -0.447 | -0.476 |       |
| -9.5e                                                                                                                                                           | -3.572                                     | -3.509 | -3.380 | -2.789 | -2.235 | -1.750 | -1.284 | -0.984 | -1.182 | 0.076 |
| f                                                                                                                                                               | -3.543                                     | -3.544 | -3.281 | -2.842 | -2.315 | -1.788 | -1.350 | -1.088 | -1.092 |       |
| rms                                                                                                                                                             | 0.054                                      | 0.085  | 0.092  | 0.050  | 0.040  | 0.057  | 0.042  | 0.101  | 0.058  | 0.065 |
| Fairing Polynomial†                                                                                                                                             |                                            |        |        |        |        |        |        |        |        |       |
| $Y \times 10^5 = -3.63 - 1160 \times v' - 8078 \times v'^3 + 1190 \times v'^2 \delta - 3.83 \times v' \delta^2 - 276.5 \times \delta^3$ $+ 277.9 \times \delta$ |                                            |        |        |        |        |        |        |        |        |       |

See footnotes at end of table.

(Table continues)

Table 3 (Cont'd)

| Drift Angle*<br>(deg)                                                                                                                                            | Nondimensional Moment, $N \times 10^{-3}$ |       |       |        |        |        |        |        |        |       |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------|-------|-------|--------|--------|--------|--------|--------|--------|-------|
|                                                                                                                                                                  | Rudder Angle in deg:                      |       |       |        |        |        |        |        |        | rms   |
|                                                                                                                                                                  | -40.0                                     | -30.0 | -20.0 | -10.0  | 0.0    | 10.0   | 20.0   | 30.0   | 40.0   |       |
| 9.5e                                                                                                                                                             | 0.983                                     | 1.013 | 0.846 | 0.614  | 0.363  | 0.125  | -0.099 | -0.129 | -0.164 | 0.041 |
| f                                                                                                                                                                | 0.976                                     | 0.984 | 0.860 | 0.647  | 0.390  | 0.131  | -0.086 | -0.216 | -0.217 |       |
| 7.0e                                                                                                                                                             | 0.842                                     | 0.852 | 0.744 | 0.532  | 0.298  | 0.064  | -0.137 | -0.201 | -0.217 | 0.030 |
| f                                                                                                                                                                | 0.867                                     | 0.884 | 0.770 | 0.567  | 0.320  | 0.072  | -0.134 | -0.253 | -0.242 |       |
| 4.5e                                                                                                                                                             | 0.751                                     | 0.715 | 0.649 | 0.452  | 0.233  | 0.006  | -0.200 | -0.314 | -0.274 | 0.025 |
| f                                                                                                                                                                | 0.747                                     | 0.771 | 0.663 | 0.467  | 0.227  | -0.013 | -0.211 | -0.322 | -0.302 |       |
| 2.0e                                                                                                                                                             | 0.654                                     | 0.609 | 0.539 | 0.348  | 0.146  | -0.076 | -0.289 | -0.409 | -0.395 | 0.026 |
| f                                                                                                                                                                | 0.625                                     | 0.652 | 0.547 | 0.355  | 0.120  | -0.117 | -0.309 | -0.415 | -0.390 |       |
| 1.0e                                                                                                                                                             | 0.605                                     | 0.586 | 0.497 | 0.311  | 0.103  | -0.124 | -0.335 | -0.463 | -0.414 | 0.022 |
| f                                                                                                                                                                | 0.578                                     | 0.605 | 0.501 | 0.309  | 0.074  | -0.161 | -0.353 | -0.458 | -0.432 |       |
| 0.5e                                                                                                                                                             | 0.581                                     | 0.553 | 0.485 | 0.273  | 0.073  | -0.140 | -0.363 | -0.506 | -0.444 | 0.025 |
| f                                                                                                                                                                | 0.555                                     | 0.582 | 0.478 | 0.286  | 0.051  | -0.184 | -0.375 | -0.480 | -0.453 |       |
| 0.0e                                                                                                                                                             | 0.541                                     | 0.515 | 0.443 | 0.249  | 0.037  | -0.177 | -0.398 | -0.543 | -0.465 | 0.025 |
| f                                                                                                                                                                | 0.532                                     | 0.559 | 0.455 | 0.263  | 0.028  | -0.207 | -0.398 | -0.502 | -0.475 |       |
| -0.5e                                                                                                                                                            | 0.525                                     | 0.505 | 0.432 | 0.235  | 0.029  | -0.199 | -0.426 | -0.579 | -0.512 | 0.027 |
| f                                                                                                                                                                | 0.510                                     | 0.536 | 0.432 | 0.240  | 0.005  | -0.230 | -0.421 | -0.525 | -0.498 |       |
| -1.0e                                                                                                                                                            | 0.483                                     | 0.489 | 0.396 | 0.198  | -0.004 | -0.231 | -0.449 | -0.593 | -0.519 | 0.022 |
| f                                                                                                                                                                | 0.488                                     | 0.514 | 0.409 | 0.218  | -0.018 | -0.253 | -0.444 | -0.548 | -0.521 |       |
| -2.0e                                                                                                                                                            | 0.453                                     | 0.449 | 0.345 | 0.134  | -0.074 | -0.299 | -0.528 | -0.651 | -0.586 | 0.030 |
| f                                                                                                                                                                | 0.447                                     | 0.472 | 0.366 | 0.173  | -0.063 | -0.299 | -0.491 | -0.595 | -0.569 |       |
| -4.5e                                                                                                                                                            | 0.379                                     | 0.407 | 0.306 | 0.059  | -0.170 | -0.415 | -0.637 | -0.757 | -0.667 | 0.028 |
| f                                                                                                                                                                | 0.359                                     | 0.378 | 0.267 | 0.070  | -0.171 | -0.411 | -0.606 | -0.714 | -0.691 |       |
| -7.0e                                                                                                                                                            | 0.344                                     | 0.353 | 0.268 | -0.011 | -0.268 | -0.524 | -0.749 | -0.864 | -0.773 | 0.042 |
| f                                                                                                                                                                | 0.299                                     | 0.309 | 0.190 | -0.015 | -0.263 | -0.510 | -0.713 | -0.828 | -0.810 |       |
| -9.5e                                                                                                                                                            | 0.303                                     | 0.288 | 0.215 | -0.088 | -0.345 | -0.620 | -0.859 | -0.979 | -0.871 | 0.044 |
| f                                                                                                                                                                | 0.274                                     | 0.273 | 0.142 | -0.074 | -0.333 | -0.591 | -0.803 | -0.928 | -0.920 |       |
| rms                                                                                                                                                              | 0.024                                     | 0.035 | 0.035 | 0.021  | 0.019  | 0.027  | 0.026  | 0.047  | 0.028  | 0.029 |
| Fairing Polynomial†                                                                                                                                              |                                           |       |       |        |        |        |        |        |        |       |
| $N \times 10^5 = + 2.83 - 263.5 \times v + 1636 \times v'^3 - 489.0 \times v'^2 \delta + 12.54 \times v' \delta^2 + 136.7 \times \delta^3 - 138.8 \times \delta$ |                                           |       |       |        |        |        |        |        |        |       |

\*e = experimental points

f = faired points

rms = root mean square of differences.

† $v'$  = nondimensional sway velocity $\delta$  = rudder angle in radians.



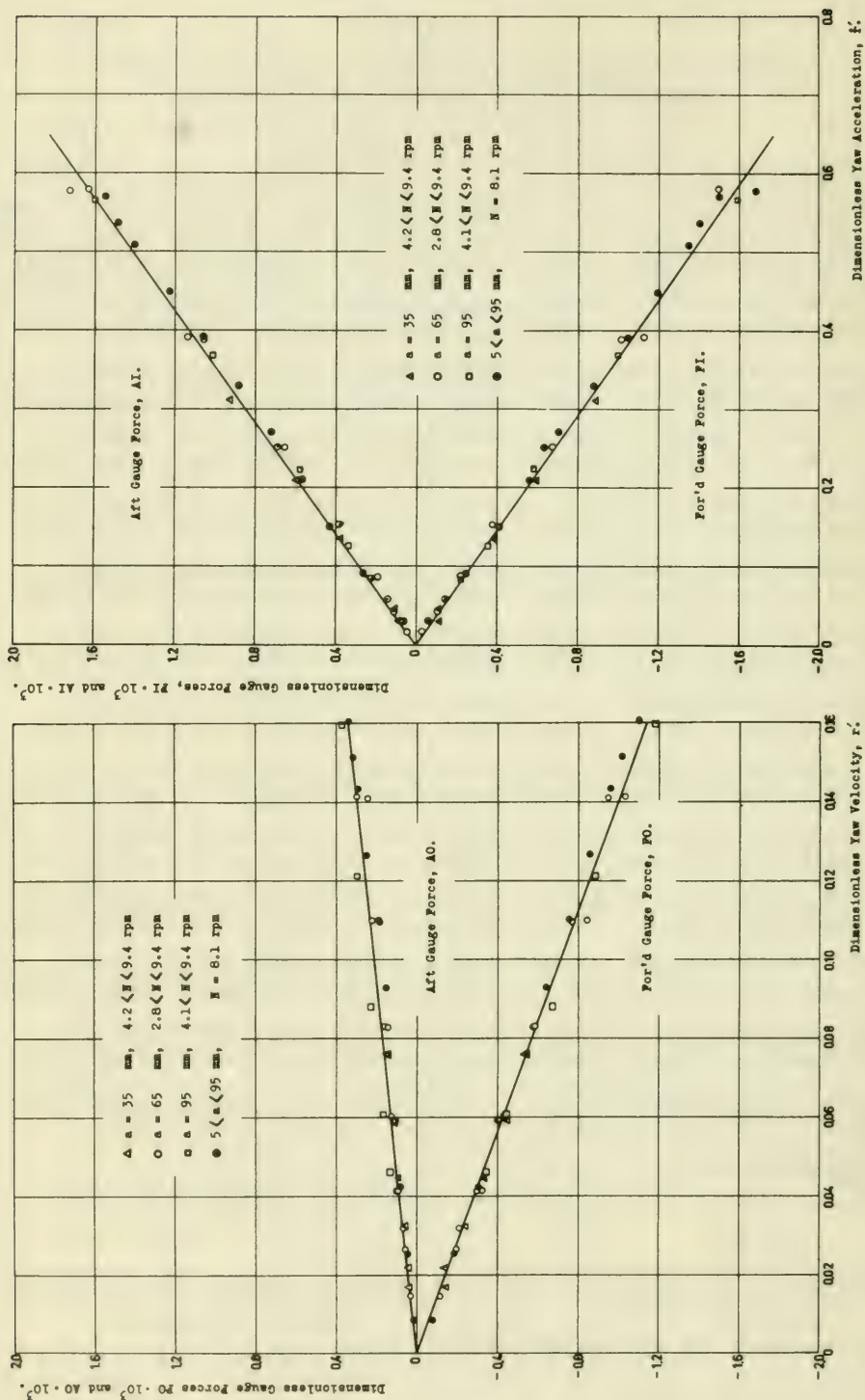


Fig. 24 - Example of results from a pure-yaw test: transverse in-phase and out-of-phase forces as functions of yaw velocity and acceleration

# Prediction of Steering and Manoeuvring of Ships

## Table 4 Example of Analysis of a Pure Yaw Test

| Hydro- and Aerodynamics Laboratory<br>Lyngby, Denmark           |                               |                                             |                                             | PMM Reduced Experimental Data<br>from Pure Yaw Tests |                                             |                                             |                                             | Page 1     |                 |
|-----------------------------------------------------------------|-------------------------------|---------------------------------------------|---------------------------------------------|------------------------------------------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|------------|-----------------|
| Order No.: 6427                                                 |                               | Test Made: 10-7-1964                        |                                             | Model Speed: 1.533 m/sec                             |                                             | Ship Speed: 17.003 knots                    |                                             |            |                 |
| Propeller No.: 6127                                             |                               | Model Scale: 1/32.550                       |                                             |                                                      |                                             |                                             |                                             |            |                 |
| Experimental Data                                               |                               |                                             |                                             |                                                      |                                             |                                             |                                             |            |                 |
| Run No.                                                         | N PMM<br>(1/min)              | a PMM<br>(mm)                               | Phase Angle<br>(deg)                        | Carr. Speed<br>(m/sec)                               | FO<br>(kg)                                  | AO<br>(kg)                                  | FI<br>(kg)                                  | AI<br>(kg) | XF + XA<br>(kg) |
| 27                                                              | 8.07                          | 95                                          | 60.0                                        | 1.531                                                | -13.24                                      | 4.31                                        | -16.14                                      | 14.01      | -2.44           |
| 28                                                              | 8.06                          | 75                                          | 60.0                                        | 1.532                                                | -10.44                                      | 3.53                                        | -12.46                                      | 11.08      | -1.15           |
| 29                                                              | 8.07                          | 55                                          | 60.0                                        | 1.532                                                | -7.56                                       | 2.57                                        | -9.15                                       | 8.12       | -0.58           |
| 30                                                              | 8.07                          | 35                                          | 60.0                                        | 1.532                                                | -4.82                                       | 1.82                                        | -6.02                                       | 5.30       | -0.29           |
| 31                                                              | 8.08                          | 25                                          | 60.0                                        | 1.531                                                | -3.47                                       | 1.31                                        | -4.37                                       | 4.02       | -0.14           |
| 32                                                              | 8.03                          | 15                                          | 60.0                                        | 1.531                                                | -2.19                                       | 0.84                                        | -2.70                                       | 2.77       | 0.00            |
| 52                                                              | 8.06                          | 45                                          | 60.0                                        | 1.533                                                | -6.35                                       | 2.31                                        | -7.74                                       | 6.81       | -0.43           |
| 53                                                              | 8.06                          | 28                                          | 60.0                                        | 1.533                                                | -3.91                                       | 1.46                                        | -4.93                                       | 4.30       | -0.29           |
| Run No.                                                         | Angular Velocity<br>(nondim.) | Y Force from Pure Yaw Velocity              |                                             |                                                      | N Moment from Pure Yaw Velocity             |                                             |                                             |            |                 |
|                                                                 |                               | Y(r)<br>Expt.<br>$\times 10^3$<br>(nondim.) | Y(r)<br>Fair.<br>$\times 10^3$<br>(nondim.) | Y(r)<br>Diff.<br>$\times 10^3$<br>(nondim.)          | N(r)<br>Expt.<br>$\times 10^3$<br>(nondim.) | N(r)<br>Fair.<br>$\times 10^3$<br>(nondim.) | N(r)<br>Diff.<br>$\times 10^3$<br>(nondim.) |            |                 |
| 27                                                              | 0.1735                        | -1.5496                                     | -1.5141                                     | -0.0355                                              | -0.5059                                     | -0.5114                                     | 0.0055                                      |            |                 |
| 28                                                              | 0.1367                        | -1.1965                                     | -1.1931                                     | -0.0033                                              | -0.4022                                     | -0.4030                                     | 0.0008                                      |            |                 |
| 29                                                              | 0.1004                        | -0.8643                                     | -0.8760                                     | 0.0117                                               | -0.2918                                     | -0.2959                                     | 0.0041                                      |            |                 |
| 30                                                              | 0.0639                        | -0.5208                                     | -0.5574                                     | 0.0365                                               | -0.1911                                     | -0.1883                                     | -0.0029                                     |            |                 |
| 31                                                              | 0.0457                        | -0.3755                                     | -0.3991                                     | 0.0236                                               | -0.1379                                     | -0.1348                                     | -0.0031                                     |            |                 |
| 32                                                              | 0.0273                        | -0.2334                                     | -0.2379                                     | 0.0045                                               | -0.0875                                     | -0.0804                                     | -0.0071                                     |            |                 |
| 52                                                              | 0.0820                        | -0.7004                                     | -0.7154                                     | 0.0151                                               | -0.2490                                     | -0.2416                                     | -0.0074                                     |            |                 |
| 53                                                              | 0.0501                        | -0.4240                                     | -0.4373                                     | 0.0133                                               | -0.1544                                     | -0.1477                                     | -0.0067                                     |            |                 |
| rms = 0.0216                                                    |                               |                                             |                                             |                                                      | rms = 0.0052                                |                                             |                                             |            |                 |
| Fairing Expressions: $Y(r) \times 10^5 = -872.77 \times r$      |                               |                                             |                                             |                                                      | $N(r) \times 10^5 = -294.79 \times r$       |                                             |                                             |            |                 |
| Run No.                                                         | Angular Velocity<br>(nondim.) | X Force from Pure Yaw Velocity              |                                             |                                                      |                                             |                                             |                                             |            |                 |
|                                                                 |                               | X(r)<br>Expt.<br>$\times 10^3$<br>(nondim.) | X(r)<br>Fair.<br>$\times 10^3$<br>(nondim.) | X(r)<br>Diff.<br>$\times 10^3$<br>(nondim.)          |                                             |                                             |                                             |            |                 |
| 27                                                              | 0.1735                        | -0.4239                                     | -0.3866                                     | -0.0372                                              |                                             |                                             |                                             |            |                 |
| 28                                                              | 0.1367                        | -0.1992                                     | -0.2401                                     | 0.0409                                               |                                             |                                             |                                             |            |                 |
| 29                                                              | 0.1004                        | -0.0996                                     | -0.1294                                     | 0.0298                                               |                                             |                                             |                                             |            |                 |
| 30                                                              | 0.0639                        | -0.0498                                     | -0.0524                                     | 0.0026                                               |                                             |                                             |                                             |            |                 |
| 31                                                              | 0.0457                        | -0.0249                                     | -0.0269                                     | 0.0019                                               |                                             |                                             |                                             |            |                 |
| 32                                                              | 0.0273                        | 0.0000                                      | -0.0095                                     | 0.0095                                               |                                             |                                             |                                             |            |                 |
| 52                                                              | 0.0820                        | -0.0746                                     | -0.0863                                     | 0.0117                                               |                                             |                                             |                                             |            |                 |
| 53                                                              | 0.0501                        | -0.0498                                     | -0.0323                                     | -0.0175                                              |                                             |                                             |                                             |            |                 |
| rms = 0.0237                                                    |                               |                                             |                                             |                                                      |                                             |                                             |                                             |            |                 |
| Fairing Expressions: $X(r) \times 10^5 = -1284.75 \times r + 2$ |                               |                                             |                                             |                                                      |                                             |                                             |                                             |            |                 |

(Table continues)

Table 4 (Cont'd)

| Run No.                                                                       | Angular Accel-eration (nondim.) | Y Force from Pure Yaw Acceleration    |                                       |                                       | N Moment from Pure Yaw Acceleration                       |                                       |                                       |                         |
|-------------------------------------------------------------------------------|---------------------------------|---------------------------------------|---------------------------------------|---------------------------------------|-----------------------------------------------------------|---------------------------------------|---------------------------------------|-------------------------|
|                                                                               |                                 | Y(rdot) Expt. $\times 10^3$ (nondim.) | Y(rdot) Fair. $\times 10^3$ (nondim.) | Y(rdot) Diff. $\times 10^3$ (nondim.) | N(rdot) Expt. $\times 10^3$ (nondim.)                     | N(rdot) Fair. $\times 10^3$ (nondim.) | N(rdot) Diff. $\times 10^3$ (nondim.) |                         |
| 27                                                                            | 0.6653                          | -0.3690                               | -0.3329                               | -0.0361                               | -0.8692                                                   | -0.8732                               | 0.0040                                |                         |
| 28                                                                            | 0.5233                          | -0.2400                               | -0.2619                               | 0.0219                                | -0.6780                                                   | -0.6868                               | 0.0088                                |                         |
| 29                                                                            | 0.3847                          | -0.1774                               | -0.1925                               | 0.0151                                | -0.4974                                                   | -0.5049                               | 0.0076                                |                         |
| 30                                                                            | 0.2447                          | -0.1247                               | -0.1225                               | -0.0022                               | -0.3260                                                   | -0.3212                               | -0.0047                               |                         |
| 31                                                                            | 0.1756                          | -0.0614                               | -0.0879                               | 0.0265                                | -0.2420                                                   | -0.2305                               | -0.0115                               |                         |
| 32                                                                            | 0.1040                          | 0.0128                                | -0.0521                               | 0.0648                                | -0.1577                                                   | -0.1365                               | -0.0211                               |                         |
| 52                                                                            | 0.3136                          | -0.1614                               | -0.1569                               | -0.0045                               | -0.4183                                                   | -0.4116                               | -0.0068                               |                         |
| 53                                                                            | 0.1917                          | -0.1101                               | -0.0960                               | -0.0141                               | -0.2657                                                   | -0.2517                               | -0.0140                               |                         |
| rms = 0.0299                                                                  |                                 |                                       |                                       |                                       | rms = 0.0112                                              |                                       |                                       |                         |
| Fairing Expressions: $Y(\text{rdot}) \times 10^5 = -50.04 \times \text{rdot}$ |                                 |                                       |                                       |                                       | $N(\text{rdot}) \times 10^5 = -131.26 \times \text{rdot}$ |                                       |                                       |                         |
| Hydrodynamic Derivatives, Mass, and Inertia Terms for Model and Ship          |                                 |                                       |                                       |                                       |                                                           |                                       |                                       |                         |
| Item                                                                          |                                 | Model (nondim.)                       | Ship (nondim.)                        |                                       | Nondimensionalising Factors                               |                                       |                                       |                         |
| Y(r)                                                                          |                                 | $308.4 \times 10^{-5}$                | $308.4 \times 10^{-5}$                |                                       | $\frac{1}{2} \rho L_{pp}^3 U$                             |                                       |                                       |                         |
| mU(O)                                                                         |                                 | $1181.2 \times 10^{-5}$               | $1181.3 \times 10^{-5}$               |                                       |                                                           |                                       |                                       |                         |
| Y(r) - mU                                                                     |                                 | $-872.8 \times 10^{-5}$               | $-872.9 \times 10^{-5}$               |                                       |                                                           |                                       |                                       |                         |
| N(r)                                                                          |                                 | $-281.0 \times 10^{-5}$               | $-281.0 \times 10^{-5}$               |                                       | $\frac{1}{2} \rho L_{pp}^4 U$                             |                                       |                                       |                         |
| mx(G)U                                                                        |                                 | $13.8 \times 10^{-5}$                 | $13.8 \times 10^{-5}$                 |                                       |                                                           |                                       |                                       |                         |
| N(r) - mx(G)U                                                                 |                                 | $-294.8 \times 10^{-5}$               | $-294.8 \times 10^{-5}$               |                                       |                                                           |                                       |                                       |                         |
| Y(rdot)                                                                       |                                 | $-36.3 \times 10^{-5}$                | $-36.3 \times 10^{-5}$                |                                       | $\frac{1}{2} \rho L_{pp}^4$                               |                                       |                                       |                         |
| mx(G)                                                                         |                                 | $13.8 \times 10^{-5}$                 | $13.8 \times 10^{-5}$                 |                                       |                                                           |                                       |                                       |                         |
| Y(rdot) - mx(G)                                                               |                                 | $-50.0 \times 10^{-5}$                | $-50.1 \times 10^{-5}$                |                                       |                                                           |                                       |                                       |                         |
| N(rdot)                                                                       |                                 | $-66.4 \times 10^{-5}$                | $-66.4 \times 10^{-5}$                |                                       | $\frac{1}{2} \rho L_{pp}^5$                               |                                       |                                       |                         |
| I(z)                                                                          |                                 | $64.8 \times 10^{-5}$                 | $60.7 \times 10^{-5}$                 |                                       |                                                           |                                       |                                       |                         |
| N(rdot) - I(z)                                                                |                                 | $-131.3 \times 10^{-5}$               | $-127.1 \times 10^{-5}$               |                                       |                                                           |                                       |                                       |                         |
| Nondimensional Coefficients for PMM-Prediction Program                        |                                 |                                       |                                       |                                       |                                                           |                                       |                                       |                         |
| X Equation                                                                    |                                 |                                       | Y Equation                            |                                       |                                                           | N Equation                            |                                       |                         |
| Identi-fier                                                                   | Vari-able                       | Coefficient                           | Identi-fier                           | Vari-able                             | Coefficient                                               | Identi-fier                           | Vari-able                             | Coefficient             |
| f                                                                             | rr                              | $-1284.7 \times 10^{-5}$              | k3                                    | rdot                                  | $50.1 \times 10^{-5}$                                     | k4                                    | rdot                                  | $127.1 \times 10^{-5}$  |
|                                                                               |                                 |                                       | B1                                    | r                                     | $-872.9 \times 10^{-5}$                                   | B2                                    | r                                     | $-294.8 \times 10^{-5}$ |

### Calculation of Coefficients in the $X$ Equation

When the ship is sailing straight ahead with constant velocity  $u_1$ , the propeller thrust, modified by the thrust-deduction effect, exactly equals the resistance of the ship:

$$X = T(1 - t) - R_T = 0.$$

This equilibrium condition defines the initial propeller thrust and the corresponding propeller torque and revolutions.

As soon as a manoeuvre is initiated, the equilibrium condition is disturbed and the  $X$  force, which represents the difference between the propeller thrust and the ship resistance, will then vary as a function of speed  $u$  and propeller revolutions. Approximating the  $X$  force by a third-order polynomial,

$$X(u) = a_0 + a_1 \Delta u + a_2 \Delta u^2 + a_3 \Delta u^3,$$

where  $\Delta u = (u - u_1)/U$ , the dimensionless hydrodynamic coefficients  $X_*$ ,  $X_u$ ,  $X_{uu}$ , and  $X_{uuu}$  can be obtained directly as the coefficients of the approximating polynomial, as follows:

$$X_* = a_0 \sim 0; \quad X_u = a_1; \quad X_{uu} = a_2; \quad X_{uuu} = a_3.$$

It has been found that the  $X$  force and corresponding coefficients are computed most accurately on the basis of the results from the open-water, resistance, and self-propulsion tests. Table 5 shows examples of such calculations. The thrust deduction coefficient  $t = 0.136$  and wake coefficient  $w = 0.160$  used in the computation but not indicated in the table are taken from the self-propulsion test. These values are assumed to be constant for all speeds. The corresponding propeller thrust values are computed from the open-water propeller curves assuming constant wake, and taking the type of engine and engine setting to be maintained during the manoeuvre into account. As shown in the example, the propeller thrust can be calculated assuming constant propeller revolutions, or assuming the propeller torque to vary proportionally to the revolutions to a certain power. If torque is assumed to vary inversely proportionally to propeller revolutions, the thrust corresponding to a turbine powerplant capable of maintaining constant power output would be obtained. If torque is assumed to be constant during the manoeuvre, the corresponding condition for a diesel engine would be obtained. Figure 25 illustrates the relationship between propeller revolutions, torque and speed for these various conditions.

The variation of propeller revolutions with speed as derived from this computation can conveniently be used to obtain the correct propeller revolutions, which have to be used when tests are executed at reduced speeds to obtain the  $\Delta u$ - and revolution-dependent coefficients ("rudder angle and speed" tests).

The three coefficients,  $X_u$ ,  $X_{uu}$ , and  $X_{uuu}$ , can also be obtained experimentally by fairing  $X$ -force measurements made for zero rudder angle in a "rudder angle and speed" test. Such experimentally derived values have agreed well with the calculated coefficients.



Table 5  
Derivation of Nondimensional Coefficients  $X_u$ ,  $X_{uu}$ , and  $X_{uuu}$ : Calculation Examples for Constant Power, Constant Propeller Revolutions per Minute, and Equilibrium Speed (Approach Speed) of 15.00 knots

| J      | KT<br>Input | KT<br>Fair. | 10×KQ<br>Input | 10×KQ<br>Fair. |
|--------|-------------|-------------|----------------|----------------|
| 0.0000 |             | 0.4734      |                | 0.7181         |
| 0.1000 |             | 0.4334      |                | 0.6576         |
| 0.2000 |             | 0.3927      |                | 0.5983         |
| 0.3000 |             | 0.3515      |                | 0.5397         |
| 0.4000 |             | 0.3098      |                | 0.4819         |
| 0.4400 | 0.2930      | 0.2931      | 0.4590         | 0.4589         |
| 0.4900 | 0.2720      | 0.2720      | 0.4300         | 0.4303         |
| 0.5400 | 0.2510      | 0.2509      | 0.4020         | 0.4017         |
| 0.5900 | 0.2300      | 0.2297      | 0.3730         | 0.3732         |
| 0.6400 | 0.2080      | 0.2085      | 0.3450         | 0.3447         |
| 0.6900 | 0.1870      | 0.1872      | 0.3160         | 0.3163         |
| 0.7400 | 0.1660      | 0.1659      | 0.2880         | 0.2878         |
| 0.7900 | 0.1450      | 0.1446      | 0.2590         | 0.2593         |
| 0.8400 | 0.1230      | 0.1233      | 0.2310         | 0.2307         |
| 0.8900 | 0.1020      | 0.1020      | 0.2020         | 0.2021         |

| Speed<br>(knots) | EHP<br>(hp) | Resist.<br>$\times 10^{-5}$<br>(kg) | Thrust<br>$\times 10^{-5}$<br>(kg) | Torque<br>$\times 10^{-5}$<br>(kg-m) | J     | PHP<br>(hp) | Prop.<br>Revs.<br>(rpm) |
|------------------|-------------|-------------------------------------|------------------------------------|--------------------------------------|-------|-------------|-------------------------|
| 7.00             | 286.0       | 0.0596                              | 0.8550                             | 0.8913                               | 0.396 | 8504.7      | 68.3                    |
| 8.00             | 414.0       | 0.0755                              | 0.7899                             | 0.8314                               | 0.453 | 7933.8      | 68.3                    |
| 9.00             | 582.0       | 0.0943                              | 0.7244                             | 0.7718                               | 0.509 | 7365.2      | 68.3                    |
| 10.0             | 789.0       | 0.1150                              | 0.6587                             | 0.7124                               | 0.566 | 6798.1      | 68.3                    |
| 11.0             | 1060        | 0.1405                              | 0.5928                             | 0.6531                               | 0.622 | 6232.1      | 68.3                    |
| 12.0             | 1391        | 0.1690                              | 0.5268                             | 0.5938                               | 0.679 | 5666.6      | 68.3                    |
| 13.0             | 1815        | 0.2036                              | 0.4607                             | 0.5346                               | 0.735 | 5100.9      | 68.3                    |
| 14.0             | 2318        | 0.2414                              | 0.3945                             | 0.4752                               | 0.792 | 4534.4      | 68.3                    |
| 15.0             | 2919        | 0.2837                              | 0.3284                             | 0.4157                               | 0.849 | 3966.7      | 68.3                    |
| 15.0             | 2919        | 0.2837                              | 0.3284                             | 0.4157                               | 0.849 | 3966.7      | 68.3                    |

| Speed<br>(knots) | u<br>(nondim.) | X Force<br>$\times 10^{-5}$ (kg) | X $\times 10^5$<br>(nondim.) | X $\times 10^5$<br>Fair. (nondim.) |
|------------------|----------------|----------------------------------|------------------------------|------------------------------------|
| 7.00             | -1.14          | 0.679                            | 386.7                        | 386.7                              |
| 8.00             | -0.875         | 0.607                            | 264.6                        | 264.5                              |
| 9.00             | -0.667         | 0.532                            | 183.1                        | 183.2                              |
| 10.0             | -0.500         | 0.454                            | 126.7                        | 126.6                              |
| 11.0             | -0.364         | 0.372                            | 85.7                         | 85.7                               |
| 12.0             | -0.250         | 0.286                            | 55.4                         | 55.3                               |
| 13.0             | -0.154         | 0.194                            | 32.1                         | 32.2                               |
| 14.0             | -0.0714        | 0.0995                           | 14.2                         | 14.2                               |
| 15.0             | 0.00000        | -0.00002                         | 0.0                          | 0.0                                |

$$\text{nondim. } X_u \times 10^5 = -190.4$$

$$\text{nondim. } X_{uu} \times 10^5 = 122.8$$

$$\text{nondim. } X_{uuu} \times 10^5 = -5.8$$

(Table continues)

Prediction of Steering and Manoeuvring of Ships

Table 5 (Cont'd)

| J      | KT<br>Input | KT<br>Fair. | 10×KQ<br>Input | 10×KQ<br>Fair. |
|--------|-------------|-------------|----------------|----------------|
| 0.0000 |             | 0.4734      |                | 0.7181         |
| 0.1000 |             | 0.4334      |                | 0.6576         |
| 0.2000 |             | 0.3927      |                | 0.5983         |
| 0.3000 |             | 0.3515      |                | 0.5397         |
| 0.4000 |             | 0.3098      |                | 0.4819         |
| 0.4400 | 0.2930      | 0.2931      | 0.4590         | 0.4589         |
| 0.4900 | 0.2720      | 0.2720      | 0.4300         | 0.4303         |
| 0.5400 | 0.2510      | 0.2509      | 0.4020         | 0.4017         |
| 0.5900 | 0.2300      | 0.2297      | 0.3730         | 0.3732         |
| 0.6400 | 0.2080      | 0.2085      | 0.3450         | 0.3447         |
| 0.6900 | 0.1870      | 0.1872      | 0.3160         | 0.3163         |
| 0.7400 | 0.1660      | 0.1659      | 0.2880         | 0.2878         |
| 0.7900 | 0.1450      | 0.1446      | 0.2590         | 0.2593         |
| 0.8400 | 0.1230      | 0.1233      | 0.2310         | 0.2307         |
| 0.8900 | 0.1020      | 0.1020      | 0.2020         | 0.2021         |

| Speed<br>(knots) | EHP<br>(hp) | Resist.<br>× 10 <sup>-5</sup><br>(kg) | Thrust<br>× 10 <sup>-5</sup><br>(kg) | Torque<br>× 10 <sup>-5</sup><br>(kg-m) | J     | PHP<br>(hp) | Prop.<br>Revs.<br>(rpm) |
|------------------|-------------|---------------------------------------|--------------------------------------|----------------------------------------|-------|-------------|-------------------------|
| 7.00             | 286.0       | 0.0596                                | 0.4855                               | 0.5152                                 | 0.491 | 3966.8      | 55.1                    |
| 8.00             | 414.0       | 0.0755                                | 0.4667                               | 0.5021                                 | 0.547 | 3966.8      | 56.6                    |
| 9.00             | 582.0       | 0.0943                                | 0.4476                               | 0.4892                                 | 0.599 | 3966.7      | 58.1                    |
| 10.0             | 789.0       | 0.1150                                | 0.4282                               | 0.4764                                 | 0.648 | 3966.8      | 59.6                    |
| 11.0             | 1060        | 0.1405                                | 0.4085                               | 0.4639                                 | 0.694 | 3966.8      | 61.2                    |
| 12.0             | 1391        | 0.1690                                | 0.3887                               | 0.4515                                 | 0.737 | 3966.7      | 62.9                    |
| 13.0             | 1815        | 0.2036                                | 0.3687                               | 0.4393                                 | 0.777 | 3966.8      | 64.7                    |
| 14.0             | 2318        | 0.2414                                | 0.3486                               | 0.4274                                 | 0.814 | 3966.8      | 66.5                    |
| 15.0             | 2919        | 0.2837                                | 0.3284                               | 0.4157                                 | 0.849 | 3966.7      | 68.3                    |
| 15.0             | 2919        | 0.2837                                | 0.3284                               | 0.4157                                 | 0.849 | 3966.7      | 68.3                    |

| Speed<br>(knots) | u<br>(nondim.) | X Force<br>× 10 <sup>-5</sup> (kg) | X × 10 <sup>5</sup><br>(nondim.) | X × 10 <sup>5</sup><br>Fair. (nondim.) |
|------------------|----------------|------------------------------------|----------------------------------|----------------------------------------|
| 7.00             | -1.14          | 0.360                              | 204.9                            | 204.9                                  |
| 8.00             | -0.875         | 0.328                              | 142.9                            | 142.8                                  |
| 9.00             | -0.667         | 0.292                              | 100.7                            | 100.9                                  |
| 10.0             | -0.500         | 0.255                              | 71.1                             | 71.0                                   |
| 11.0             | -0.364         | 0.212                              | 49.0                             | 49.0                                   |
| 12.0             | -0.250         | 0.167                              | 32.3                             | 32.2                                   |
| 13.0             | -0.154         | 0.115                              | 19.0                             | 19.1                                   |
| 14.0             | -0.0714        | 0.0598                             | 8.5                              | 8.5                                    |
| 15.0             | 0.00000        | -0.00002                           | 0.0                              | 0.0                                    |

$$\text{nondim. } X_u \times 10^5 = -116.6$$

$$\text{nondim. } X_{uu} \times 10^5 = 48.3$$

$$\text{nondim. } X_{uuu} \times 10^5 = -5.8$$

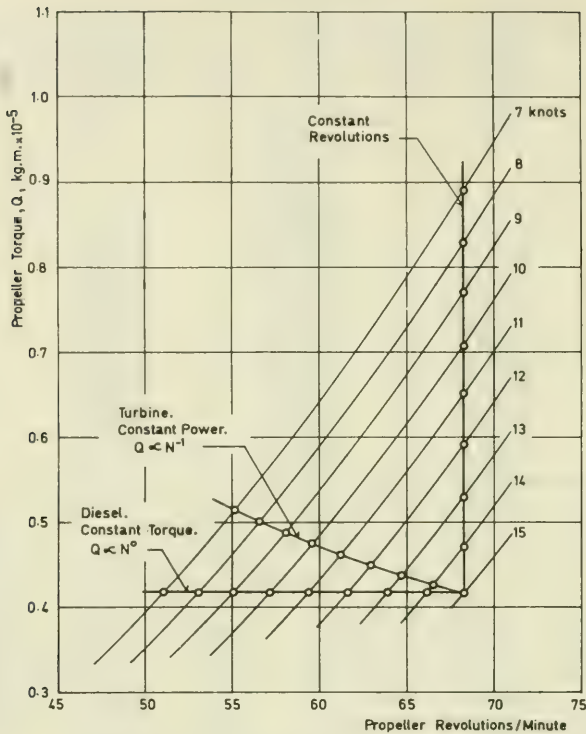


Fig. 25 - Relationship between propeller revolutions and propeller torque as a function of speed reduction encountered while manoeuvring

### Measurement of Model Polar Moment of Inertia

The term  $N_i - I_z$  measured in the "pure yaw" test contains the effect of model moment of inertia  $I_z$ . For prediction of ship-scale manoeuvres, it is necessary, as mentioned above, to correct for the difference between model and ship inertias.

The inertia of the model, including instrumentation and equipment, is found by oscillating the model in air on a torsional pendulum. The torsion rod is of 6-mm piano wire, 2 m long, clamped at the top to a stiff beam and at the bottom to a fitting which can be rigidly fastened to the yoke normally used for lifting models. Four wooden struts are used to prevent movement of the model relative to the yoke (Fig. 26).

The torsion rod is enclosed, as a safety precaution, by a brass tube which will restrain the rod if it should break. A steadying support is used to prevent the model from swinging.





Fig. 26 - Model suspended on a torsional pendulum for measurement of the polar moment of inertia

The inertia of the model is found by first measuring the period of oscillation of the model and yoke, and then the period of the yoke without the model. The period of oscillation can be measured with sufficient accuracy using a stopwatch, as the time for one complete oscillation is of the order of 2 minutes. The very slow movement of the model precludes the possibility of significant aerodynamic damping.

The inertia of the full-scale ship is normally computed on the basis of the longitudinal weight distribution and the main dimensions.

The ability to account for differences in inertia between the model and the ship is convenient, as it permits the model to be constructed without paying any regard to its inertia. Model testing can be executed for any value of model inertia, and the appropriate ship value introduced in the analysis of the force measurements.

### Scale Effects

Most of the hydrodynamic coefficients are obtained from model tests; hence it is reasonable to give some consideration to correlation between model and full-scale results before applying the coefficients to the prediction of full-scale manoeuvres.

The model tests are conducted according to Froude's law; consequently Reynolds' number is not satisfied, and the possibility of scale effects due to differences in Reynolds' number must be considered.

Results from airfoil testing are relevant in a discussion of scale effects. Such tests, covering a wide range of Reynolds' numbers, indicate that change of



Reynolds' number has no systematic effect on the lift-curve slope, whereas the variation of maximum lift might be appreciable because separation or flow breakdown occurs earlier for the relatively thicker boundary layer around a model body. As most of the hydrodynamic coefficients are due to related circulation and viscous drag effects, scale effects should not be expected for any of the first-order coefficients,  $Y_v$ ,  $Y_r$ ,  $Y_\delta$ ,  $N_v$ ,  $N_r$ ,  $N_\delta$ , etc. In the case of the nonlinear coefficients, however, scale effects could be more serious, as these coefficients are more likely to be influenced by separation or flow breakdown. Normally, nonlinear coefficients of the motion parameters  $v$  and  $r$ , for instance  $Y_{vvv}$  and  $Y_{rrr}$ , are determined for relatively small values of  $v$  and  $r$  corresponding to angles of attack below that at which separation takes place. For this reason, scale effects are probably also negligible for these coefficients. This is not true for the rudder, as rudder deflection for which rudder characteristics are measured will generally cover the range of rudder-lift breakdown. For the coefficients  $Y_{\delta\delta\delta}$  and  $N_{\delta\delta\delta}$  in particular, it has sometimes been found desirable to make corrections for this scale effect.

### Résumé of Experimental Program

A summary of a typical experimental program is given in Table 6. This program is used as a standard when testing cargo ships for which complete predictions of extreme manoeuvres are required.

The table refers to the series of planar-motion mechanism experiments which are necessary for the prediction of manoeuvres made for one given initial, or approach, speed. If other approach speeds are wanted, additional tests are necessary.

Short descriptions of each test are given in the table, together with suggested settings of the planar-motion mechanism, and the hydrodynamic coefficients obtained from each of the tests are also indicated.

Typical values for the different coefficients and examples of predicted full-scale turning circles, zigzag, and spiral manoeuvres are to be found in Refs. 4 and 5.

### INFLUENCE OF SPEED ON DIMENSIONLESS FORCE AND MOMENT COEFFICIENTS

The mathematical model (Eqs. (5)) is composed of coefficients expressing relationships between orientations, velocities, and accelerations and the resulting inertial and hydrodynamic forces and moments which act on a given hull-rudder-propeller combination. These coefficients are nondimensionalized, according to the consistent system described in Ref. 2, by dividing mass by  $\rho L_{pp}^3/2$ , length by  $L_{pp}$  and time by  $L_{pp}/U$ , where  $U$  is the instantaneous velocity of the origin of the body-axes relative to the fluid. The resulting nondimensionalizing factors used are given in Table 7.

Forces and moments resulting from surge, sway, and yaw accelerations  $\dot{u}$ ,  $\dot{v}$ , and  $\dot{r}$  have been found, in general, to be independent of the velocity of the

Table 6  
Standard Programme for Manoeuvring Tests Carried Out by Means of the Planar-Motion Mechanism

| Type and Description of Test                                                                                                                                                                                                                                                            | Range of Variables                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Number of Runs | Derivatives Obtained from Test                                                                                                                   |                                                                                                                                                                                                        | Remarks                                                                                                                                                                                                                                       |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------|--------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------|----|---|----|---|-------|----|---|---|---|-------|----|---|---|---|-------|----|---|---|----|-------|----|---|---|----|-------|----|---|---|----|-------|----|---|---|----|-------|--------------------------------------------------------------------------------------------------|--------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------|
|                                                                                                                                                                                                                                                                                         |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |                | One Variable                                                                                                                                     | Crosscoupling                                                                                                                                                                                          |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 1. Static Rudder and Drift Angle Test<br>In each run the model is traveling with a constant angle between path of origin and centerline; i.e., constant drift angle and rudder angle $\delta$ is varied.                                                                                | $\delta = 0$ to $\pm 40$ deg in steps of 10<br>$\beta = 0, \pm 1, \pm 2, \pm 4.5, \pm 7, \pm 9.5, \pm 12, \pm 14$ and for unstable ships $\pm 17$ and $\pm 20$ deg.<br>$r = 0, U = U_0 = -m/\text{sec.}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                | $X_{vv}, X_{\delta\delta}$<br>$Y_v, Y_{vv}, Y_{\delta}, Y_{\delta\delta}, Y_{\delta}$<br>$N_v, N_{vv}, N_{\delta}, N_{\delta\delta}, N_{\delta}$ | $X_{\delta}$<br>$Y_{vv}, Y_{\delta\delta}$<br>$N_{vv}, N_{\delta\delta}$                                                                                                                               | In case of a partial test, it should consist at one of the following combinations<br>$\beta \geq 0, \text{ all } \delta$<br>$\delta \geq 0, \text{ all } \beta$<br>$\beta \leq 0, \text{ all } \delta$<br>$\delta \leq 0, \text{ all } \beta$ |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 2. Pure Sway Test<br>The bow and stern are oscillated in phase, and a pure sway motion is produced. The side velocity and acceleration are varied by changing the amplitude of oscillation in between each run. The frequency of oscillation is kept constant. The phase angle is zero. | $\delta = 0, V_{\max} = a2\pi n$<br>$r = 0, V_{\max} = a(2\pi n)^2$ , where<br>$a = 35, 50, 65, 80, \text{ and } 95 \text{ mm}$<br>$n = N/60, N = - \text{rpm.}$<br>$\phi_s = 0 \text{ deg.}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                | $(m - Y_{\delta})$<br>$(mx_G - N_{\delta})$                                                                                                      |                                                                                                                                                                                                        |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 3. Yaw and Drift Angle Test<br>During the runs the bow and stern of the model are oscillated out of phase in such a way that the center line stays tangent or at a constant angle with the tangent to the path traveled by origin.                                                      | Pure Yaw and Drift<br><table><tr><th>a (mm)</th><th>a (deg)</th><th>a (mm)</th><th>a (deg)</th><th><math>\frac{1}{\cos \beta}</math></th></tr><tr><td>35</td><td>0</td><td>95</td><td>4</td><td>1.004</td></tr><tr><td>45</td><td>-</td><td>-</td><td>6</td><td>1.007</td></tr><tr><td>55</td><td>-</td><td>-</td><td>8</td><td>1.010</td></tr><tr><td>65</td><td>-</td><td>-</td><td>10</td><td>1.015</td></tr><tr><td>75</td><td>-</td><td>-</td><td>11</td><td>1.018</td></tr><tr><td>85</td><td>-</td><td>-</td><td>12</td><td>1.022</td></tr><tr><td>95</td><td>-</td><td>-</td><td>13</td><td>1.026</td></tr></table><br>$U = -U_0 = -\phi_s = -\phi_\delta = 0$<br>$N = -N_0 = -N_0/\cos \beta$<br>$\phi_s = 0$<br>$\phi_\delta = 0$ | a (mm)         | a (deg)                                                                                                                                          | a (mm)                                                                                                                                                                                                 | a (deg)                                                                                                                                                                                                                                       | $\frac{1}{\cos \beta}$ | 35 | 0 | 95 | 4 | 1.004 | 45 | - | - | 6 | 1.007 | 55 | - | - | 8 | 1.010 | 65 | - | - | 10 | 1.015 | 75 | - | - | 11 | 1.018 | 85 | - | - | 12 | 1.022 | 95 | - | - | 13 | 1.026 | $(X_{rr} = mx_G)$<br>$(Y_r - mU), Y_{rr}, (mx_G - Y_r)$<br>$(N_r - mx_G U), N_{rr}, (I_z - N_r)$ | $Y_{vv}, Y_{rr}$<br>$N_{vv}, N_{rr}$ | $(X_{rv} + m) \approx m$<br>$(X_{rr} + mx_G) \approx mx_G$<br>$X_{rr}$ is negligible since it has not been detected at the maximum obtainable yaw velocity |
| a (mm)                                                                                                                                                                                                                                                                                  | a (deg)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | a (mm)         | a (deg)                                                                                                                                          | $\frac{1}{\cos \beta}$                                                                                                                                                                                 |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 35                                                                                                                                                                                                                                                                                      | 0                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | 95             | 4                                                                                                                                                | 1.004                                                                                                                                                                                                  |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 45                                                                                                                                                                                                                                                                                      | -                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | -              | 6                                                                                                                                                | 1.007                                                                                                                                                                                                  |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 55                                                                                                                                                                                                                                                                                      | -                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | -              | 8                                                                                                                                                | 1.010                                                                                                                                                                                                  |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 65                                                                                                                                                                                                                                                                                      | -                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | -              | 10                                                                                                                                               | 1.015                                                                                                                                                                                                  |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 75                                                                                                                                                                                                                                                                                      | -                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | -              | 11                                                                                                                                               | 1.018                                                                                                                                                                                                  |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 85                                                                                                                                                                                                                                                                                      | -                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | -              | 12                                                                                                                                               | 1.022                                                                                                                                                                                                  |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 95                                                                                                                                                                                                                                                                                      | -                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | -              | 13                                                                                                                                               | 1.026                                                                                                                                                                                                  |                                                                                                                                                                                                                                               |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 4. Static Speed and Rudder Angle Test<br>For different speeds the rudder angle $\delta$ is varied. The propeller revolutions are normally kept constant.                                                                                                                                | $\delta = 0$ to $\pm 40$ deg in steps of 10 deg.<br>$\beta = - \text{deg}$ $r = 0$<br>$U = -$ and $-m/s$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                | $X_{vv}, X_{uu}, X_{uuu}$                                                                                                                        | $X_{\delta\delta}$<br>$Y_{vv}, Y_{\delta\delta}$<br>$N_{vv}, N_{\delta\delta}$                                                                                                                         | $X_v, X_{vv}$ and $X_{uu}$ may also be calculated from resistance, self-propulsion, and open-water tests.                                                                                                                                     |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| 5. Yaw and Rudder Angle Test<br>During the runs the bow and stern are oscillated out of phase in such a way that the center line stays tangent to the path traveled by origin. This is done for different rudder angles.                                                                | $\delta = -$<br>$a = -$<br>$N = -$<br>$\phi_s = -$<br>$U = U_0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |                | $X_{\delta}$<br>$Y_{\delta\delta}, Y_{rr}$<br>$N_{\delta\delta}, N_{rr}$                                                                         | $X_{\delta}$<br>$Y_{\delta\delta}, Y_{rr}$<br>$N_{\delta\delta}, N_{rr}$                                                                                                                               | This test is only made occasionally.                                                                                                                                                                                                          |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |
| $\delta =$ rudder angle in deg.<br>$\beta =$ drift angle in deg.<br>$r =$ yaw velocity radians/sec<br>$U =$ model speed in m/sec                                                                                                                                                        | $\phi_s =$ phase angle between bow and stern supports<br>$2d =$ distance between supports ( $= 2,100 \text{ m}$ )<br>$a =$ half PMM - amplitude<br>$N =$ PMM-revolutions rpm, $n = N/60$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                | $v = -\sin \beta$<br>$v_{\max} = a2\pi n$<br>$\dot{\psi}_{\max} = a(2\pi n)^2$                                                                   | $n = \tan \frac{\phi_s}{2} \frac{U}{2\pi d \cos \beta}$<br>$r_{\max} = 2\pi na \sin \frac{\phi_s}{2} \frac{1}{d \cos \beta}$<br>$r_{\max} = (2\pi n)^2 a \sin \frac{\phi_s}{2} \frac{1}{d \cos \beta}$ | yaw and drift                                                                                                                                                                                                                                 |                        |    |   |    |   |       |    |   |   |   |       |    |   |   |   |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |    |   |   |    |       |                                                                                                  |                                      |                                                                                                                                                            |

Table 7  
Nondimensionalizing Factors

| Quantity                | Dimensions               | Nondim. Factor                  |
|-------------------------|--------------------------|---------------------------------|
| Mass                    | kg-sec <sup>2</sup> /m   | $\frac{1}{2} \rho L_{pp}^3$     |
| Inertia                 | kg-sec <sup>2</sup> /m   | $\frac{1}{2} \rho L_{pp}^5$     |
| X and Y forces          | kg                       | $\frac{1}{2} \rho L_{pp}^2 U^2$ |
| N moment                | kg-m                     | $\frac{1}{2} \rho L_{pp}^3 U^2$ |
| u and v                 | m/sec                    | U                               |
| r                       | radians/sec              | $U/L_{pp}$                      |
| $\delta$                | radians                  | —                               |
| $\dot{u}$ and $\dot{v}$ | m/sec <sup>2</sup>       | $U^2/L_{pp}$                    |
| $\dot{r}$               | radians/sec <sup>2</sup> | $U^2/L_{pp}^2$                  |

origin U. Nondimensional coefficients expressing such data are then also independent of speed, since  $U^2$  occurs in the nondimensionalizing factors for both force (or moment) and acceleration. This is supported by numerous experimental results. For example, Fig. 22 of Ref. 11 shows complete agreement between side forces and turning moments measured as functions of yaw acceleration at ship speeds of 4, 6, 8, and 10 knots, for a 53-m- $L_{pp}$  trawler form.

Forces and moments resulting from drift angle, rudder angle, and yaw velocity are largely due to circulation and viscous crossflow drag. Drift-angle and rudder-angle forces may be likened to lift forces acting on an airfoil having an angle of attack, and yaw velocity is analogous to camber. It is thus reasonable to expect the resulting hydrodynamic forces to vary proportionally with the square of the low velocity as long as the flows have the same geometry, i.e., as long as sinkage, trim, wave making, flow separation, etc., do not vary appreciably with speed.

Forces and moments due to drift angle and to yaw velocity are principally dependent on the flow patterns around the hull, and the speed of the origin is therefore a good representative velocity. Nondimensionalization with this velocity results in dimensionless values which change only with flow-geometry variations.

Figure 27 shows measurements of side force and turning moment due to drift angle, made in "static drift angle" tests at speeds of 5, 10, 15, 20, and 25 knots ship speed, with the 20-ft Mariner model described in Ref. 5. The same measured values are plotted in dimensional and nondimensional form, and the faired lines are polynomials of the form



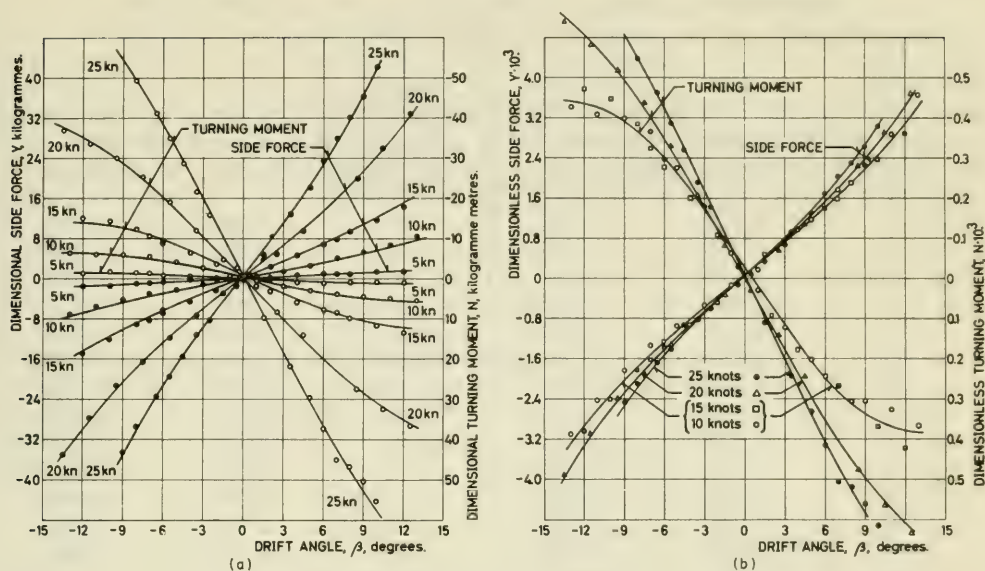


Fig. 27 - Results of static-drift-angle tests made at different speeds  
(a) dimensional plot, (b) dimensionless plot

$$Y(v) = Y_* + Y_v v + Y_{vvv} v^3$$

$$N(v) = N_* + N_v v + N_{vvv} v^3,$$

which have been fitted to the points by a least-square procedure. The numerical values and coefficients of the polynomials are given in Table 8. The coefficients are plotted against speed in Fig. 28.

It is seen that the nondimensional values are almost independent of speed up to 20 knots. Above about 20 knots, the nondimensional forces acting on the hull increase in size. This was due to sinkage of the model both forward and aft. Two effects were apparent. The model sank more as the speed increased, increasing the lateral area and hence the linear coefficients, and for a given speed, sinkage increased with drift angle, causing the cubic terms to become more pronounced. Sinkage, both with speed and drift angle, was greatest at the bow, probably because the crossflow under the forebody increased the local velocity and reduced the hydrostatic pressure more than was the case in the afterbody, where the flow had become straightened somewhat.

On the basis of similar results obtained for angular velocities, it can be concluded that  $Y'_v$ ,  $Y'_r$ ,  $N'_v$ , and  $N'_r$  are largely independent of speed over the speed ranges commonly covered by merchant ships. It has consequently not been found necessary to include tests in the standard program (Table 6) for



Table 8  
Experimental Results of Static Drift Angle and Speed Tests

| $\beta$<br>(deg)                                                                                                                    | $Y'$<br>$\times 10^3$ | $N'$<br>$\times 10^3$ | $\beta$<br>(deg)                                                                                                   | $Y'$<br>$\times 10^3$ | $N'$<br>$\times 10^3$ |
|-------------------------------------------------------------------------------------------------------------------------------------|-----------------------|-----------------------|--------------------------------------------------------------------------------------------------------------------|-----------------------|-----------------------|
| Nominal Ship Speed = 10 knots<br>Exact Model Speed = 1.048 m/sec                                                                    |                       |                       | Nominal Ship Speed = 20 knots<br>Exact Model Speed = 2.06 m/sec                                                    |                       |                       |
| 1                                                                                                                                   | 0.170                 | 0.031                 | 0.5                                                                                                                | 0.090                 | 0.031                 |
| 3                                                                                                                                   | 0.668                 | 0.121                 | 2.5                                                                                                                | 0.562                 | 0.143                 |
| 5                                                                                                                                   | 1.179                 | 0.203                 | 4.5                                                                                                                | 1.080                 | 0.244                 |
| 7                                                                                                                                   | 1.767                 | 0.258                 | 8.5                                                                                                                | 2.243                 | 0.477                 |
| 9                                                                                                                                   | 2.257                 | 0.306                 | 10.5                                                                                                               | 2.914                 | 0.565                 |
| 11                                                                                                                                  | 2.885                 | 0.328                 | 12.5                                                                                                               | 3.676                 | 0.634                 |
| 13                                                                                                                                  | 3.659                 | 0.368                 | -1.5                                                                                                               | -0.329                | -0.082                |
| -1                                                                                                                                  | -0.142                | -0.062                | -3.5                                                                                                               | -0.817                | -0.206                |
| -3                                                                                                                                  | -0.539                | -0.178                | -5.5                                                                                                               | -1.312                | -0.330                |
| -5                                                                                                                                  | -0.962                | -0.276                | -7.5                                                                                                               | -1.872                | -0.438                |
| -7                                                                                                                                  | -1.337                | -0.366                | -9.5                                                                                                               | -2.388                | -0.520                |
| -9                                                                                                                                  | -1.840                | -0.398                | -11.5                                                                                                              | -3.100                | -0.583                |
| -11                                                                                                                                 | -2.432                | -0.409                | -13.5                                                                                                              | -3.908                | -0.641                |
| -13                                                                                                                                 | -3.110                | -0.427                | Fairing Expressions:<br>$Y \times 10^5 = 4.1 - 1323v - 6967v^3$<br>$N \times 10^5 = 0.0 - 348.7v + 1293v^3$        |                       |                       |
| (Faired together with 15-knot data; see bottom of this column)                                                                      |                       |                       |                                                                                                                    |                       |                       |
| Nominal Ship Speed = 15 knots<br>Exact Model Speed = 1.534 m/sec                                                                    |                       |                       | Nominal Ship Speed = 25 knots<br>Exact Model Speed = 2.57 m/sec                                                    |                       |                       |
| 0                                                                                                                                   | -0.034                | -0.006                | 1.5                                                                                                                | 0.346                 | 0.108                 |
| 2                                                                                                                                   | 0.481                 | 0.093                 | 3.5                                                                                                                | 0.916                 | 0.243                 |
| 4                                                                                                                                   | 0.958                 | 0.178                 | 5                                                                                                                  | 1.308                 | 0.332                 |
| 6                                                                                                                                   | 1.395                 | 0.244                 | 6                                                                                                                  | 1.676                 | 0.415                 |
| 7                                                                                                                                   | 1.578                 | 0.262                 | 7                                                                                                                  | 2.016                 | 0.505                 |
| 8                                                                                                                                   | 1.914                 | 0.306                 | 8                                                                                                                  | 2.303                 | 0.520                 |
| 10                                                                                                                                  | 2.380                 | 0.370                 | 9                                                                                                                  | 2.619                 | 0.561                 |
| 12                                                                                                                                  | 2.889                 | 0.424                 | 10                                                                                                                 | 3.032                 | 0.615                 |
| -2                                                                                                                                  | -0.492                | -0.112                | -0.5                                                                                                               | -0.109                | -0.028                |
| -4                                                                                                                                  | -0.940                | -0.200                | -2.5                                                                                                               | -0.588                | -0.177                |
| -6                                                                                                                                  | -1.390                | -0.279                | -4.5                                                                                                               | -1.116                | -0.321                |
| -7                                                                                                                                  | -1.645                | -0.325                | -3.5                                                                                                               | -0.812                | -0.241                |
| -12                                                                                                                                 | -3.043                | -0.473                | -5.5                                                                                                               | -1.405                | -0.388                |
| -10                                                                                                                                 | -2.411                | -0.447                | -6.5                                                                                                               | -1.688                | -0.462                |
| -8                                                                                                                                  | -1.823                | -0.386                | -8                                                                                                                 | -2.108                | -0.549                |
| -6                                                                                                                                  | -1.260                | -0.297                | -9                                                                                                                 | -2.484                | -0.659                |
| Fairing Expressions (10 and 15 knots):<br>$Y' \times 10^5 = 7.1 - 1225v' - 5049v'^3$<br>$N' \times 10^5 = 2.9 - 280.0v' + 1905v'^3$ |                       |                       | Fairing Expressions:<br>$Y' \times 10^5 = 5.5 - 1413v' - 9535v'^3$<br>$N' \times 10^5 = -1.0 - 414.5v' - 1357v'^3$ |                       |                       |

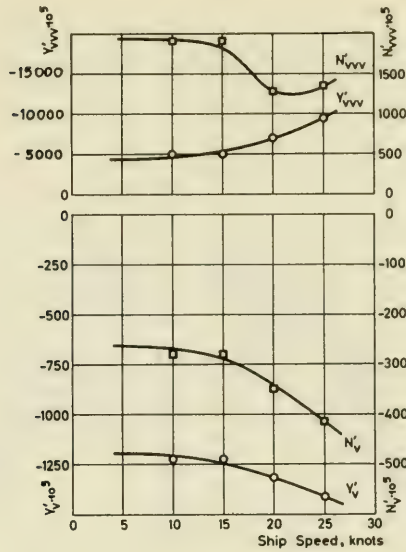


Fig. 28 - Variation of dimensionless drift-angle coefficients with speed

measurement of the coefficients  $Y'_{vu}$ ,  $Y'_{vuu}$ ,  $Y'_{ru}$ ,  $Y'_{ruu}$ ,  $N'_{vu}$ ,  $N'_{vuu}$ ,  $N'_{ru}$ , and  $N'_{ruu}$ .

Forces and moments due to rudder angle depend on the flow velocity around the rudder. This is influenced by ship speed, and, for a rudder in a propeller slip stream, by propeller revolutions. The speed of the origin is then not a representative velocity, since the flow velocity can vary markedly with propeller revolutions for constant ship speed.

Figures 29 to 31 inclusive show measurements of side force, turning moment, and longitudinal force due to rudder angle, made in "static-rudder-angle" tests at 7 to 15 knots in steps of one knot, with the Mariner model described in Ref. 5. The tests were made with constant propeller revolutions corresponding to the ship self-propulsion point at 15 knots. The numerical values and coefficients of the polynomials are given in Table 9.

In contrast to the results of the "static-drift-angle" tests described above, the nondimensional forces and moments vary markedly with speed, becoming infinite at zero ship speed. The coefficients of the fairing polynomials are plotted against nondimensional speed  $\Delta u' = (u - u_1)/U$ , in Fig. 32.

The crosscoupling terms  $Y_{\delta u}$ ,  $Y_{\delta \delta \delta u}$ ,  $N_{\delta u}$ ,  $N_{\delta \delta \delta u}$ , and  $X_{\delta \delta u}$  obtained from these data are of importance in the prediction of radical manoeuvres when speed loss is appreciable. Since the propeller slip stream has a strong influence on

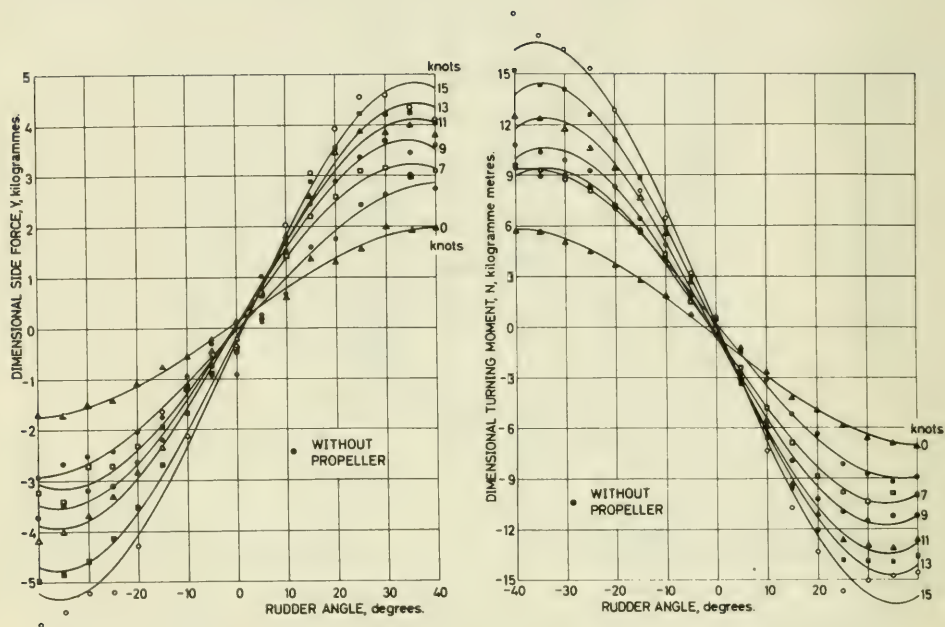


Fig. 29 - Results of static-rudder-angle tests made at different speeds  
(a) dimensional plot of side force  $Y'$ , (b) dimensional plot of turning moment  $N'$

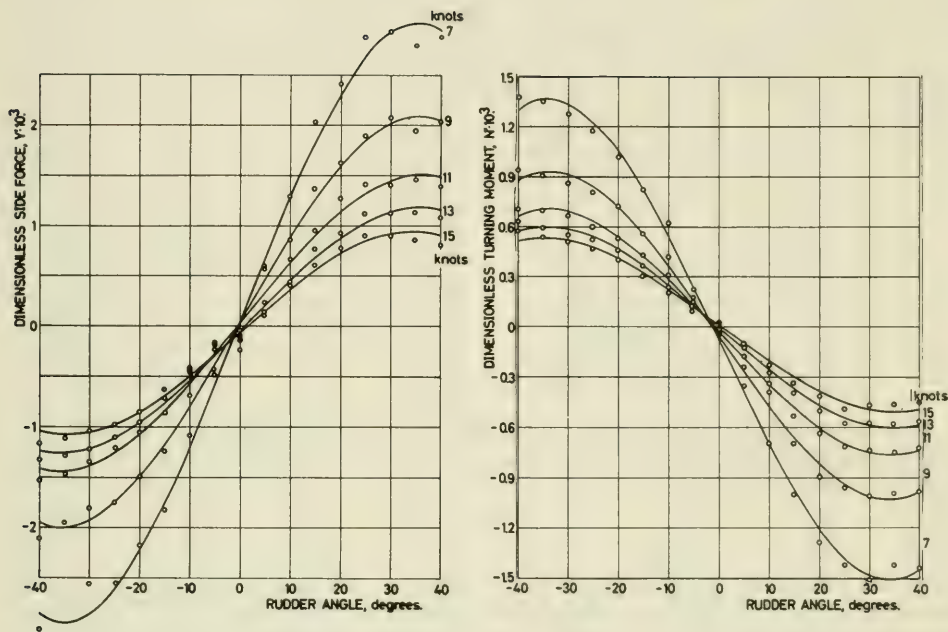


Fig. 30 - Results of static-rudder-angle tests made at different speeds (a) dimensionless plot of side force  $Y'$ , (b) dimensionless plot of turning moment  $N'$



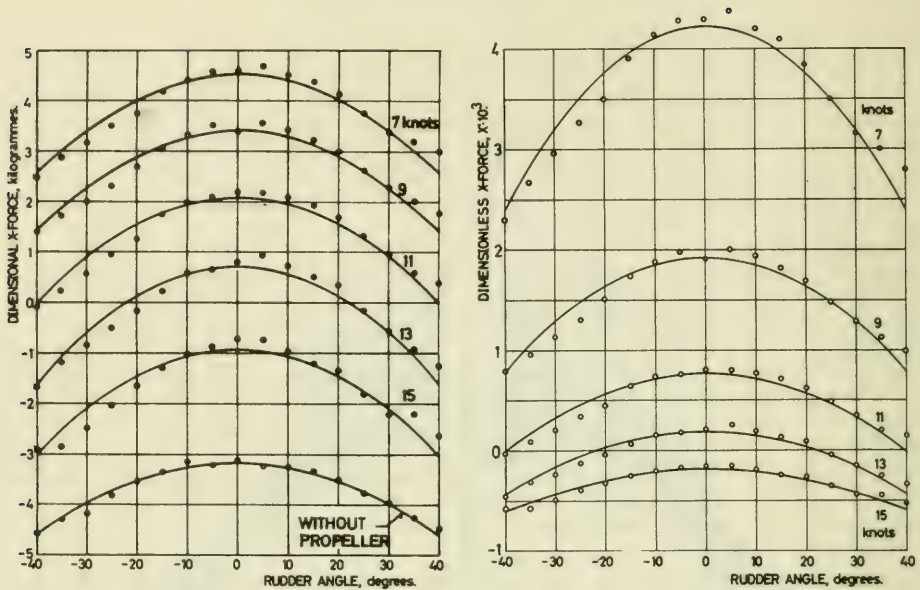


Fig. 31 - Results of static-rudder-angle tests made at different speeds (a) dimensional plot of longitudinal force  $X$ , (b) dimensionless plot of longitudinal force  $X$

the forces generated by a rudder situated within it, it can be seen that these crosscoupling terms will vary with engine characteristics and control settings. Values corresponding to a particular type of engine or control arrangement can conveniently be obtained, however, by carrying out model tests at predetermined revolution and speed values which have been obtained from calculations of the type described in the foregoing section.

The experimental results presented in Figs. 27 to 32 were measured with the model free to sink and trim but restrained in roll.

A distinction should be drawn between "speed-reduction" tests of the type described above and tests at different speeds in which the propeller revolutions correspond to the propulsion point for the speed in question. The nondimensional results of such tests vary very little with speed, which may be explained in a rough way by considering the flow geometry at different speeds to be similar in such tests, as the revolutions increase uniformly with ship speed.

# Prediction of Steering and Manoeuvring of Ships

Table 9  
Experimental Results of Static Rudder Angle and Speed Tests

| <p>The following experimental measurements were made with constant propeller revolutions, 345 revs./min model scale, corresponding to the ship self-propulsion point at 15 knots.<br/>Y and X forces have been nondimensionalized with <math>\rho L^2 U^2/2</math>, N moments with <math>\rho L^3 U^2</math>, where U is the exact model speed given in Table 8<sup>PP</sup></p> |                  |                  |                  |          |                  |                  |                  |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------|------------------|------------------|----------|------------------|------------------|------------------|
| $\delta$                                                                                                                                                                                                                                                                                                                                                                         | $Y' \times 10^3$ | $N' \times 10^3$ | $X' \times 10^3$ | $\delta$ | $Y' \times 10^3$ | $N' \times 10^3$ | $X' \times 10^3$ |
| 15 knots (without propeller)                                                                                                                                                                                                                                                                                                                                                     |                  |                  |                  | 13 knots |                  |                  |                  |
| -40                                                                                                                                                                                                                                                                                                                                                                              | -0.590           | 0.292            | -0.910           | -40      | -1.331           | 0.629            | -0.448           |
| -30                                                                                                                                                                                                                                                                                                                                                                              | -0.508           | 0.279            | -0.837           | -30      | -1.222           | 0.553            | -0.228           |
| -20                                                                                                                                                                                                                                                                                                                                                                              | -0.409           | 0.219            | -0.705           | -20      | -0.947           | 0.461            | -0.043           |
| -10                                                                                                                                                                                                                                                                                                                                                                              | -0.192           | 0.122            | -0.630           | -10      | -0.445           | 0.242            | 0.155            |
| 0                                                                                                                                                                                                                                                                                                                                                                                | 0.000            | 0.000            | -0.624           | 0        | -0.118           | 0.021            | 0.208            |
| 10                                                                                                                                                                                                                                                                                                                                                                               | 0.133            | -0.097           | -0.649           | 10       | 0.445            | -0.274           | 0.189            |
| 20                                                                                                                                                                                                                                                                                                                                                                               | 0.350            | -0.198           | -0.702           | 20       | 0.938            | -0.499           | 0.091            |
| 30                                                                                                                                                                                                                                                                                                                                                                               | 0.526            | -0.270           | -0.794           | 30       | 1.126            | -0.573           | -0.153           |
| 40                                                                                                                                                                                                                                                                                                                                                                               | 0.549            | -0.276           | -0.897           | 40       | 1.087            | -0.563           | -0.336           |
| -35                                                                                                                                                                                                                                                                                                                                                                              | -0.537           | 0.278            | -0.860           | -35      | -1.292           | 0.597            | -0.317           |
| -25                                                                                                                                                                                                                                                                                                                                                                              | -0.484           | 0.260            | -0.745           | -25      | -1.104           | 0.523            | -0.133           |
| -15                                                                                                                                                                                                                                                                                                                                                                              | -0.350           | 0.178            | -0.672           | -15      | -0.719           | 0.366            | 0.060            |
| -5                                                                                                                                                                                                                                                                                                                                                                               | -0.064           | 0.056            | -0.644           | -5       | -0.235           | 0.122            | 0.171            |
| 5                                                                                                                                                                                                                                                                                                                                                                                | 0.052            | -0.046           | -0.648           | 5        | 0.116            | -0.135           | 0.247            |
| 15                                                                                                                                                                                                                                                                                                                                                                               | 0.315            | -0.159           | -0.669           | 15       | 0.766            | -0.394           | 0.132            |
| 25                                                                                                                                                                                                                                                                                                                                                                               | 0.485            | -0.252           | -0.753           | 25       | 1.126            | -0.573           | -0.044           |
| 35                                                                                                                                                                                                                                                                                                                                                                               | 0.602            | -0.284           | -0.853           | 35       | 1.134            | -0.577           | -0.247           |
| 15 knots                                                                                                                                                                                                                                                                                                                                                                         |                  |                  |                  | 12 knots |                  |                  |                  |
| -40                                                                                                                                                                                                                                                                                                                                                                              | -1.171           | 0.576            | -0.587           | -40      | -1.422           | 0.664            | -0.208           |
| -30                                                                                                                                                                                                                                                                                                                                                                              | -1.043           | 0.510            | -0.493           | -30      | -1.280           | 0.611            | 0.019            |
| -20                                                                                                                                                                                                                                                                                                                                                                              | -0.857           | 0.400            | -0.331           | -20      | -0.997           | 0.505            | 0.266            |
| -10                                                                                                                                                                                                                                                                                                                                                                              | -0.426           | 0.201            | -0.207           | -10      | -0.479           | 0.265            | 0.447            |
| 0                                                                                                                                                                                                                                                                                                                                                                                | -0.094           | 0.017            | -0.147           | 0        | -0.076           | 0.003            | 0.533            |
| 10                                                                                                                                                                                                                                                                                                                                                                               | 0.407            | -0.227           | -0.193           | 10       | 0.516            | -0.313           | 0.490            |
| 20                                                                                                                                                                                                                                                                                                                                                                               | 0.786            | -0.413           | -0.269           | 20       | 1.062            | -0.586           | 0.350            |
| 30                                                                                                                                                                                                                                                                                                                                                                               | 0.914            | -0.468           | -0.440           | 30       | 1.307            | -0.664           | 0.090            |
| 40                                                                                                                                                                                                                                                                                                                                                                               | 0.814            | -0.452           | -0.525           | 40       | 1.327            | -0.654           | -0.082           |
| -35                                                                                                                                                                                                                                                                                                                                                                              | -1.119           | 0.537            | -0.571           | -35      | -1.384           | 0.657            | -0.112           |
| -25                                                                                                                                                                                                                                                                                                                                                                              | -0.979           | 0.465            | -0.405           | -25      | -1.138           | 0.562            | 0.122            |
| -15                                                                                                                                                                                                                                                                                                                                                                              | -0.629           | 0.303            | -0.259           | -15      | -0.733           | 0.406            | 0.374            |
| -5                                                                                                                                                                                                                                                                                                                                                                               | -0.186           | 0.098            | -0.176           | -5       | -0.264           | 0.109            | 0.492            |
| 5                                                                                                                                                                                                                                                                                                                                                                                | 0.139            | -0.100           | -0.148           | 5        | 0.186            | -0.161           | 0.511            |
| 15                                                                                                                                                                                                                                                                                                                                                                               | 0.611            | -0.332           | -0.243           | 15       | 0.846            | -0.451           | 0.457            |
| 25                                                                                                                                                                                                                                                                                                                                                                               | 0.907            | -0.486           | -0.363           | 25       | 1.241            | -0.656           | 0.230            |
| 35                                                                                                                                                                                                                                                                                                                                                                               | 0.867            | -0.460           | -0.439           | 35       | 1.269            | -0.668           | 0.000            |
| 14 knots                                                                                                                                                                                                                                                                                                                                                                         |                  |                  |                  | 11 knots |                  |                  |                  |
| -40                                                                                                                                                                                                                                                                                                                                                                              | -1.218           | 0.588            | -0.549           | -40      | -1.536           | 0.710            | -0.029           |
| -30                                                                                                                                                                                                                                                                                                                                                                              | -1.103           | 0.535            | -0.371           | -30      | -1.354           | 0.666            | 0.208            |
| -20                                                                                                                                                                                                                                                                                                                                                                              | -0.864           | 0.440            | -0.187           | -20      | -1.047           | 0.536            | 0.455            |
| -10                                                                                                                                                                                                                                                                                                                                                                              | -0.442           | 0.228            | -0.014           | -10      | -0.440           | 0.314            | 0.729            |
| 0                                                                                                                                                                                                                                                                                                                                                                                | -0.124           | 0.007            | 0.053            | 0        | -0.131           | -0.016           | 0.800            |
| 10                                                                                                                                                                                                                                                                                                                                                                               | 0.387            | -0.238           | 0.011            | 10       | 0.666            | -0.335           | 0.764            |
| 20                                                                                                                                                                                                                                                                                                                                                                               | 0.815            | -0.461           | -0.082           | 20       | 1.270            | -0.631           | 0.619            |
| 30                                                                                                                                                                                                                                                                                                                                                                               | 0.944            | -0.524           | -0.291           | 30       | 1.416            | -0.736           | 0.345            |
| 40                                                                                                                                                                                                                                                                                                                                                                               | 0.965            | -0.531           | -0.465           | 40       | 1.395            | -0.716           | 0.149            |
| -35                                                                                                                                                                                                                                                                                                                                                                              | -1.212           | 0.570            | -0.473           | -35      | -1.472           | 0.699            | 0.088            |
| -25                                                                                                                                                                                                                                                                                                                                                                              | -1.021           | 0.511            | -0.285           | -25      | -1.217           | 0.602            | 0.345            |
| -15                                                                                                                                                                                                                                                                                                                                                                              | -0.646           | 0.344            | -0.079           | -15      | -0.867           | 0.434            | 0.639            |
| -5                                                                                                                                                                                                                                                                                                                                                                               | -0.183           | 0.117            | 0.000            | -5       | -0.177           | 0.149            | 0.768            |
| 5                                                                                                                                                                                                                                                                                                                                                                                | 0.135            | -0.118           | 0.053            | 5        | 0.230            | -0.171           | 0.800            |
| 15                                                                                                                                                                                                                                                                                                                                                                               | 0.597            | -0.370           | -0.032           | 15       | 0.949            | -0.531           | 0.706            |
| 25                                                                                                                                                                                                                                                                                                                                                                               | 0.945            | -0.522           | -0.192           | 25       | 1.417            | -0.716           | 0.482            |
| 35                                                                                                                                                                                                                                                                                                                                                                               | 0.979            | -0.535           | -0.388           | 35       | 1.459            | -0.743           | 0.215            |

(Table continues)

Table 9 (Continued)

| $\delta$                                                                                 | $Y' \times 10^3$                   | $N' \times 10^3$   | $X' \times 10^3$        |                                       | $\delta$           | $Y' \times 10^3$        | $N' \times 10^3$                      | $X' \times 10^3$                |
|------------------------------------------------------------------------------------------|------------------------------------|--------------------|-------------------------|---------------------------------------|--------------------|-------------------------|---------------------------------------|---------------------------------|
| 10 knots                                                                                 |                                    |                    |                         |                                       | 8 knots            |                         |                                       |                                 |
| -40                                                                                      | -1.639                             | 0.814              | 0.183                   |                                       | -40                | -2.234                  | 1.101                                 | 1.484                           |
| -30                                                                                      | -1.541                             | 0.751              | 0.514                   |                                       | -30                | -2.022                  | 0.970                                 | 1.979                           |
| -20                                                                                      | -1.211                             | 0.602              | 0.759                   |                                       | -20                | -1.615                  | 0.858                                 | 2.425                           |
| -10                                                                                      | -0.585                             | 0.360              | 1.034                   |                                       | -10                | -0.871                  | 0.475                                 | 2.968                           |
| 0                                                                                        | -0.199                             | 0.000              | 1.112                   |                                       | 0                  | -0.023                  | -0.050                                | 3.133                           |
| 10                                                                                       | 0.684                              | -0.378             | 1.062                   |                                       | 10                 | 1.428                   | -0.606                                | 3.036                           |
| 20                                                                                       | 1.158                              | -0.700             | 0.923                   |                                       | 20                 | 2.514                   | -1.105                                | 2.706                           |
| 30                                                                                       | 1.452                              | -0.834             | 0.581                   |                                       | 30                 | 2.469                   | -1.182                                | 2.124                           |
| 40                                                                                       | 1.404                              | -0.771             | 0.372                   |                                       | 40                 | 2.361                   | -1.171                                | 1.789                           |
| -35                                                                                      | -1.552                             | 0.798              | 0.334                   |                                       | -35                | -2.127                  | 1.059                                 | 1.707                           |
| -25                                                                                      | -1.407                             | 0.678              | 0.595                   |                                       | -25                | -1.958                  | 0.919                                 | 2.207                           |
| -15                                                                                      | -0.990                             | 0.500              | 0.884                   |                                       | -15                | -1.576                  | 0.703                                 | 2.725                           |
| -5                                                                                       | -0.343                             | 0.151              | 1.079                   |                                       | -5                 | -0.448                  | 0.181                                 | 3.046                           |
| 5                                                                                        | 0.180                              | -0.189             | 1.131                   |                                       | 5                  | 0.509                   | -0.309                                | 3.094                           |
| 15                                                                                       | 0.989                              | -0.558             | 1.029                   |                                       | 15                 | 1.959                   | -0.912                                | 2.915                           |
| 25                                                                                       | 1.354                              | -0.803             | 0.767                   |                                       | 25                 | 2.447                   | -1.194                                | 2.415                           |
| 35                                                                                       | 1.392                              | -0.778             | 0.495                   |                                       | 35                 | 2.318                   | -1.171                                | 1.920                           |
| 9 knots                                                                                  |                                    |                    |                         |                                       | 7 knots            |                         |                                       |                                 |
| -40                                                                                      | -2.117                             | 0.940              | 0.795                   |                                       | -40                | -3.013                  | 1.378                                 | 2.294                           |
| -30                                                                                      | -1.803                             | 0.866              | 1.133                   |                                       | -30                | -2.548                  | 1.275                                 | 2.953                           |
| -20                                                                                      | -1.491                             | 0.726              | 1.515                   |                                       | -20                | -2.171                  | 1.018                                 | 3.500                           |
| -10                                                                                      | -0.685                             | 0.424              | 1.879                   |                                       | -10                | -1.081                  | 0.623                                 | 4.116                           |
| 0                                                                                        | -0.051                             | -0.039             | 1.913                   |                                       | 0                  | -0.222                  | -0.040                                | 4.283                           |
| 10                                                                                       | 0.866                              | -0.487             | 1.939                   |                                       | 10                 | 1.299                   | -0.692                                | 4.196                           |
| 20                                                                                       | 1.635                              | -0.890             | 1.688                   |                                       | 20                 | 2.409                   | -1.280                                | 3.842                           |
| 30                                                                                       | 2.081                              | -1.006             | 1.286                   |                                       | 30                 | 2.926                   | -1.502                                | 3.158                           |
| 40                                                                                       | 2.032                              | -0.949             | 0.994                   |                                       | 40                 | 2.873                   | -1.433                                | 2.798                           |
| -35                                                                                      | -1.968                             | 0.908              | 0.971                   |                                       | -35                | -3.207                  | 1.353                                 | 2.673                           |
| -25                                                                                      | -1.755                             | 0.809              | 1.305                   |                                       | -25                | -2.551                  | 1.176                                 | 3.258                           |
| -15                                                                                      | -1.247                             | 0.561              | 1.733                   |                                       | -15                | -1.819                  | 0.825                                 | 3.898                           |
| -5                                                                                       | -0.427                             | 0.173              | 1.977                   |                                       | -5                 | -0.488                  | 0.227                                 | 4.277                           |
| 5                                                                                        | 0.575                              | -0.242             | 2.003                   |                                       | 5                  | 0.594                   | -0.346                                | 4.371                           |
| 15                                                                                       | 1.375                              | -0.693             | 1.823                   |                                       | 15                 | 2.034                   | -0.993                                | 4.091                           |
| 25                                                                                       | 1.900                              | -0.955             | 1.481                   |                                       | 25                 | 2.874                   | -1.413                                | 3.500                           |
| 35                                                                                       | 1.949                              | -0.982             | 1.133                   |                                       | 35                 | 2.791                   | -1.418                                | 2.997                           |
| Nondimensional Coefficients Obtained from Fairing<br>the Above Experimental Measurements |                                    |                    |                         |                                       |                    |                         |                                       |                                 |
| Nominal<br>Ship<br>Speed<br>(knots)                                                      | Exact<br>Model<br>Speed<br>(m/sec) | $Y'_* \times 10^5$ | $Y'_\delta \times 10^5$ | $Y'_{\delta\delta\delta} \times 10^5$ | $N'_* \times 10^5$ | $N'_\delta \times 10^5$ | $N'_{\delta\delta\delta} \times 10^5$ | $X'_{\delta\delta} \times 10^5$ |
| 15*                                                                                      | 1.54                               | -0.7               | 120.5                   | -77.0                                 | 0.6                | -67.3                   | 54.9                                  | -58.5                           |
| 15                                                                                       | 1.54                               | -6.2               | 254.5                   | -234.7                                | 1.0                | -127.6                  | 114.7                                 | -85.4                           |
| 14                                                                                       | 1.43                               | -6.1               | 256.3                   | -205.1                                | 0.2                | -140.9                  | 129.0                                 | -171.1                          |
| 13                                                                                       | 1.334                              | -4.0               | 294.0                   | -248.6                                | -0.3               | -150.1                  | 137.9                                 | -127.1                          |
| 12                                                                                       | 1.214                              | -0.1               | 315.3                   | -247.5                                | -2.2               | -169.9                  | 159.3                                 | -143.5                          |
| 11                                                                                       | 1.138                              | 3.3                | 353.6                   | -298.3                                | -2.8               | -186.8                  | 178.7                                 | -159.3                          |
| 10                                                                                       | 1.064                              | -5.0               | 379.7                   | -342.9                                | -2.1               | -209.1                  | 202.6                                 | -176.4                          |
| 9                                                                                        | 0.917                              | 5.1                | 496.5                   | -429.9                                | -5.1               | -251.6                  | 244.2                                 | -231.7                          |
| 8                                                                                        | 0.807                              | 18.4               | 650.4                   | -703.7                                | -8.4               | -305.2                  | 309.4                                 | -316.2                          |
| 7                                                                                        | 0.713                              | 4.1                | 729.5                   | -642.9                                | -7.3               | -364.4                  | 346.4                                 | -374.4                          |

\*Without propeller.

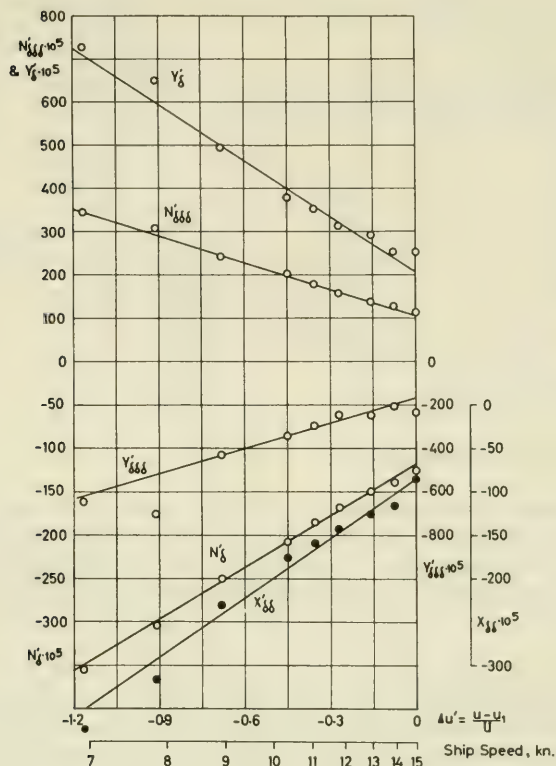


Fig. 32 - Variation of dimensionless rudder-angle coefficients with speed

## FREQUENCY AND TANK-INTERFERENCE EFFECTS

### Resonant Standing Waves

The following consists of a brief description of some aspects of the influence of frequency-dependent surface waves on force measurements made with the HyA planar-motion mechanism used in the dynamic mode. Not much information is available, as the policy adopted has been that of avoiding these effects, but in order to be able to do this, two series of exploratory tests were made soon after the mechanism was completed, and it is felt that the results of these tests are of sufficient interest to warrant description.

It is not unnatural that oscillation of a body in a free surface should give rise to waves which are dependent on frequency and amplitude of oscillation as well as on the speed of the body through the water. When the frequency of such waves corresponds to the natural frequency of the water in the towing tank, then tank resonance may be expected. Such resonance can result in the buildup of



significant standing waves which completely preclude the possibility of making useful measurements.

The frequencies at which tank resonance occurs are solely dependent on tank dimensions (12) and governed by the relation

$$\text{period} = \frac{2\pi}{\sigma}$$

where

$$\sigma^2 = gk \tanh kh$$

$$2\pi/k = \text{wavelength} = 2b, b, (2/3)b, (2/4)b, \text{ etc.}$$

$$h = \text{depth of tank water}$$

$$b = \text{breadth of tank}$$

$$g = \text{acceleration due to gravity.}$$

For the HyA towing tank, when  $b = 2h$ , the first four resonant standing-wave systems, according to these relationships, have periods of 4.07, 2.78, 2.26, and 1.98 seconds. The corresponding revolutions of the HyA planar-motion mechanism at which tank resonance may be expected are then 14.8, 20.5, 26.5, and 30.3 revolutions per minute.

"Pure sway" tests and "pure yaw" tests were made with a 6-m model of the Mariner class of vessels, for revolution of the planar-motion mechanism covering the range from 0 to 35 rpm. The object of the tests was to find the low-frequency area in which the measured forces ceased to be frequency dependent. The model used in the tests was the same as that described in Ref. 5, and the tests were made predominantly at 15 knots ship speed and with 65 mm amplitude at the scotch yokes. The raw forces measured at the Y gauges in the "pure sway" tests are shown plotted in Fig. 33. Above about 12 rpm the forces measured during a run down the tank varied in a manner akin to the "beating" phenomenon resulting from the interference of two wave trains having nearly equal frequencies. The ranges of variation of forces measured during a run down the tank are shown by heavy lines, and when the forces were repeatable, the values are indicated by single points in the usual manner.

It is evident that resonant standing waves occurred at about 15, 20, and 30 rpm exactly as predicted by theory.

At the higher frequencies, standing waves were clearly visible, and in an additional test at zero speed, they built up to alarming proportions, the whole water surface in the portion of the tank adjacent to the model being set into violent motion.

Below 12 rpm, however, no standing waves were observable, and it can be said that if they did exist, then the wave height was less than 1 mm. This was ascertained by close examination of the water level on the tank side when the model was oscillating.

## Prediction of Steering and Manoeuvring of Ships

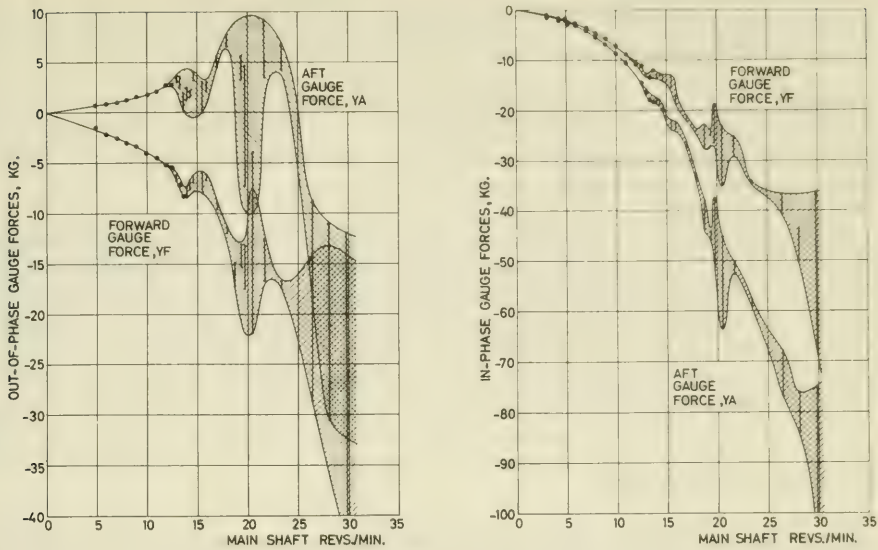


Fig. 33 - Results of pure sway tests: dimensional gauge forces as functions of planar-motion mechanism revolutions per minute

In the "pure yaw" tests, results similar to those shown in Fig. 33 were obtained, the resonant ranges occurring at about the same frequencies, but the variation of the measured forces in the resonant areas being less marked. This is explained by the fact that the model was a more effective wavemaker when oscillating with sway motions than when oscillating with yaw motions.

To sum up, for the model and speed tested, undesirable effects due to tank resonance were found to be negligible below about 12 rpm.

While tests have been made with only one model, at 0 and 15 knots, it is considered improbable that differences in speed or model proportions such as block-coefficient, length-to-draught, and length-to-breadth ratios would precipitate the onset of resonant waves. Based on these considerations, oscillatory tests made with the HyA planar-motion mechanism used in the dynamic mode are normally carried out at about 8 rpm, thus allowing a reasonable safety margin.

### Frequency Effects

A more insidious source of differences between measurements made using a steady-state technique such as a rotating arm and measurements made with a planar-motion mechanism stems from unsteady flow effects (independent of the free surface) which invalidate the assumption of quasi-steady conditions. This is a similar phenomenon to the well-known unsteady lift forces which act on an

airfoil having an oscillatory angle of attack. Waves generated as a result of the oscillatory motions of the model (independent of the tank boundaries) and reflections of such waves from the tank walls are two other sources of differences.

Since such effects, while repeatable, are frequency dependent, they can be assessed by comparing force measurements made at different frequencies, the agreement of which may safely be assumed to preclude the possibility of significant unsteady-flow effects.

The results of "pure sway" tests and "pure yaw" tests previously reported (Figs. 4 and 5 and Tables 7 and 8 of Ref. 5), showed that forces resulting from yaw velocity, yaw acceleration, and sway acceleration are quite independent of frequency of oscillation, within the low-frequency range used during the tests. Forces resulting from sway velocity, i.e., drift angle, were found to vary with frequency but to approach the steady-state results of the "static-drift-angle" test as frequency tended to zero.

### Higher Order Frequencies

The periodic motions generated by the HyA planar-motion mechanism are of almost perfect sine and cosine character, and so during a given run the resulting hydrodynamic forces at the Y gauges must necessarily be of the same frequency as the motions. There are no grounds to expect periodic forces having frequencies which are integer multiples of the frequency of oscillation, as opposed, for example, to analyses of fluctuating thrust and torque in a propeller shaft, when integer multiples of the basic shaft frequency would depend on number of blades, deadwood, etc. It is not therefore considered a disadvantage of the synchronous-switch technique for integration of periodic forces, that the Oscil. program is sensitive to uneven integer multiples of the frequency of the motion and insensitive to even multiples, and vice versa for the Const. program. Neither is it thought helpful to consider an alternative method of integration, in which the force signals are multiplied by  $\sin \omega t$  and  $\cos \omega t$ , as yielding the first harmonics of a Fourier analysis (13).

No difficulties have been encountered due to vibration of the HyA towing carriage and planar-motion mechanism, presumably because of their stiff construction, which ensures that any small vibrations are of such low amplitude and high frequency that the resulting high-frequency forces acting at the Y gauges are negligible in comparison with the forces under investigation.

### CONCLUSIONS

The method adopted at HyA for investigating steering and manoeuvring qualities of surface ships has been described. A planar-motion mechanism system is used for the experimental measurement of hydrodynamic coefficients, and predictions of manoeuvres are obtained from solutions of the equations of motion using a digital computer.

The mathematical model at present in use at HyA for simulation of steering and manoeuvring of surface ships is based on a third-order Taylor expansion of



forces and moments in the equations of motion. The mathematical model has been reduced to a manageable form by retaining only those hydrodynamic coefficients which experimental experience has shown not to be negligible. Roll and heel have also been neglected in the mathematical model, since they are felt to have little influence on prediction of manoeuvres, with the possible exception of fast warships.

The planar-motion mechanism system designed and built at HyA is used in a conventional towing tank and designed to oscillate the same large 6-to-7-metre wax models also used for resistance and propulsion tests. The mechanism is designed as a low-frequency, high-amplitude oscillator in order to reduce possible frequency problems and to cover adequately the various velocity and acceleration parameters encountered in ship manoeuvres.

The various tests which can be executed by the mechanism in its static and dynamic modes of operation (outlined in Figs. 2 to 4) make the planar-motion mechanism a versatile instrument for measurement of all of the hydrodynamic coefficients, both linear and nonlinear, which appear in the mathematical model.

The periodic acceleration-dependent and velocity-dependent forces measured by the force gauges in the dynamic mode of operation of the planar-motion mechanism are separated and recorded by integration using a synchronous-switch technique. The integration is controlled by programming circuits in a processing unit, and the principles of the measuring technique have been described in detail.

Considerations in the design of the different planar-motion mechanism tests and the analysis of force measurements to determine the hydrodynamic coefficients have been discussed. Results from resistance, self-propulsion, and open-water tests are used to compute coefficients in the X equation and to obtain the important relationship between propeller rpm and speed reduction encountered in manoeuvres, for different types of engines and engine settings. An experimental program used as standard at HyA for testing cargo ships is shown in Table 6.

The influence of speed on the nondimensional coefficients has been discussed, and the results which supplement those previously reported by HyA for the Mariner form (Ref. 5) have been presented as examples. It is found that the nondimensional coefficients  $Y'_v$ ,  $Y'_r$ ,  $N'_v$ , and  $N'_r$  are almost independent of speed over the speed range covered by normal merchant ships, and measurement of the coefficients  $Y'_{vu}$ ,  $Y'_{ru}$ ,  $N'_{vu}$ , and  $N'_{ru}$  has not been recommended as standard in the experimental program. The coefficients  $Y'_{\delta}$ ,  $Y'_{\delta\delta\delta}$ ,  $N'_{\delta}$ , and  $N'_{\delta\delta\delta}$  are shown to vary considerably with speed, forces on the rudder being strongly influenced by the propeller slip stream. The crosscoupling terms  $Y'_{\delta u}$ ,  $Y'_{\delta\delta\delta u}$ ,  $N'_{\delta u}$ , and  $N'_{\delta\delta\delta u}$  are consequently of importance in prediction of radical manoeuvres where speed loss is appreciable, and they should be measured as standard.

It has been found that resonance with a system of standing waves built up in the towing tank precludes the possibility of oscillating models above a certain critical frequency. The resonance is easily avoided, however, at the low frequency of oscillation recommended for experiments, but the range of yaw velocities which can be generated by the present HyA planar-motion mechanism is



consequently limited. Whereas an acceptable coverage can be obtained for the 6-to-7-metre cargo ship models, the range of yaw velocities which can be obtained for the smaller naval ship models is more limited than desirable, except at lower speed values.

Forces measured in the low-frequency range have been found either independent of frequency or to vary with frequency but approach steady-state results as frequency tends to zero.

It is finally concluded that the semitheoretical technique combining planar-motion mechanism model-testing and computer-prediction of manoeuvres allows scaling problems to be treated satisfactorily. Large models can be used in the testing, whereby scale effects in general are reduced. Tests can be carried out at the ship propulsion point by applying a towing force via the rigid connection to the mechanism, the correct propeller rpm being obtained by taking the particular engine type and control arrangement into account. The individual measurement of the various coefficients gives an insight into the hydrodynamic phenomena involved and consequently allows corrections to coefficients for difference in Reynolds' number for model and ship to be considered.

#### ACKNOWLEDGMENTS

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# WAVE EXCITATIONLESS SHIP FORMS

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## INTRODUCTION

It has been deduced theoretically by Ursell (1) and Hishida (2) that prisms of certain sectional shapes create no waves when they roll in a still water surface, and experimental check has also been made by McLeod and Hsieh (3). Bessho has extended this theme into motions of six degree of freedom and developed the theory of "Wave-free distributions" (4).

In addition, Newman (5) has shown, on the basis of Haskind relation, that the amplitude of radiated wave by an oscillating body in a free surface is directly related to the exciting force acting on the same body in waves.

These results indicate that there must be bodies which are free from exciting forces in waves.

The authors, being interested in the possible existence of such wave-excitationless bodies, have been carrying out experimental research for such bodies, and have found that there were a group of bodies which are free from wave-induced heaving force in waves of specified frequencies.

## 1. APPROXIMATE FORMULA OF HEAVING FORCE

Froude-Krylov hypothesis has long been used in estimating wave-induced heaving force until body-wave interaction was clarified in recent years. According to the Froude-Krylov hypothesis, heaving force becomes nil only when the waterplane area is zero. However, the authors were aware that the body-wave interaction is reverse in sign of that of Froude-Krylov force and that there must be a possibility that a body could be designed to have greater body-wave interaction in relation to the Froude-Krylov force so that they will cancel each other.

As Motora has shown (6), the heaving force is expressed approximately as a summation of an inertia term, a damping term, and a buoyancy term as shown in Eq. (1).

$$F_{Zw} = \gamma_1 k_z \rho V \ddot{Z}_w + \gamma_2 N_z \dot{Z}_w + \gamma_3 \rho g A Z_w, \quad (1)$$



where

$k_z$  = added mass coefficient,

$\rho$  = density of surrounding fluid,

$V$  = volume of a body,

$Z_w$  = wave elevation,

$N_z$  = damping coefficient,

$g$  = acceleration of gravity,

$A$  = waterplane area, and

$\gamma_1, \gamma_2, \gamma_3$  are correction factors for the orbital motion of the wave.

The first two terms correspond to the body-wave interaction, and the third term corresponds to the Froude-Krylov buoyancy. The first term is also an inertia term due to added mass effect. Factor  $\gamma_3$  has been known as Smith Correction Factor.

If the wave elevation is of the form

$$Z_w = \bar{Z}_w e^{i\omega t}$$

where  $\omega$  is the circular frequency, then

$$\ddot{Z}_w = -\omega^2 Z_w.$$

Therefore, Eq. (1) is rewritten as

$$F_{Z_w} = -\gamma_1 k_z \rho V \omega^2 Z_w + \gamma_2 N_z \dot{Z}_w + \gamma_3 \rho g A Z_w. \quad (2)$$

The inertia term is reverse in sign in relation to the buoyancy term and will increase rapidly as the frequency  $\omega$  increases. In case of ordinary ship form however, the first term is not large enough to cancel the buoyancy term in a frequency range of ordinary wave encounter. However, if the form of a body is chosen so that the inertia term is large enough compared to the buoyancy term, it will be possible to eliminate the heaving force in a relatively low frequency range.

According to Newman (5), heaving force is related to the damping coefficient by the following formulas:

$$N_z = \frac{\omega}{\frac{1}{4} \rho g^2 h^2} F_{Z_w}^2 \quad (\text{two-dimensional case})$$

$$N_z = \frac{\omega K}{\frac{1}{4} \pi \rho g^2 h^2} \int_0^{2\pi} F_{Z_w}(\theta) d\theta \quad (\text{three-dimensional case}) \quad (3)$$

where

$K$  = wave number,

$\theta$  = encounter angle of an incident wave, and

$h$  = the wave height.

Substituting Eq. (3) into (2), we can get approximate value of the amplitude of the heaving force  $\bar{F}_{Zw}$  as follows:

$$\bar{F}_{Zw} \doteq (-\gamma_1 k_z \rho \bar{V} \omega^2 + \gamma_3 \rho g A) \bar{Z}_w \sqrt{1 + (-\gamma_1 k_z \rho \bar{V} \omega^2 + \gamma_3 \rho g A)^2 \left( \frac{4\gamma_2 \omega^2}{\rho g^2 h^2} \right)^2 \bar{Z}_w^4} . \quad (4)$$

Therefore, heaving force will vanish when the frequency  $\omega$  takes the following value:

$$\omega_0 = \sqrt{\frac{\gamma_3 g A}{\gamma_1 k_z \bar{V}}} \quad (5)$$

or

$$K_0 = \frac{\gamma_3 A}{\gamma_1 k_z \bar{V}} . \quad (6)$$

Let us call  $\omega_0$  as "excitationless frequency" for convenience. To bring this excitationless frequency toward low-frequency range of probable wave encounter, it will be necessary to make  $A/\bar{V}$  smaller than usual proportion. It will be noted that results of measured external force to oscillate different bow section models by Paulling (7) indicate the similar tendency, though in his case the inertia force includes the mass of a model itself.

## 2. TWO-DIMENSIONAL CASE

### 2.1 Submerged Circular Cylinder with a Strut

#### 1) Theoretical consideration

In seeking a body which has relatively greater inertia force, one may be aware that an extreme case of such a body is a completely submerged body. The main part of the heaving force acting to a submerged body is an inertia force which is reverse in sign of the wave elevation. Therefore, if a vertical strut of narrower width is attached to a submerged body so that it gives small amount of buoyancy, it will be possible to eliminate the heaving force at a specified frequency of waves.

Therefore, let us choose a combination of a circular cylinder of radius  $a$ , depth  $f$ , and a vertical strut of breadth  $B$  as shown in Fig. 1.

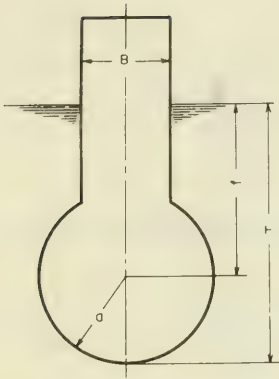


Fig. 1 - Geometry of circular cylinder type models

Though the heaving force acting on a circular cylinder has been calculated by Cummins (8), it is necessary for us to calculate the inertia term and buoyancy term separately. An approximate calculation was done making use of the Eq. (4), i.e., it was assumed approximately that a summation of the inertia term for a cylinder and for a strut would give the inertia term of the total body. The same assumption was applied for the buoyancy term.

For the inertia coefficient  $k_z$  of a circular cylinder, values given by Yamamoto (9) were used, and for that of a strut, values given by Tasai (10) for full section were employed.

Calculated inertia force and buoyancy for different breadth of the strut are as shown in Fig. 2.

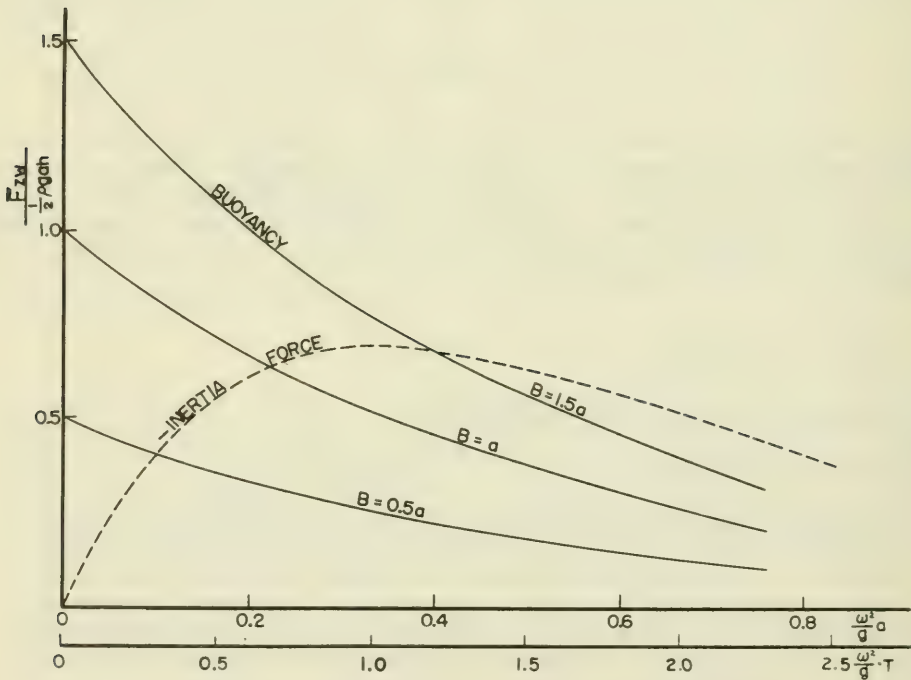


Fig. 2 - Calculated inertia force and buoyancy for different breadths of the strut

The approximate values of the heaving force is obtained by Eq. (4) as shown in Fig. 3.

At relatively low frequency range, the buoyancy is greater than the inertia force (or under cancel), and the heaving force will be inphase with the wave elevation. At  $\omega_0$  given by Eq. (5), heaving force is zero, and at higher frequency than  $\omega_0$ , the inertia force exceeds (over cancels) the buoyancy, i.e., the heaving force at this frequency range will be 180 degrees out-of-phase with the wave elevation.

It is also shown in Fig. 3 that  $\omega_0$  shifts toward lower frequency when the breadth of the strut decreases.

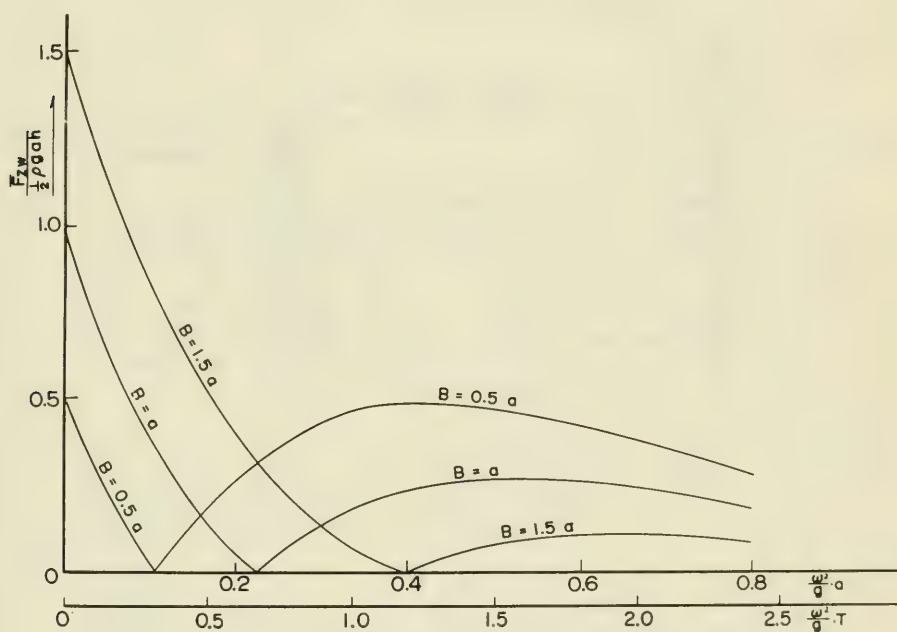


Fig. 3 - Calculated heaving force of circular cylinder type models

2) To check the above described results, experiments were conducted on a model of the following size:

Diameter -  $2a = 20$  cm

Breadth of the strut -  $B = 15$  cm ( $1.5a$ ),  $10$  cm ( $a$ ), and  $5$  cm ( $0.5a$ )

Length =  $50$  cm (with end plates).

To prevent the sway and pitch of the model a guide is attached to the model as shown in Fig. 4. The heaving force was measured by a rigid spring attached by a linear transformer as a pick-up.



A typical result for  $B = a$  and depth  $f = 2a$  is shown in Fig. 5 together with the computed one. The base of Fig. 5 is a non-dimensionalized wave frequency and the ordinate is the non-dimensionalized heaving force. The thick solid line shows the heaving force of a semisubmerged circular cylinder of radius  $a/2$ .

A remarkable decrease of the heaving force at a frequency approximately equal to the theoretical  $\omega_0$  will be recognized.

Figure 6 is a typical oscillogram of the heaving force and the wave elevation as  $(\omega^2/g)a = 0.124, 0.238, \text{ and } 0.402$ . At  $(\omega^2/g)a = 0.124$ , where the buoyancy is greater than the inertia force, the heaving force is in phase with the

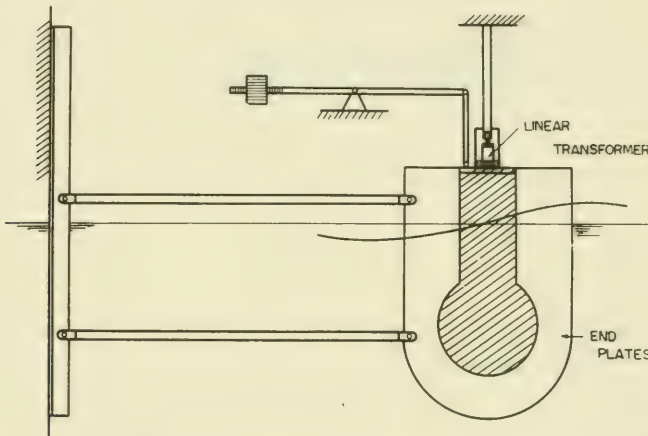


Fig. 4 - Set-up of heaving force measurement

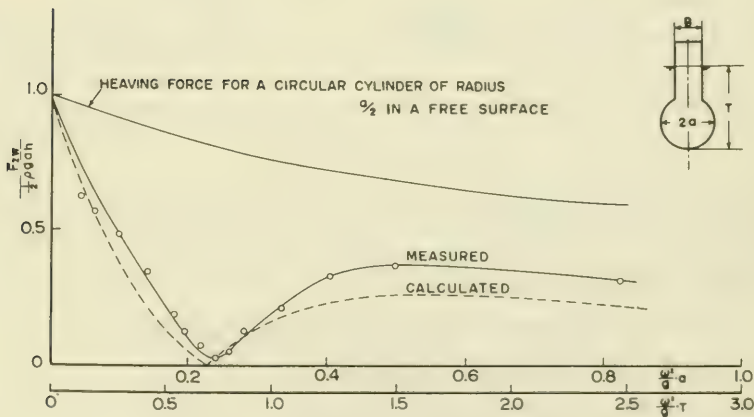


Fig. 5 - A typical result

# Wave Excitationless Ships Forms

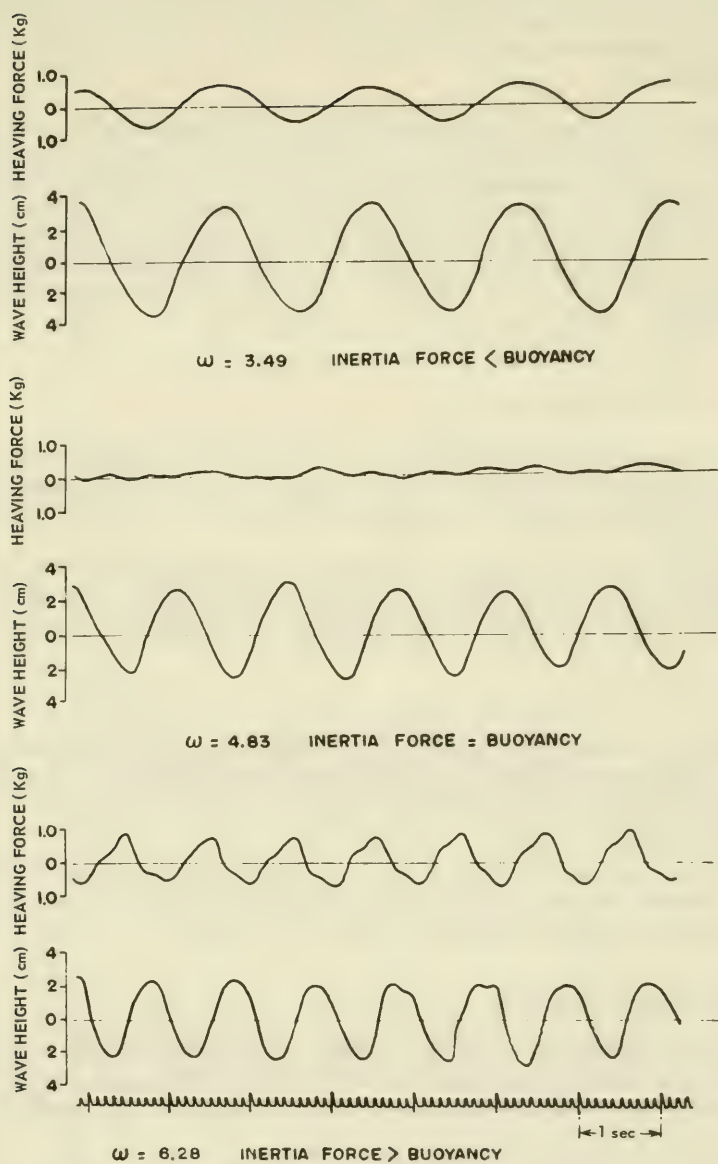


Fig. 6 - Typical oscillogram of the heaving force and wave elevation

wave elevation (under cancel). At  $(\omega^2/g)a = 0.238$ , the heaving force almost vanishes. At  $(\omega^2/g)a = 0.402$ , where the buoyancy is less than the inertia force, the heaving force is 180 degrees out of phase with the wave elevation (over cancel). These oscillograms gave valuable information in determining a suitable breadth of the strut.

To examine the effect of the breadth of the strut as well as the depth of the cylinder, results are shown in Fig. 7. As predicted by theory,  $\omega_0$  shifts to lower frequency as the breadth of the strut decreases. However, change of depth does not affect  $\omega_0$ . Frequency  $\omega_0$  for  $a = 10$  cm,  $T = 30$  cm is 5.03, which corresponds to  $\omega_0 = 0.873$  for a full-scale ship of length 140 m and draft 10 m. This frequency corresponds to wavelength 61 m for beam seas. However, in case of longitudinal waves, this frequency corresponds to wavelength 200 m at ship speed 18 kts.

## 2.2 Elliptic Cylinder with Strut

To make the draft relatively small, an elliptic cylinder was chosen as the main body. The ratio of major axis and minor axis was varied from 2 to 4, where the breadth of the strut was kept to be a half of the major axis (Fig. 8).

Results are as shown in Fig. 9. Different from the result of the former case,  $\omega_0$  seems to shift toward lower frequency as the depth increases. Absolute value of the heaving force is about the same as the former case.

## 2.3 Fins with Strut

To make the inertia term greater, it is necessary to increase the added mass  $k_z \rho V$ . The virtual mass for a thin plate of breadth  $2a$  is  $\rho \pi a^2$  per unit length. Therefore, it will be possible to replace ellipse by adequate size of fins as shown in Fig. 10.

A model in which  $2a = 25$  cm,  $B = 15$  cm,  $f' = 7$  cm was tested in the same technique. Results are as shown in Fig. 11. In general, it can be said that effectiveness of fins are almost equivalent to a thin ellipse.

## 3. HEAVING AMPLITUDE

As shown in Section 2, there are several variations of bodies which are not acted on by wave-induced heaving force at specified frequency. However, it will be premature to conclude that these bodies do not heave at the specified frequency.

As shown in Eq. (3), damping factor is proportional to the square of the heaving force. This means that a wave-excitationless body is also a dampingless body.

In fact, in the case of the circular cylinder as described in 2) of Section 2.1, measured heaving amplitude is quite large as shown in Fig. 12. There appears a high peak at resonant frequency and a minimum point at a frequency about  $\omega_0$ .

# Wave Excitationless Ships Forms

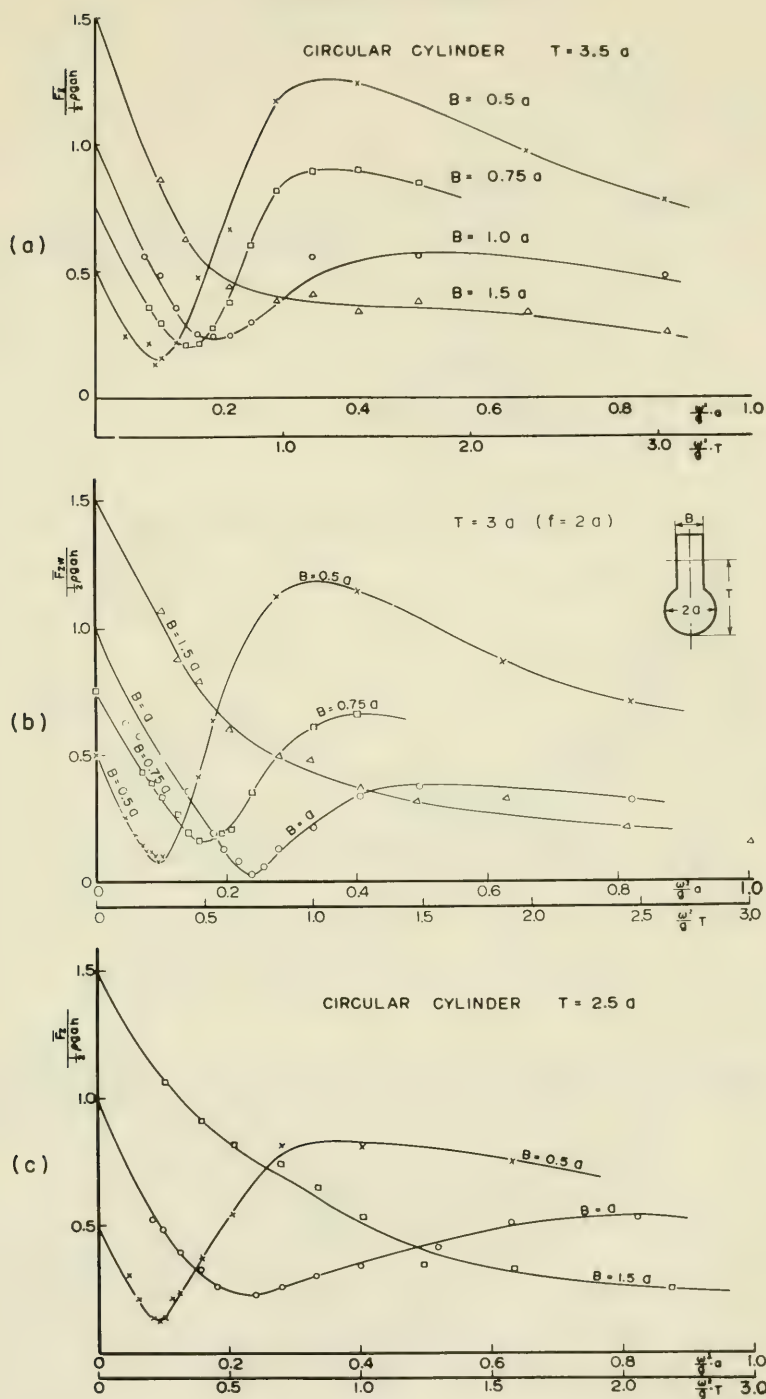


Fig. 7 - Heaving force of the circular cylinder type models



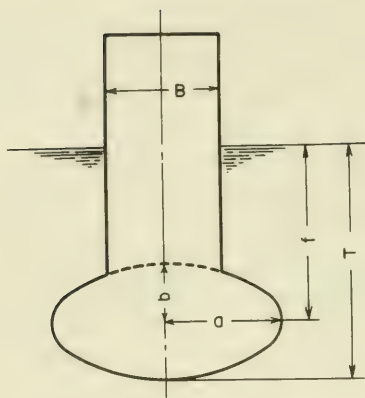


Fig. 8 - Geometry of elliptic cylinder type models

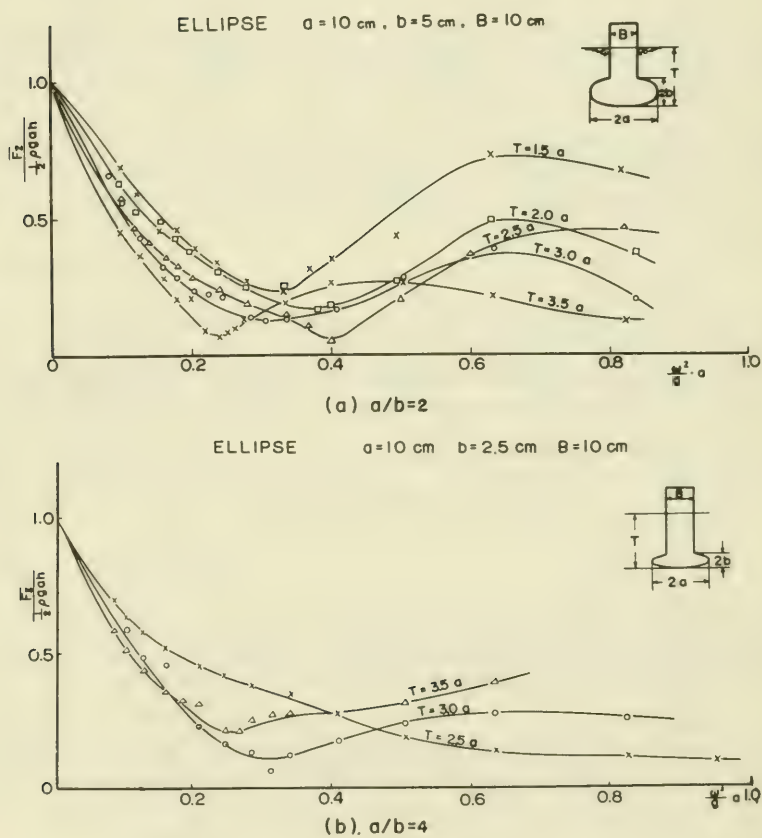


Fig. 9 - Heaving force of elliptic cylinder types

Fig. 10 - Geometry of fin type models

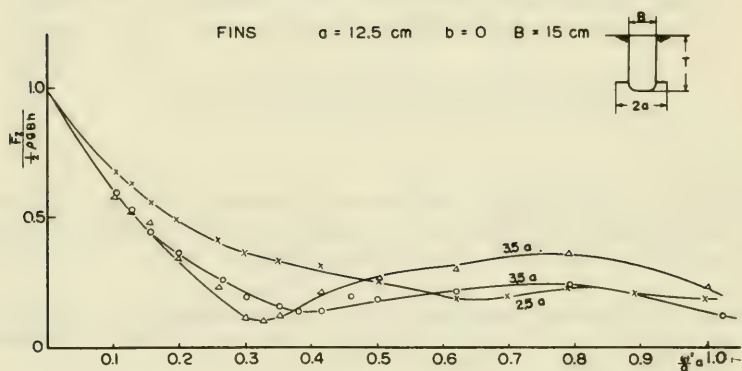
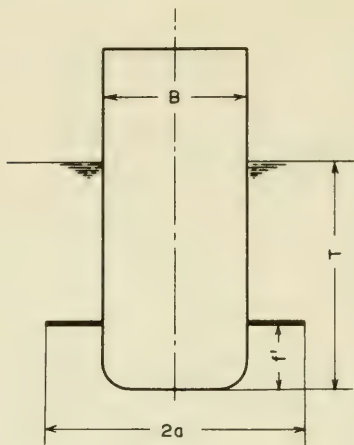


Fig. 11 - Heaving force of fin type models

It will be interesting to bring the resonant frequency equal to  $\omega_0$ . However, this is proved not to be practicable for the following reason:

$$\text{resonant frequency of heaving} = \omega_s = \sqrt{\frac{\rho g A}{m + m_z}}, \quad (7)$$

where  $m$  is the mass of a body.

$$\text{excitationless frequency} = \omega_0 = \sqrt{\frac{\gamma_3 \rho g A}{\gamma_1 m_z}}. \quad (8)$$

As the ratio  $\gamma_3/\gamma_1$  is almost equal to unity,  $\omega_s$  cannot be equal to  $\omega_0$  unless the mass of the body is zero.

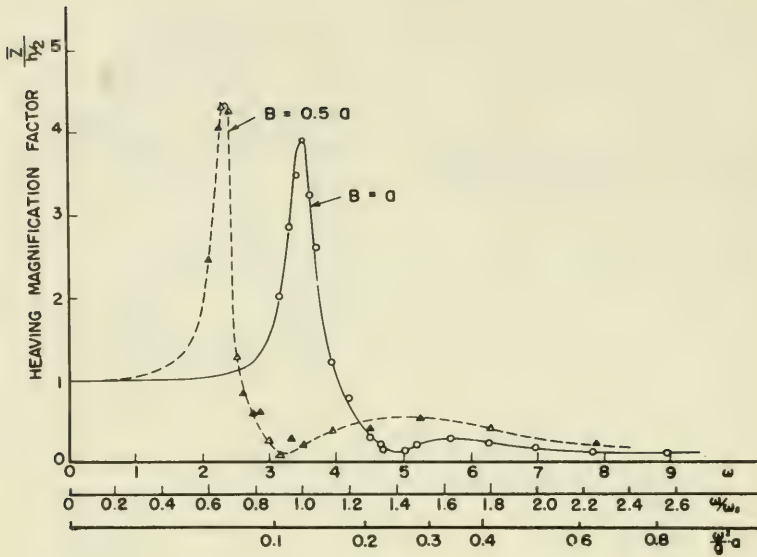


Fig. 12 - Heaving magnification factor of circular cylinder type models

As far as the wave damping is concerned, it seems hopeless to decrease the heaving amplitude. However, it should be noted that eddy damping or artificial damping are not included in Haskind-Newman relation.

Therefore, eddy-making damping or artificial damping such as by a passive tank are given to a wave-excitationless body, it will be possible to minimize the heaving.

Heaving of an ellipse ( $a/b = 2$ ) with a strut is measured as shown in Fig. 13(a) in which less heaving amplitude will be recognized than the case of a circular cylinder due to eddy damping. Thinner ellipse ( $a/b = 4$ ) with a strut is also tested as shown in Fig. 13(b). Remarkable decrease of heaving due to eddy damping will be noted.

Heaving of a body with fins as described in Section 2.3 is also tested. Results are as shown in Fig. 14. Though the magnification factor diagram looks like that of critical damping, actual damping is about one-third the critical damping. Apparent critical damping of the magnification factor diagram is due to rapid decrease of the exciting force as the frequency approaches  $\omega_0$ .

#### 4. A TRIAL TO ELIMINATE WAVE EXCITATION IN WIDER RANGE OF FREQUENCY

According to the above described method, the exciting force is eliminated at only one specified frequency. The following is a trial to eliminate the exciting force at two or more frequencies.

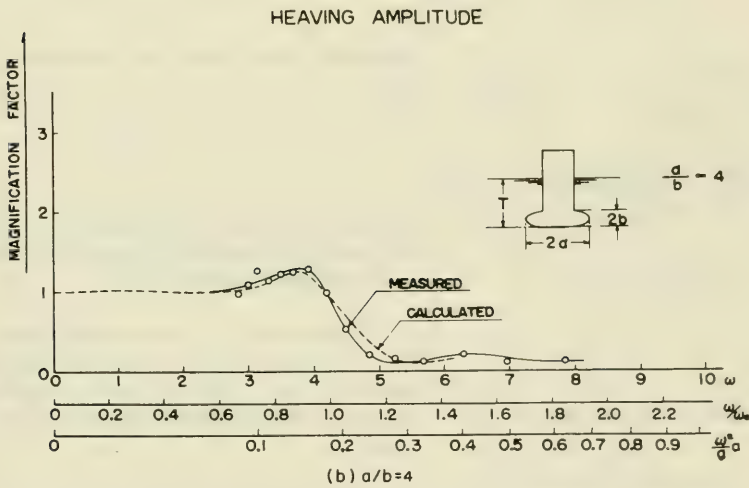
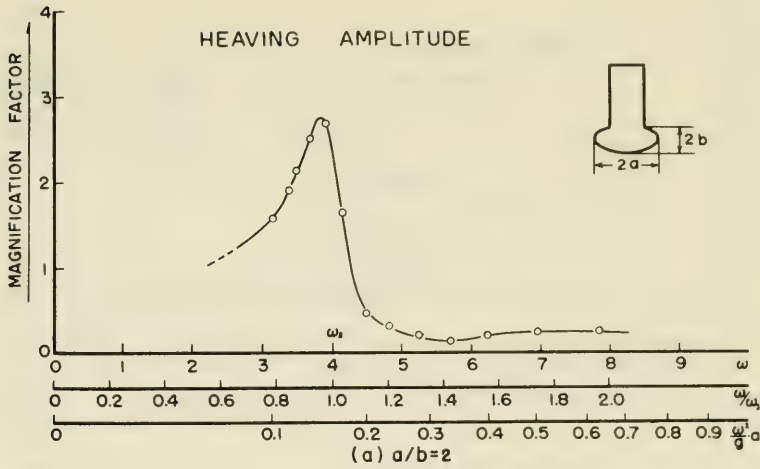


Fig. 13 - Heaving magnification factor of elliptic cylinder type models

As shown in Fig. 15, an inner solid strut of breadth  $B_1$  is covered by a tank of breadth  $B_2$ . The tank is provided by small holes of total area  $S$ , depth  $H$ .

If we denote the excitationless frequency for  $B_1$  as  $\omega_{01}$ , the heaving force will vanish at frequency  $\omega_{01}$  provided the area  $S$  is chosen adequately so that the water level in the tank is always equal to that of outer surface at this frequency.

On the other hand, according to a theory of antipitching tank (12,13), the movement of water in the tank, as the outer water level changes periodically,



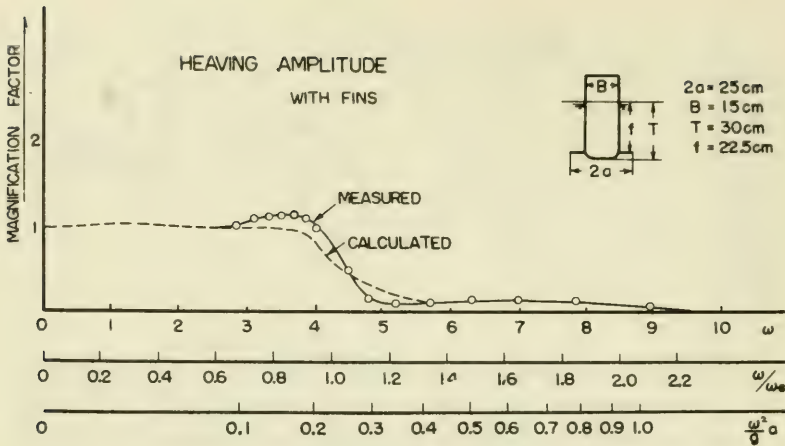


Fig. 14 - Heaving magnification factor of a fin type model

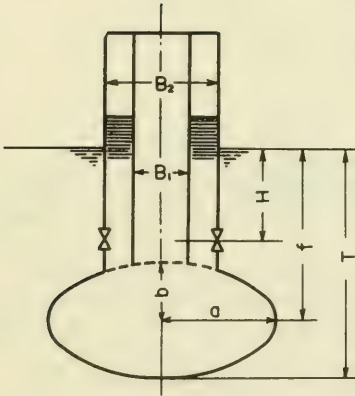


Fig. 15 - Geometry of double caisson type models

becomes zero when the frequency coincides with

$$\omega'_0 = \sqrt{\frac{g}{H}} \quad (9)$$

Therefore, at the frequency  $\omega'_0$ , tank water is practically solidified and the breadth of the strut is practically  $B_2$ .

Therefore, if the excitationless frequency for  $B_2$  is denoted  $\omega_{02}$ , heaving force will vanish at  $\omega_{02}$ . Thus, the heaving force will vanish at two different frequencies  $\omega_{01}$ , and  $\omega_{02}$ .

Experiments were conducted on a model for which  $B_1$ ,  $B_2$ , and  $H$  were chosen as shown in Fig. 15. Results are as shown in Fig. 16 in which three results for different

size of the holes are indicated. By Fig. 16, it will easily be seen that the heaving force almost vanishes at predicted  $\omega_{01}$  and  $\omega_{02}$ , and is very small in a frequency range founded by  $\omega_{01}$  and  $\omega_{02}$ .

## 5. THREE-DIMENSIONAL PROBLEMS

### 5.1 A Sphere with Vertical Cylinder

As a most simple three-dimensional case, let us consider a sphere with vertical cylinder as shown in Fig. 17. As the Eq. (2) is applicable for a

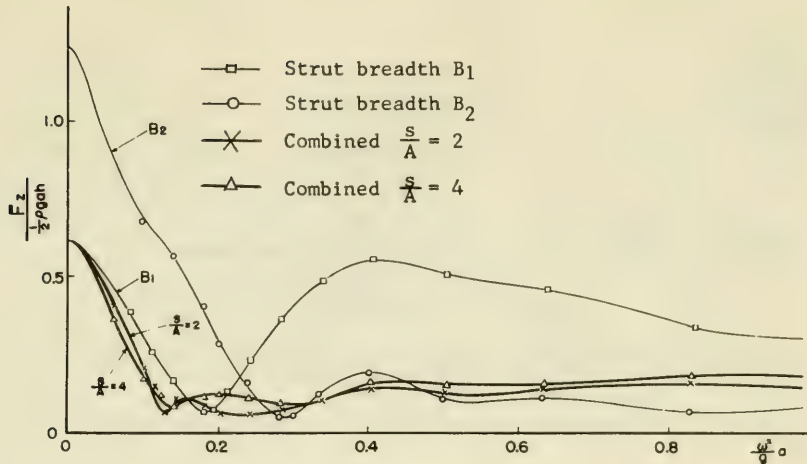
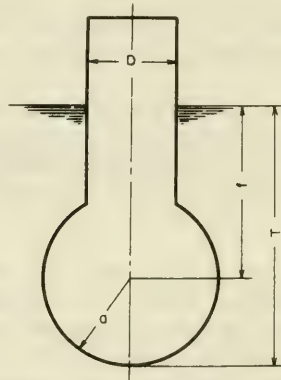


Fig. 16 - Heaving force of a double caisson type model

Fig. 17 - Geometry of sphere type models



three-dimensional case provided adequate values for  $\gamma$  are chosen, the amplitude of heaving force of such a body is calculated by Eq. (2) as shown in Fig. 18.

Heaving-force measurements have been made of models with the following sizes:  $a = 14.3$  cm,  $D = a$  and  $\sqrt{2a}$ ,  $T = 1.5a$  and  $2a$ . A typical record obtained is shown in Fig. 19 which shows almost the same tendency as the two-dimensional case shown in Fig. 6.

In Fig. 20, heaving forces for different depths are shown. Reasonable agreement with theoretical calculations will be noticed.

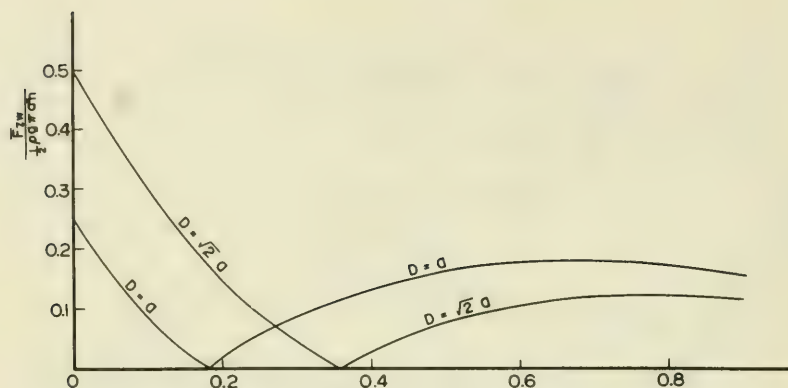


Fig. 18 - Calculated heaving force of sphere type models

## 5.2 A Prism in Longitudinal Wave

In relation to the possibility of applying the strip method for three-dimensional cases, it is important to know how a wave acts to a section of unit length of a prism fixed in longitudinal wave train.

This problem was treated by Okumura and Sugiura (11) for a prism of a section shown in Fig. 13(b). A unit length of prism was cut off and kept free from the main body and was supported by a rigid spring attached with linear transformer as shown in Fig. 21, so that heaving force acting on this segment was measured.

Results are as shown in Fig. 22. In Fig. 23, heaving forces in beam seas as well as longitudinal waves are compared with calculated values. Reasonable agreement between them will suggest that the strip method will be applicable for obtaining heaving force as well as pitching moment of ships in longitudinal waves. Therefore, there will be a possibility to eliminate pitching moment of ships by the same technique.

## 6. COMPARISON WITH THE "WAVE-FREE DISTRIBUTION" THEORY

Bessho has given in his paper "wave-free distribution" (4), a group of two-dimensional bodies which are wave-free as they heave in a free surface. They are derived by the following procedure:

Supposing  $m(t)$  is a function which is regular at infinity and is vertically skew-symmetry. Then a function

$$f(t) = Km(t) + i \frac{d}{dt} m(t) \quad (10)$$

# Wave Excitationless Ships Forms

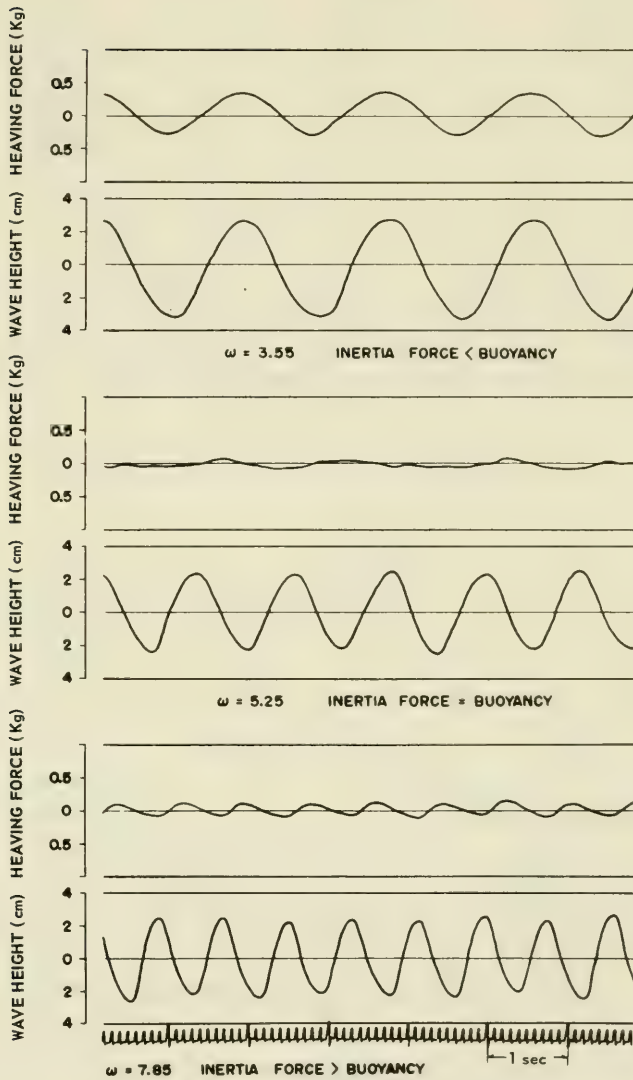


Fig. 19 - Typical record obtained for a sphere type model



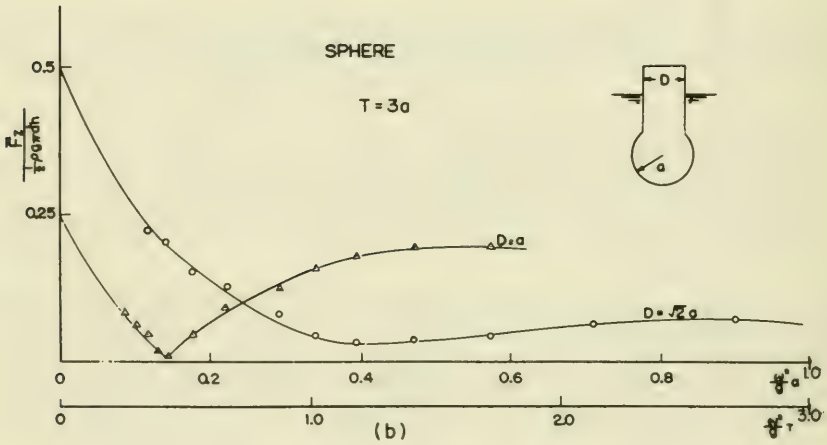
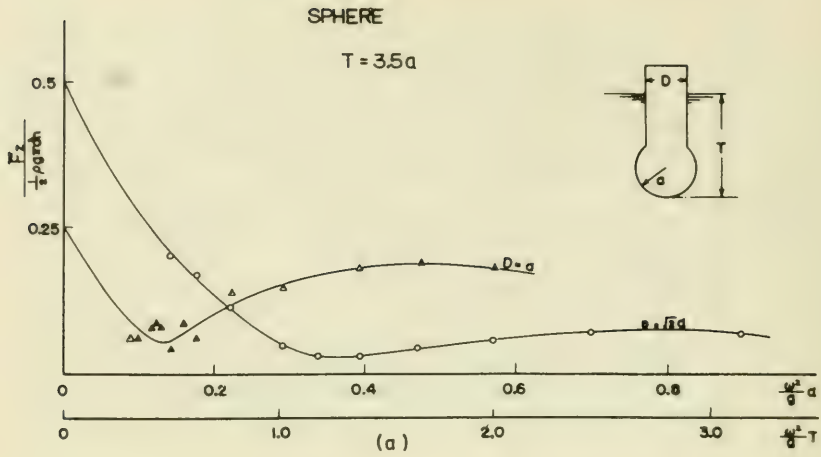


Fig. 20 - Heaving force of sphere type models

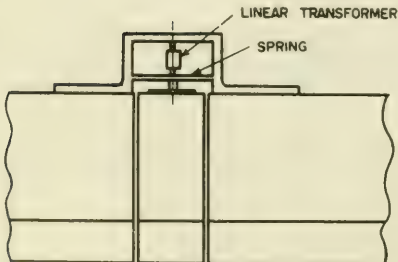


Fig. 21 - Set-up of heaving force measurement

# Wave Excitationless Ships Forms

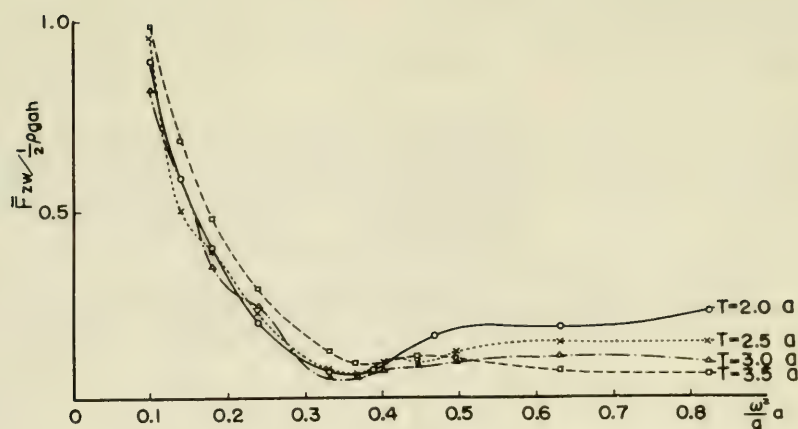


Fig. 22 - Heaving force per unit length of a elliptic-type prism in longitudinal waves

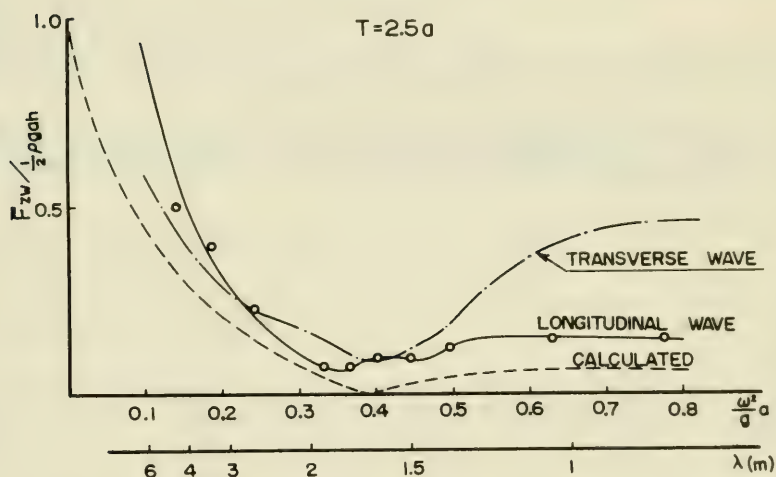


Fig. 23 - Comparison of heaving force in longitudinal waves with that in transverse waves

will give a velocity potential which satisfies the free surface condition and should not associate surface waves. Therefore,  $f(t)$  is a velocity potential around a body which is free from surface wave as it heaves in a free surface.

If a singular point at depth  $h$  is taken to be  $m(t)$ , viz,

$$m(t) = \log \left( \frac{t + ih}{t - ih} \right), \quad (11)$$

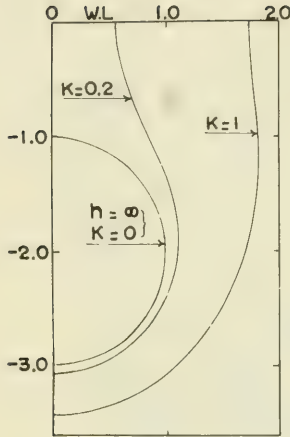


Fig. 24 - Typical stream lines of two-dimensional wave-free distributions

then

$$f(t) = \frac{i}{t + ih} - \frac{i}{t - ih} + K \log \left( \frac{t + ih}{t - ih} \right) \quad (12)$$

Stream lines given by this function are as shown in Fig. 24. As seen in Fig. 24, one of them is similar to the circular cylinder type model of the present paper. It is quite interesting to see that two different approaches reached similar results.

An experiment was conducted to measure the heaving force for such a body. Results are as shown in Fig. 25 where heaving force for a circular cylinder type is also shown for comparison. From Fig. 25 it will be noticed that the wave-free body shows quite similar heaving force vs frequency curve compared with the circular cylinder type model.

## CONCLUSIONS

1. The Froude-Krylov buoyancy and the inertia force which is induced by the added mass effect are reverse in sign. Therefore, it is possible to make them cancel each other so that wave excitation is reduced.
2. At a frequency at which the absolute value of the inertia force is equal to that of buoyancy, the heaving force becomes exactly zero theoretically, and is proved experimentally to be almost zero.

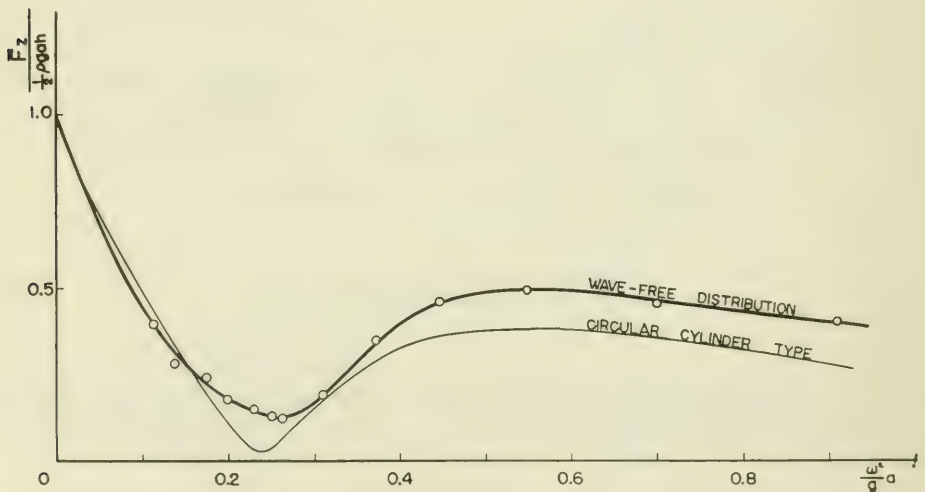


Fig. 25 - Comparison between the heaving force of a wave-free body obtained by Bessho and that of the circular cylinder type model

3. To bring the excitationless frequency into low-frequency range such as ordinary wave encounter frequency, it will be necessary to make the underwater volume large enough in relation to the water plane area. The underwater volume can be replaced by a flat plate which has the same amount of added mass as the underwater body.

4. Wave-damping also vanishes as the heaving force vanishes. Therefore, it will be necessary to give damping other than wave-damping to a body to reduce heaving or pitching motion.

5. In the case of a two-dimensional body, for a segment of unit length, heaving force due to beam seas is almost equal to that due to longitudinal waves. Therefore, strip method can be applied for three-dimensional bodies. It will also be possible to apply the present technique to the pitching motion.

6. It is also possible to extend this technique to the three-dimensional problem.

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\* \* \*

## DISCUSSION

Capt. William R. Porter, USN  
*Institute for Defense Analyses*  
*Arlington, Virginia*

I want to congratulate the authors for presenting a paper which includes not only theoretical observations of interest, but also includes experimental evidence that the observations in theory are reasonably valid in reality. I will confine my technical remarks now to the case of two-dimensional bodies only. The points on which I will comment are:

1. Shiplike sections which do not generate waves when forced in vertical oscillation at critical frequencies.
2. Shiplike sections which are free of vertical excitation in waves at critical frequencies.
3. Calculation of the hydrodynamic coefficients by exact, in lieu of approximate, analysis.

In their paper the authors report approximate techniques to calculate the results for sections synthesized from simple shapes and which are free of vertical excitation in waves. The same sections do not generate waves when forced at critical vertical frequencies. In order to study such forms with shiplike sections, I have generated a number of sections within those mapped by the transformation

$$Z = \zeta + \sum_{i=0}^N a_{2i+1} / \zeta^{2i+1}.$$

This transform has been used by myself and others for other purposes, and with the proper choices of the coefficients  $a_i$  and the number of coefficients, this can also generate more or less smooth bulbous sections which display the particular hydrodynamic behavior of interest here. The hydrodynamic coefficients for these transform sections in forced vertical, horizontal, or rolling motion in fluid of arbitrary depth can be calculated exactly (in the numerical technique sense) by the methods I have previously used. The point is that exact procedures of analysis of continuous shiplike forms tend to confirm the results of the authors' approximate calculations for their forms.

As an example, I have results for a family of forms illustrated in Fig. D1. The family varies strut length and strut width. The geometric coefficients  $a_i$  were found by a procedure numerically mapping the offsets of bulbous shapes selected as a family. The form with the widest strut, section 1, with the beam of the strut equal to the beam of the bulb, that is, the form with vertical sides and without reverse curvature, does not show zero vertical force in the finite frequency range. The other forms show the desired phenomena and the critical frequencies; and, as a family, the trend of behavior is smooth. For clarity of the figures, I have illustrated the wave amplitude ratio  $\bar{A}$  (Fig. D2) and the free surface correction coefficient  $k_4$  (Fig. D3) for only three of the forms. The zeroes of generated wave amplitude are easily seen. The approximate zero wave-excitation according to the estimate by the authors is not obvious in my figures, but it is clearly dependent on the added mass coefficient, and its variation with frequency is displayed by the free-surface correction coefficient.

Unfortunately, I can only report these theoretical observations without further experimental results such as the authors have reported. On this note, I will close with repeated congratulations to the authors for both the theoretical estimate and experimental report.

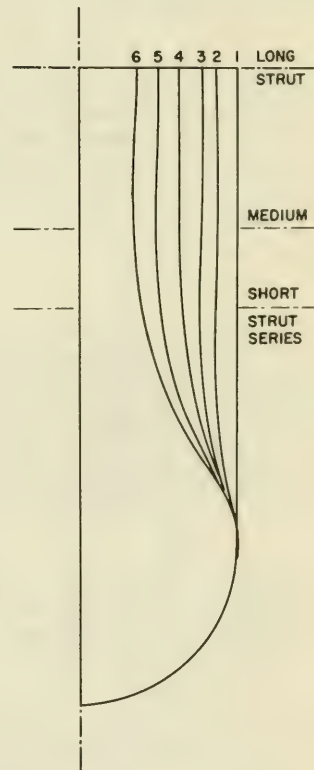


Fig. D1 - Family of ship-like forms

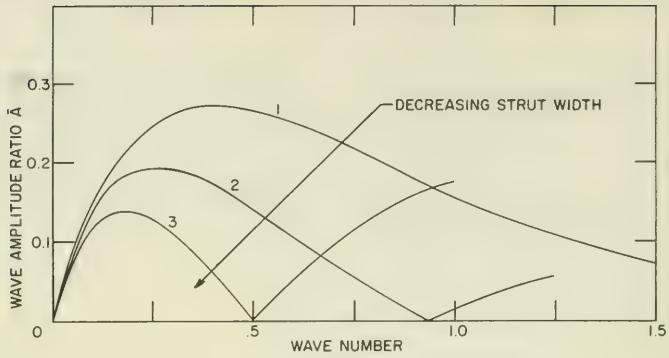


Fig. D2 - Wave amplitude ratio (damping coefficient) for forms showing zero generated wave height. Short-strut forms.

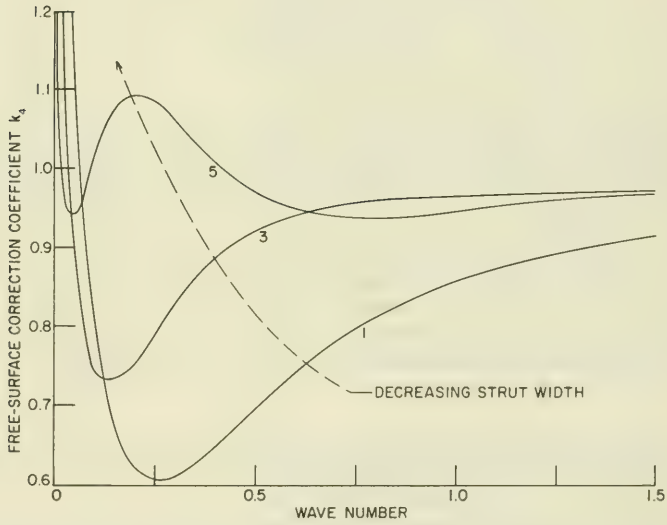


Fig. D3 - Free surface correction coefficient  $k_4$  for the added-mass coefficient for short-strut forms

\* \* \*

## DISCUSSION

W. Frank

*David Taylor Model Basin  
Washington, D.C.*

After hearing of the authors' remarkable and extensive investigations on the heaving force of bulbous bodies in regular waves, we at the David Taylor Model Basin were in a position to compare some of their results with results obtained by a computer program which calculates the added mass and damping of two-dimensional bodies.

The added mass and damping coefficient are evaluated by distributing wave sources over the cross section of the submerged body, the source strengths being determined from an integral equation obtained from the kinematic boundary condition on the body. The amplitude of the heaving force — in the authors' normalized form — was evaluated by the Newman-Haskind relation between the damping coefficient of a sinusoidally oscillating two-dimensional body and the exciting force on the same body restrained in regular waves, as expressed by Eq. (3) in the paper.

We have considered some of the geometrics investigated by the authors. Figure D4 shows our results for three circular cylinders with strut as compared with the authors' experimental data in their Fig. 7(b). The circles represent the experimental data for  $B = 1.5a$ , the squares give the measured results for the case  $B = a$ , and the triangles designate the authors' measurements for  $B = 0.5a$ . We see that our theory predicts the occurrence of minimum force amplitudes fairly well, and we notice good agreement between our theory and the experimental data in the lower end of the frequency range.

Figure D5 compares our calculations for an elliptic cylinder with strut with Motora and Koyama's measurements taken from their Fig. 9(a). Again we notice good agreement between the curve and the measurements for the lower frequencies, and we see that the zero of the curve occurs at the frequency of lowest measurement.

We have calculated the damping of circular cylinders with various strut lengths and we have found, just as Motora and Koyama have, that, for fixed beam-to-diameter ratio, zero damping occurs at nearly the same frequency for all draft-to-diameter ratios considered. This result holds also for the limiting case of ogival cylinders — that is, circular cylinders without strut but more than half submerged.



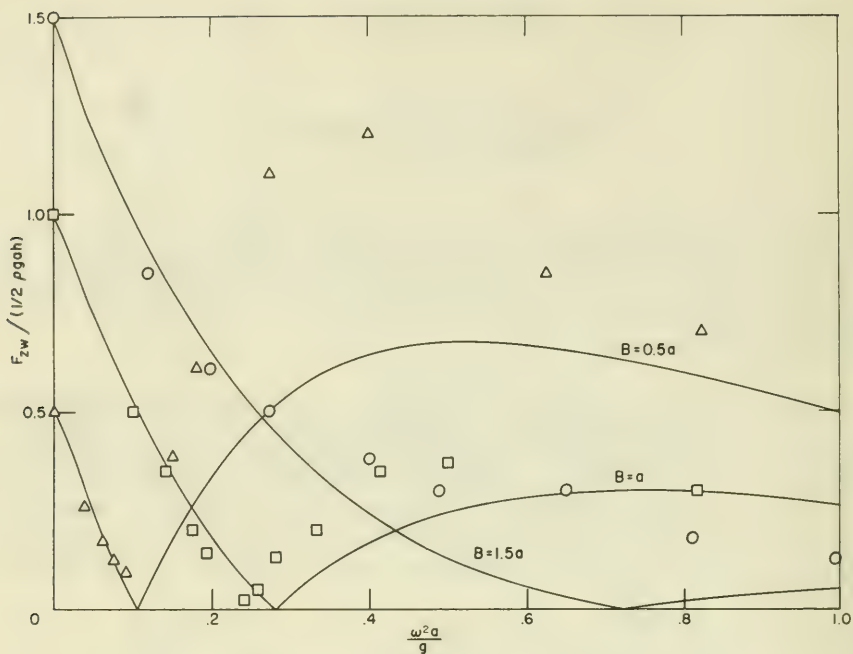


Fig. D4 - Circular cylinder with strut,  $T = 3a$

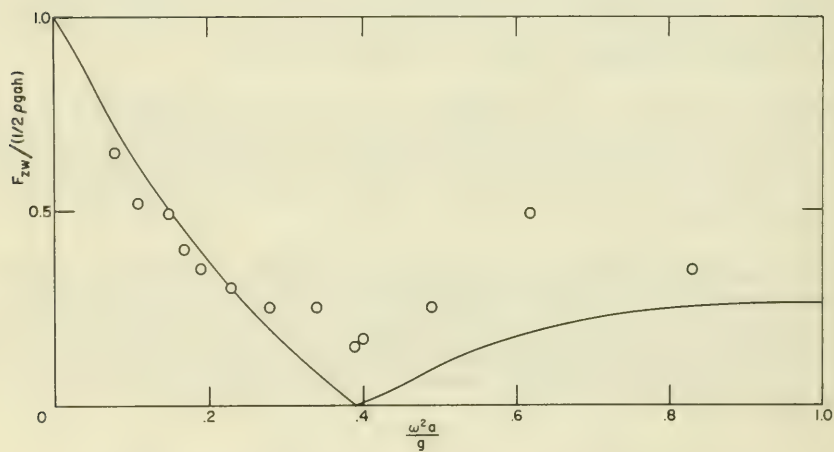


Fig. D5 - Elliptic cylinder with strut,  $T = 2a$ ,  $B = a$ ,  $a/b = 2$

\* \* \*

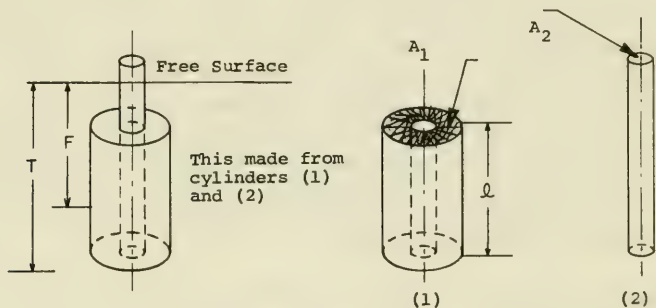
## DISCUSSION

Jerome H. Milgram

*Massachusetts Institute of Technology  
Cambridge, Massachusetts*

During the oral discussion of this paper at the Sixth Symposium on Naval Hydrodynamics, one person mentioned the possibility that wave-excitationless forms might be used for a stable base from which waves could be measured in the open sea. This possibility has been extensively explored with successful results by a number of researchers; see e.g. Kerr (1) or Milgram (2). Both Kerr and Milgram calculated the heaving force on their buoys by neglecting the scattered waves and approximately the force caused by the incident wave alone. A modified form of this calculation is given here.

Consider a body formed by two cylinders, one of which pierces the free surface (Sketch 1).



Sketch 1

The vertical force on cylinder (1) is called  $F_1$  and the vertical force on cylinder (2) is called  $F_2$ .

The pressure in an incident sinusoidal wave of amplitude  $h$  in deep water is

$$P = \rho g h e^{ky} \cos(kx - \omega t) \quad (1)$$

$$F_1 = \rho g h A_1 e^{-kF} \left[ e^{-k\ell/2} - e^{k\ell/2} \right] \cos(kx - \omega t). \quad (2)$$

For

$$k \frac{\ell}{2} \ll 1, \quad F_1 = -\rho g h e^{-kF} kV \cos(kx - \omega t), \quad (3)$$

where  $V = A_1 \ell$  is the volume of cylinder (1).

$$F_2 = \rho g h A_2 e^{-kT} \cos(kx - \omega t) \quad (4)$$

The total force is  $F_1 + F_2$

$$F_1 + F_2 = \rho g h \cos(kx - \omega t) \left[ A_2 e^{-kT} - V e^{-kF} \right] \quad (5)$$

Note that if  $e^{-kT} \approx e^{-kF}$ , the wave-excitationless circular wave number is

$$K = \frac{A_2}{V} \quad (6)$$

which is just Matora and Koyama's result without all their "correction coefficients."

When the body is not made of cylinders, it can be analyzed by subdividing it into "elemental cylinders" and Eq. (5) is still correct.

The results from the simple, straightforward theory I have given above should be accurate when the scattered wave is small. With fixed, surface-piercing two-dimensional bodies like those of Matora and Koyama, the scattered wave will not be small so the simple theory would be inaccurate for them. However, when a two-dimensional body is totally submerged with only small appendages piercing the surface, the simplified theory may be accurate. Ogilvie (3) has considered a submerged two-dimensional horizontal circular cylinder in waves with the top of the cylinder more than one cylinder diameter below the free surface, and he has shown that the forces when considering the scattered wave agree within 10 percent error with the forces found when neglecting the scattered wave. His results for no scattered wave reduce to Eq. (5) when  $ka$  is small,  $a$  being the cylinder diameter.

The above facts as well as the very good agreement between the simple theory and experiments with the FLIP buoy shown by Kerr (4) have prompted me to calculate by the simple theory the heaving force on the sphere with vertical cylinder used by Matora and Koyama. The results are shown in Fig. D6. The simple theory does not fit the experimental results of Matora and Koyama as well as their theory, but their theory is "adjustable" by means of their four arbitrary constants  $\gamma_1$ ,  $\gamma_2$ ,  $\gamma_3$ , and  $K_z$ .

The disagreement of the simple theory with the experimental result is probably not due to the scattered wave alone. The diameter of the thinnest vertical strut considered here is half the diameter of the sphere so there is a large asymmetry about a horizontal plane through the center of the sphere. Therefore, the body can develop vertical lift associated with circulation, in the presence of horizontal currents, and the horizontal fluid velocity in the wave is in phase with the pressure forces thereby maximizing the possible discrepancy due to lift. There are many structures in use today where the bulk of the buoyancy is wholly submerged and the surface-piercing elements are relatively small. For structures of this type the simple theory should give predictions suitable for use in the engineering office.

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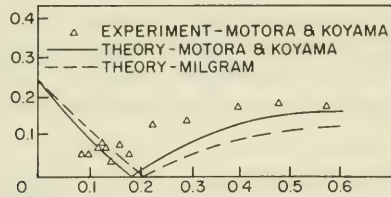


Fig. D6 - Heaving force  
for sphere type models,  
 $T = 3.5a$

\* \* \*

## REPLY TO DISCUSSION

Seizo Motora and Takeo Koyama

### TO CAPT. PORTER'S DISCUSSION

The authors would like to express their warmest thanks for Capt. Porter's kind remarks. It is very interesting and encouraging to see that Capt. Porter's exact solution for bulbous bodies showed quite similar tendency as the authors' rather intuitive observation.

The authors are most interested in measuring heaving forces of those ship-like forms and in treating three-dimensional problem by strip method using these sections.



# TO MR. FRANK'S DISCUSSION

The authors appreciate Mr. Frank's discussion showing results of his fine calculation at DTMB. It is quite encouraging to see the agreement between his results and the authors' measurements. There are some discrepancies between the theory and the measurement at higher frequencies, and the authors are interested in checking their measurement at those points.

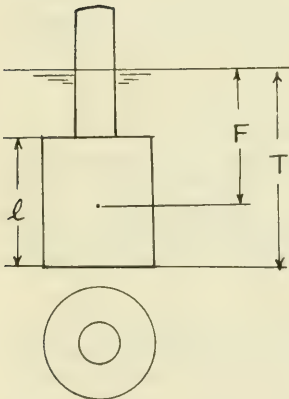
In comparing Fig. 3 of the paper to Mr. Frank's Fig. D4, there are some discrepancies between the authors' approximation and the exact solution given by Mr. Frank for greater strut width. The authors hope that those results will give valuable information to refine their approximation.

# TO MR. MILGRAM'S DISCUSSION

Mr. Milgram showed that heaving excitationless phenomena can be explained by a simple theory neglecting the wave diffraction.

It is very interesting to know that the Froude-Krylov force itself has negative sign to the wave elevation when a body is entirely submerged. This means that the correction factor  $\gamma_3$  in the authors' theory is less than they have estimated, and therefore the excitationless frequency will shift toward a lower frequency.

For example, let us take a body shown by Mr. Milgram (Sketch 2).



Sketch 2

The Froude-Krylov force is

$$FK = \rho gh \left[ A_2 e^{-kT} - KV e^{-kF} \right] \quad (1)$$

where notations are as given in Mr. Milgram's discussion.

Inertia force induced by wave diffraction is

$$I = -\rho gh \gamma_1 k_z K \bar{V}. \quad (2)$$

Therefore, total force is

$$F = FK - I = \rho gh \left[ A_2 e^{-kT} - KV (e^{-kF} + \gamma_1 k_z) \right]. \quad (3)$$

The excitationless wave number  $K_0$  is

$$K_0 = \frac{A_2 e^{-K_0 T}}{\bar{V}(e^{-K_0 F} + \gamma_1 k_z)}. \quad (4)$$

Comparing this with Mr. Milgram's theory

$$K'_0 = \frac{A_2 e^{-K_0 T}}{\bar{V} e^{-K_0 F}} \doteq \frac{A_2}{\bar{V}} \quad (5)$$

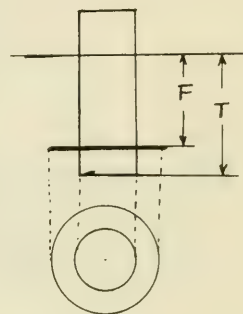
and the authors' theory:

$$K''_0 = \frac{\gamma_3 A_1}{\gamma_1 k_z \bar{V}} \quad (6)$$

we know  $K_0$  is less than  $K'_0$  and  $K''_0$ .

Mr. Milgram has shown that Eq. (5) agreed very well with measured one. In fact, in case of three dimensional problem where  $v$  is sufficiently large, and the inertia coefficient is not so large, Eq. (5) will give considerably reasonable value. However, if  $\ell$  (the height of the submerged cylinder) becomes zero (thin collar piece), Eq. (5) will give  $K \rightarrow \infty$  with  $\bar{V} \rightarrow 0$ . On the other hand, since  $k_z \bar{V}$  does not become zero when  $v$  tends to zero, Eq. (4) and Eq. (6) will give finite  $K$  value.

Therefore simplified Eq. (5) will result in considerably higher value of excitationless frequency when the underwater bulb is flattened.



Sketch 3

\* \* \*



# PROJECT MOHOLE DRILLING PLATFORM: MANEUVERING, PROPULSION, AND STATION-KEEPING

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*Project Mohole - Brown and Root, Inc.*  
*Houston, Texas*

## THE DRILLING PLATFORM

Project Mohole is this nation's contribution to the International Upper Mantle Study and is the most ambitious part of this program. The major scientific objective of Project Mohole is to obtain a core record of the layers of the earth's crust and the upper part of the underlying mantle. In addition to core samples, valuable information will be acquired from measurements made in the hole both during drilling and after completion.

The boundary between the crust of the earth and the mantle is defined in terms of seismic measurements, indicated by a sharp increase in acoustic velocity at the interface known as the Mohorovicic Discontinuity. This phenomenon occurs at much shallower depths under the oceans than under the continents; hence, the decision was made for the drilling to be carried out at sea. At the selected drilling site, northeast of the island of Maui, Hawaii, the water is 14,700 ft deep and the mantle is another 16,000 to 18,000 ft beneath the ocean floor.

The drilling operation at the selected site requires an accurately positioned stable platform. The steadiness of the working vessel is of critical importance in every phase of activity. The most critical items regarding steadiness are the riser system and the drill pipe which connect the vessel to the ocean floor. The manipulations for reentry demand minimum surface motions and precise maneuvering capabilities.

Positioning of the vessel must be accomplished by propulsive means rather than by anchoring because of the great depth of water at the proposed drilling site. The platform cannot be allowed to drift out of the prescribed area with the riser pipe attached or the drill pipe in the hole. Precise and dependable positioning control is required in the performance of all drilling and reentry operations.

A high degree of steadiness is designed into the platform by developing a configuration which has very weak coupling with surface waves, and consequently minimal response to the sea. Low response to wave action is inherent in the



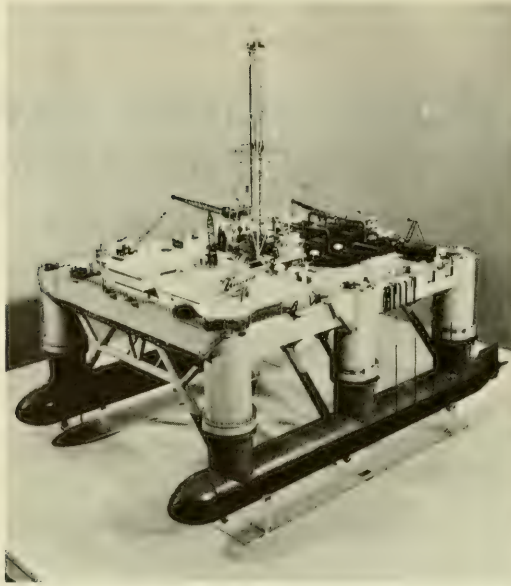


Fig. 1 - Drilling platform model

design of the vessel with small water plane area and deeply submerged main hulls. These characteristics make it difficult to obtain low resistance which is a desirable characteristic for station-keeping. The photograph, Fig. 1, shows that the configuration chosen will yield relatively low drag for the hulls without sacrificing desirable wave transparency (1).

It is mandatory to the life of the drill string and riser that the platform be kept within a prescribed distance of the hole, dependent upon water depth. At the Mohole site, in 14,700 ft of water, the maximum safe distance off the hole is 400 ft. A circle of this radius is clearly too small for conventional ship-type maneuvering. The platform is equipped with propulsion equipment which enables it to turn at any radius, or to move in a straight line in any direction. While on station it is intended that the vessel will be headed into the current if the current is one knot or more, or into the wind if wind predominates. If both wind and current are strong, a heading between the two forces will be chosen to minimize propulsive power. At times it will be necessary to move the platform transversely through the water or at some large angle to the centerline in order to hold station in a weak current from one direction while a strong wind is blowing from another direction. As conditions change, a new heading will be selected where propulsive power and resistance can be most favorably matched. The platform will not automatically change heading but rather will maintain a predetermined gyrocompass heading. Environmental conditions for design of the positioning system were of necessity selected before the Mohole site was chosen. Wind and current conditions were established which would permit operation in any of the potentially suitable areas. The design conditions do not

necessarily represent expected sea conditions in the Hawaiian area. Design limiting conditions are simultaneous winds of 33 knots and current of 3 knots with commensurate seas. This results in enough power so that squalls of much greater intensity can be withstood in combination with nominal ocean current.

Two propulsion systems are provided: The main propulsion system consists of a propeller at the stern of each lower hull powered by a 7500-HP DC motor. The positioning propulsion system consists of six right-angle drive units, one in each of the six main columns. These units each consist of a 5 ft 9 in. shrouded propeller driven through right-angle gearing by a 1000-HP DC motor. The propellers are trainable and can be continuously rotated in either direction to provide thrust at any desired angle. Both the main propulsion and the positioning propulsion systems are controlled automatically in the station-keeping mode.

The positioning system includes a position reference system which determines location relative to the hole by sonar ranging to bottom-mounted transponders. A positioning computer calculates the relative platform position, determines the thrust required to correct position errors, and directs the propellers accordingly. Two identical computers are installed to insure a high degree of reliability. These systems are explained in more detail in Appendix D. A semiautomatic mode is also provided in which the positioning units are coupled to a joy-stick.

In moving from one site to another the platform will be operated at light draft, with the lower hulls on the surface. Since ample propulsive power is available as a result of positioning requirements, the platform will be operated self-propelled in transit. In this mode it will operate very much like a ship in straight and curved maneuvers. No rudders are provided, only the main propellers will be used, controlled manually from a console on the bridge either at the same speed for straight line travel or at different speeds to develop turning moment.

It can be seen that propulsive power, rotation, and resistance in all directions are of interest. As an essential step in design of propulsion and control systems, it was necessary to determine characteristics of the platform in straight-line motion fore and aft, athwartships, and obliquely to the center line and in curved maneuvers varying from turning about its own vertical center line to turns at normal turning radius. These characteristics were required at the normal drilling draft of 65 ft, and at the light or transit draft of about 29 ft. Since the analysis and test procedures differ between the deep draft and light draft conditions, these are treated separately. The most vital criterion of the drilling platform is that it must remain on station within the prescribed limits. This facet of operation received the most attention and is discussed first and in most detail.

## STATION-KEEPING

In order to verify the platform's ability to hold station, the resistance and moment data were combined with the propulsion data in a comprehensive mathematical model for single plane performance. The mathematical route was

chosen over the experimental route due to the multiplicity of possible environmental conditions, practical difficulties and expense in equipping a model with the six right-angle drive units and automatic controls. The analysis served another vital purpose; besides verifying design expectations, it has contributed to design of the control system by providing time constants and values of system constants to be set into the on-board positioning computer.

Drag and moment coefficients and other platform characteristics were determined with the aid of model tests. Four types of tests were conducted:

1. Resistance and moments in straight-line motion
2. Resistance and moments in curved maneuvers
3. Aerodynamic resistance and moments
4. Propeller performance.

#### Resistance and Moments in Straight-Line Motion

Platform propulsion systems are capable of providing thrust in whatever direction is required to move the platform toward the drill hole. Also combinations of propulsive power and wind may move the platform through the water at any angle to the center line. In positioning maneuvers (station-keeping maneuvers) it is as important to know the moment and side force developed by oblique motion as to know the resistance in the direction of the motion. Principal reliance in obtaining these data was placed on model tests which were conducted at two model scales in two different facilities: 1:45 in the David Taylor Model Basin main towing tank and 1:27-1/2 at the University of Michigan Naval Tank (2,3) (Appendix A).

The model was towed over a range of speeds at headings from zero to 180 degrees measured from the fore and aft center line. Figures 2, 3, and 4 give drag force in the direction of motion, side force perpendicular to the direction of motion, and yaw moment about the vertical axis as functions of speed and drift angle measured from the bow.

These data are scaled directly from the model basin tests and are not corrected for roughness and minor appendages. These tests were conducted early in the program and the model incorporated elliptical cross-section diagonal bracing in place of the present circular bracing. The effects of all of these items are incorporated in the equations for the positioning system analysis.

In addition to the towing tests a composite test was used to estimate platform resistance. To eliminate Reynolds number effect, tests were conducted at high Reynolds number in a wind tunnel to determine the resistance of the fully immersed body to which was added the wave-making component of drag measured by towing a cylinder in a water basin. Although a rigorous comparison is not possible, agreement between the two methods is well within the limits of required accuracy and the towing tests which yield the more extensive data are considered entirely satisfactory for design purposes.



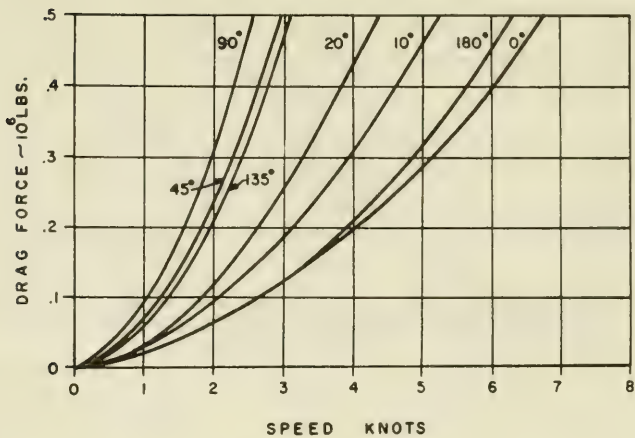


Fig. 2 - Resistance vs speed  
for various drift angles

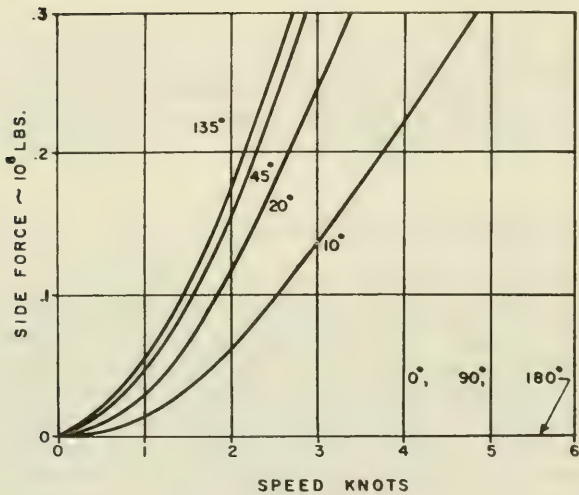


Fig. 3 - Side force vs speed for  
various drift angles (deep draft)



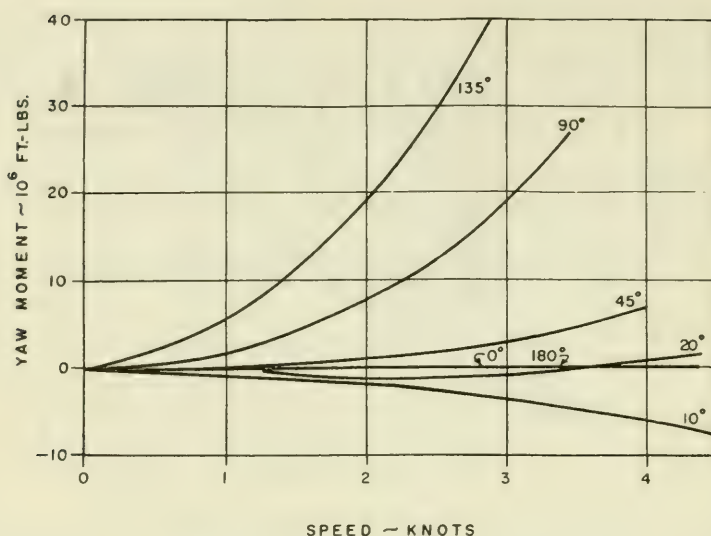


Fig. 4 - Yaw moment vs speed for various drift angles (deep draft)

Extrapolation of the data to full scale has been done by the simple lambda-cubed relationship. Extrapolation by the customary ATTC friction line resulted in negligible differences, indicating that friction is a minor part of the total, as one would expect.

#### Resistance and Moments in Curved Maneuvers

During positioning maneuvers within the 400-ft-radius circle the platform will be obliged to turn with very short radii. When the turning radius is in the same order as, or smaller than, the dimensions of the platform, resistance and moment cannot be accurately determined by applying straight-line resistance data to the instantaneous direction of motion of the center of gravity. In such maneuvers the two hulls see quite different relative flow angles. Rotating arm tests were conducted in an effort to shed some light on this problem. Unfortunately the facilities were limited to a minimum radius several times the model dimension. Therefore, data in the critical range of very small radii could not be obtained. Instead of mapping the moment and forces as functions of heading angle and radius of turn, it was found expedient to consider the six columns and two hulls separately, computing the forces on each part according to local relative velocities, then adding up the contributions to obtain total forces and moments. The manner in which this was done is explained in detail in Appendix C.

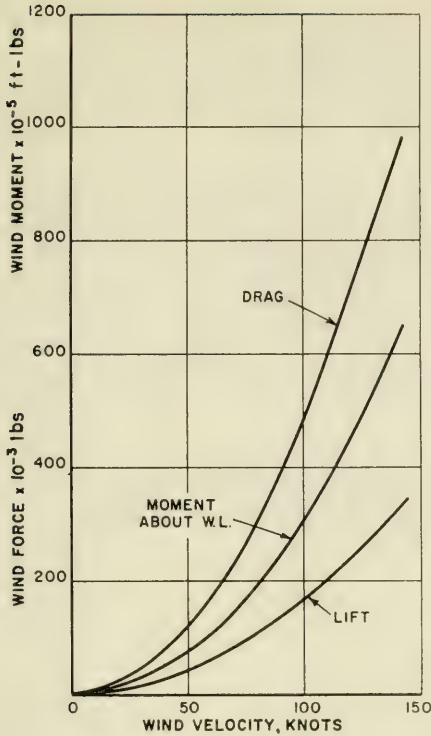


Fig. 5 - Wind force lift, moment (70 ft draft)

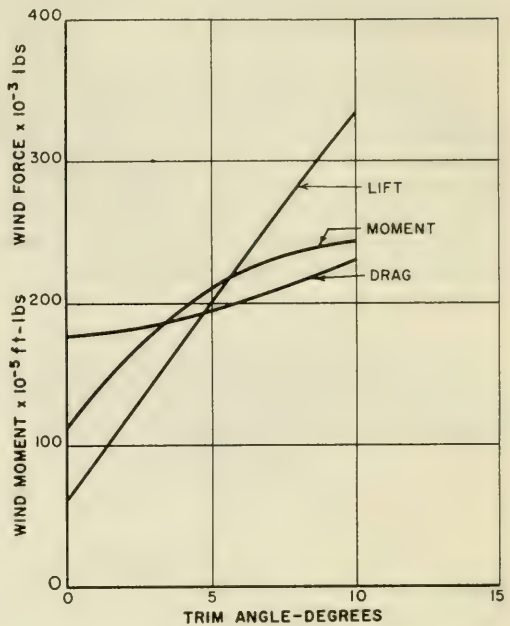


Fig. 6 - Variation of wind forces with trim angle (70 ft draft)

## Aerodynamic Resistance and Moment

The effects of wind on the platform were measured in the David Taylor Basin low-speed wind tunnel using a 1:60 scale model (4). The tests encompassed not only heading angles of 0 degrees, 45 degrees, 90 degrees, and 135 degrees, but also several angles of trim and heel. Superstructure and major items of deck equipment were included in the model. Fairied test results are given in Figs. 5 and 6. It will be noted that although drag force increases slightly with trim angle or heel angle, the lift force increases rapidly with trim or list angle. The lift force, usually neglected in wind heeling calculations, is seen to contribute substantially to the heeling moment. Its relative effect decreases at large angle of heel as the center of pressure moves inboard.

## Propeller Performance

A great deal of published information exists concerning propellers in open water and in various wake patterns. Selection of the main propellers was aided by the work of Nordstrom (5) which gives thrust and torque coefficients for

propellers with both ahead and astern rotation and ahead and astern velocity. Self-propelled tests were conducted to determine propeller-hull interaction. Aside from the greater than usual importance of knowledge of propeller thrust at negative speeds, there is nothing particularly unusual in selection of the main propeller characteristics.

The shrouded steering propellers can be directed at any angle to the vessel's center line. When correcting an adverse heading, the positioning unit will frequently be directed at a large angle to the direction of motion. Performance data for these conditions were not available in the literature, so it was necessary to conduct tests with varying inflow angles. The propeller is a modified Kaplan type with four blades similar to the van Manen K 4-55 series. The nozzle is a van Manen No. 18 modified by opening up the inlet to compensate for the hub nacelle. The positioning unit propeller and nozzle are described more fully in Appendix B.

Tests in both open water and in the presence of a hull segment were conducted at the University of Michigan Naval Tank using a 1:7.5 scale model. Equations were fitted to the data (6) to complete the performance map and to facilitate positioning analysis.

$$K_T = (0.50 + 0.45 J_a) \cos \theta - J_a$$

$$K_S = (0.50 + 0.45 J_a) \sin \theta$$

where

$J_a$  = advance coefficient

$\theta$  = angle of propeller shaft to the flow.

As defined here  $K_T$  is in the direction of flow and  $K_S$  is at right angles to it. Figures 7, 8, and 9 are plots of the longitudinal force, side force, and torque coefficients in open water.

An additional factor affecting positioning unit performance is the proximity of the hull. When the platform is moving ahead, only a slight frictional boundary layer influences the propeller. An athwartship's velocity component around the hull creates an accelerated region in the vicinity of the propeller. The effect of the hull is equivalent to an increase in transverse velocity component of about 25 percent.

For general application, Taggart (7) gave the following equations for propeller-nozzle performance in open water as a function of pitch ratio. These are based on the Mohole test data and tests by van Manen with the assumption that pitch ratio effects measured in axial flow hold proportionately at other angles.

# Project Mohole Drilling Platform

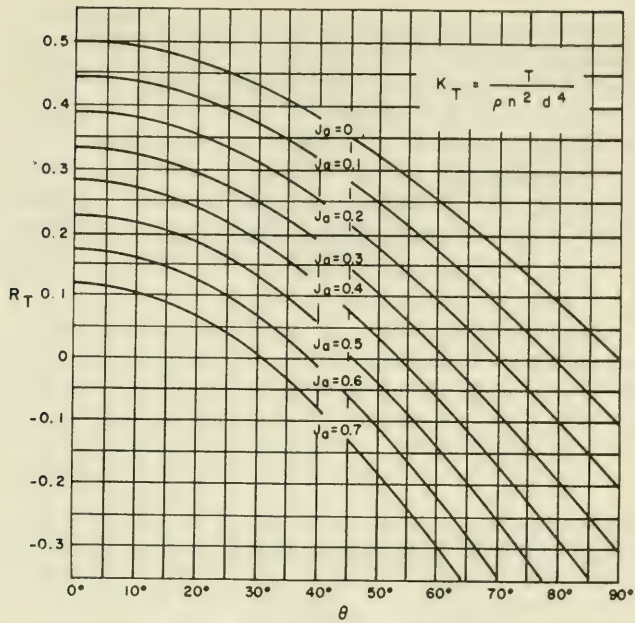


Fig. 7 - Thrust coefficient vs flow angle  $\theta$

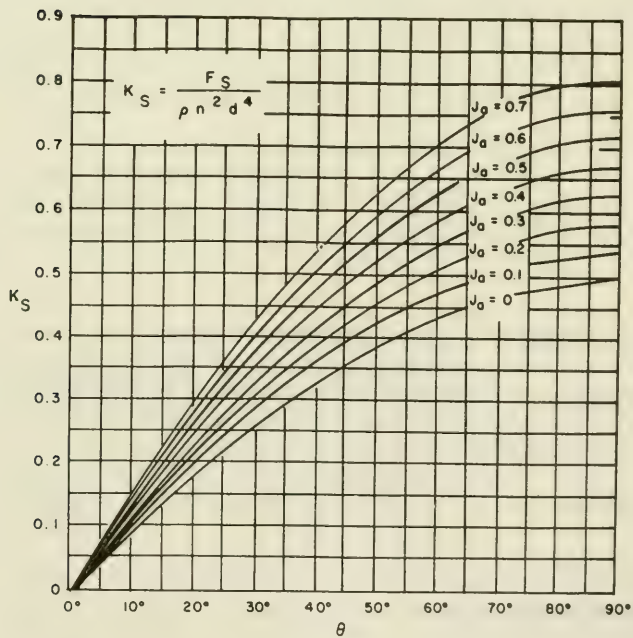


Fig. 8 - Side force coefficient vs flow angle  $\theta$



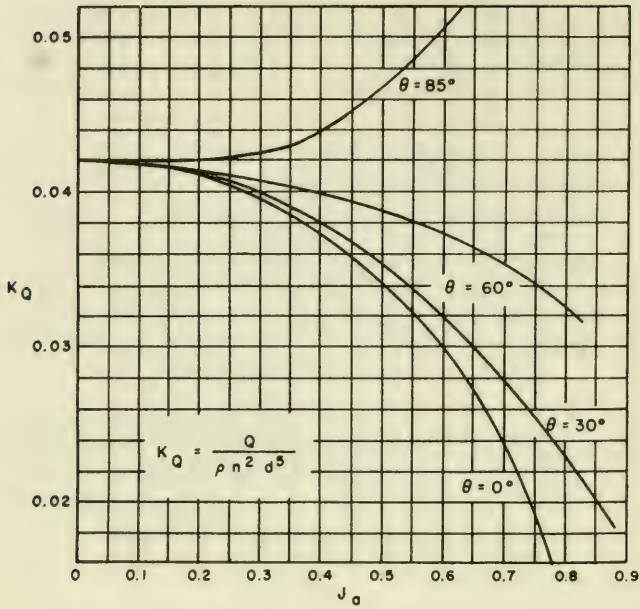


Fig. 9 - Torque coefficient  
vs advance coefficient  $J_a$

$$K_T = (0.725 P/d - 0.20 + 0.45 J_a) \cos \theta - J_a$$

$$K_S = (0.725 P/d - 0.20 + 0.45 J_a) \sin \theta$$

$$K_Q = 10^{-3} \left[ 45 (P/d)^2 + 1 - e^{4.25 J_a} + J_a^3 e^{(1.25 + 2.22 \theta)} \right]$$

where

$K_T$  = thrust coefficient in the direction of inflow

$K_S$  = side force coefficient at right angles to the direction of inflow

$K_Q$  = torque coefficient

$\theta$  = angle between the propeller shaft and the direction of inflow

$P/d$  = pitch ratio

$J_a$  = advance coefficient.

## Station-Keeping Analysis

Behavior of the platform acted upon by wind, current, and propulsive forces was simulated mathematically with the aid of a digital computer program. Written in Fortran IV the program has sufficient flexibility that any foreseeable combination of wind and current can be imposed including time variations of velocity and direction. Resistance, propulsion, and mass acceleration terms are added vectorially to obtain equilibrium in the horizontal plane. In addition, any desired propulsive thrust control formulation can be inserted. Movement of the platform is calculated at frequent intervals affording a close approximation to dynamic equilibrium. The equations and treatment of platform parameters are stated in Appendix C.

Platform parameters such as mass moments of inertia, resistance, side force, and yaw moment are built into the program based on the best available information from model test analysis. Some flexibility is available for varying thrust programming and environmental forces from run to run. Two examples requiring the following definitive information are presented to illustrate the capability of the positioning maneuvers program:

1. **Translational Thrust** — Thrust is calculated as a linear function of distance from the hole. It has been found desirable to specify a dead band within which no thrust is applied in order to minimize "hunting." In the example this is specified as a 50-ft radius from the hole. At this distance any desired amount of thrust may be called for, increasing linearly to a maximum at some specified distance. For the example we have chosen, thrust equals  $1070 D$  minus 37,500 pounds where  $D$  is the distance from the hole. This results in 16,000 pounds thrust at 50 ft and 390,000 pounds at 400 ft which is approximately the maximum available thrust. Positioning unit thrust is built into the program and depends upon orientation and velocity. Main propeller thrust is a variable and this example is limited to a maximum of 240,000 pounds.

Another variable in the thrust program is the instruction for the integrator circuit. This device notes the distance off the hole and biases the range information so that the platform can be moved toward the hole while maintaining the thrust required for equilibrium with the environment. For the examples, we have chosen a 60-second sampling interval and a correction of 1 percent of the distance off hole.

2. **Turning Thrust** — In order to correct errors in heading, power is supplied to the positioning units in amounts and directions needed to apply torque to the platform. A dead band of 3 degrees is specified, beyond which thrust is applied varying as the sine of the heading error angle up to a maximum moment at 30 degrees. The maximum thrust for turning moment is specified in the program as 40,000 pounds ahead on one side and astern on the other which results in a maximum torque of 40,000 times the 215 ft transverse spacing of the units or 8,600,000 ft-pounds.

3. **Wind** — Wind may be specified at any desired velocity and direction, either as a constant value or as a ramp function approximated by a series of small step increases. For the examples, we have chosen wind blowing in the

direction of the negative  $y$  axis increasing in a step function from zero to 33 knots.

4. Current — Current may also be specified in terms of direction and velocity and may be either a steady value or ramp function approximated by small increments. In the first example, we have chosen zero current. In the second example current is 3 knots in the same direction as the wind.

5. Initial Conditions — Initial conditions for the problem may be specified in terms of platform position, velocity, and heading. In this way a previous problem can be utilized as input for a succeeding problem. In the two examples, we have chosen to start with the vessel at rest. In the first case the initial heading is in the  $x$  axis direction and a 90-degree change of heading is ordered. In the second case the heading is in the  $y$  axis direction and is to be maintained.

The first example is a simple case of a step increase in wind. The vessel is assumed to be holding position on-station and the wind arises from abeam. At the start of the problem a command is given to change the heading of the vessel by 90 degrees so that it will head into the wind. The vessel, therefore, starts with zero velocity and with a 33-knot wind from abeam. As the problem proceeds, the platform is turned into the direction of the wind and is moved back toward the hole after being forced off by the wind. The equilibrium point is a function of wind force versus thrust. The integrating circuit gradually shifts the  $x$ - $y$  coordinate of the equilibrium point from an off-hole position to a position over the hole.

Figure 10 is an  $x$ - $y$  plot of the maneuvering path of the platform as it regains position. The heavy line is the path of the center of the vessel. The short arrows along the line indicate the heading of the platform at each moment of time. The number at each arrow is the time in minutes from the start of the problem. This is a plausible situation which the vessel might encounter in the event of a sudden squall. Often the exact wind direction is not known until the wind actually strikes.

The second example involves both wind and current. The initial conditions for the platform are again zero velocity and zero position error. In this case the vessel is assumed to be heading at 90 degrees, that is, in the direction of the  $y$  axis. The wind is 33 knots and the current 3 knots, both in the direction of the negative  $y$  axis. Since the vessel is headed into the wind and current no heading change is called for. It is assumed that the integrator circuit has been operating and that the thrust is nearly in equilibrium over the hole. It will be noted in Fig. 11 that the vessel initially drifts only a short distance from the hole, since the integrator has previously displaced the 50-ft-radius dead band away from the hole. The integrator circuit is working and gradually displaces the equilibrium point so that the vessel returns slowly to the hole.

These two examples are not sufficient to demonstrate satisfactory maneuvering performance of the platform. A great many cases have been investigated with variations in thrust programming, integrator circuit, environmental conditions, and other variables. It is concluded that the platform can maintain station in the design conditions under automatic control. It can, of course, maintain equilibrium at stronger wind with presence of less current and vice versa.



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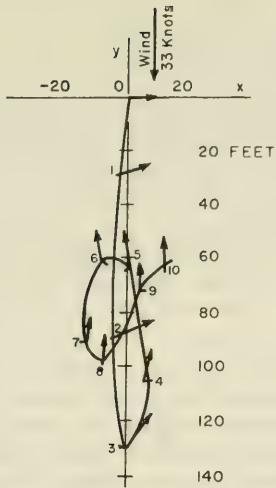


Fig. 10 - Positioning maneuvers in 0 to 33 knot step increase in wind from abeam

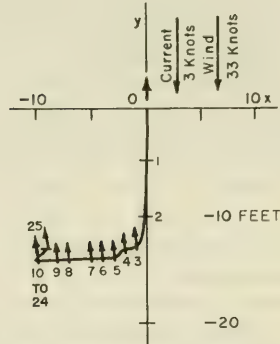


Fig. 11 - Positioning maneuvers in 3-knot current and 33-knot wind, both from ahead

In the event that conditions are more severe than the design conditions, it is intended that the vessel would be under manual control. Under manual control the main shafts can be rotated at different speeds or in different directions to apply thrust and torque as required. Additional power may be obtained by reassigning generator capacity from drilling functions to the main propulsion motors. The power plant is designed so the entire 13,000-kw generating capacity of the DC electric power system can be applied to the positioning units and main propellers.

### TRANSIT

Self-propelled transit from site to site was not initially a basic requirement for the platform. A large main propulsion system was found to be needed to insure that the platform could remain on-station through storm conditions which might occur frequently and with little warning. It was advantageous to design for utilization of this propulsion system for moving to and from drill sites rather than having to rely upon auxiliary vessels. More important is the added freedom and safety afforded the platform. In the event of a severe storm forcing it to leave the site, it can be moved into sheltered waters, into the open sea, or possibly out of the storm track entirely. It is desirable to know the speed-power characteristics of the platform at shallow draft although precise determination is not necessary.

It is most essential to insure that sufficient directional stability is built into the design so it can be steered with ease and precision. Principal reliance



to establish these facts was placed on model testing. Towing tests were conducted to determine resistance, turning moment, and lateral force in straight-line maneuvers. Also, steering tests were conducted with a self-propelled model. From a qualitative standpoint it was found that the model could easily be steered in a straight path down the model basin using manual remote control. Spiral and zig-zag maneuvers were conducted to determine quantitative directional stability and control as illustrated in Fig. 12. In the zig-zag tests the model was run in a straight path before a predetermined RPM variation was applied causing the vessel to turn. After a specified angle was reached, the RPM variation was reversed. The turns were repeated in both directions and the overshoot and time to complete the cycle were noted. Results show that the platform is about as controllable as a conventional ship hull of the same size.

One of the measures of controllability is the overshoot angle or the amount the vessel continues to swing in a certain direction after the turning force has changed to the opposite direction. A small overshoot angle signifies good controllability. To investigate the sensitivity of the vessel to control changes, small fins or trim tabs were fitted aft of the vertical stern fins. The overshoot angles were 2.5 degrees and slightly greater with the adjustable trim tabs. This does not necessarily mean that the platform is more controllable without the

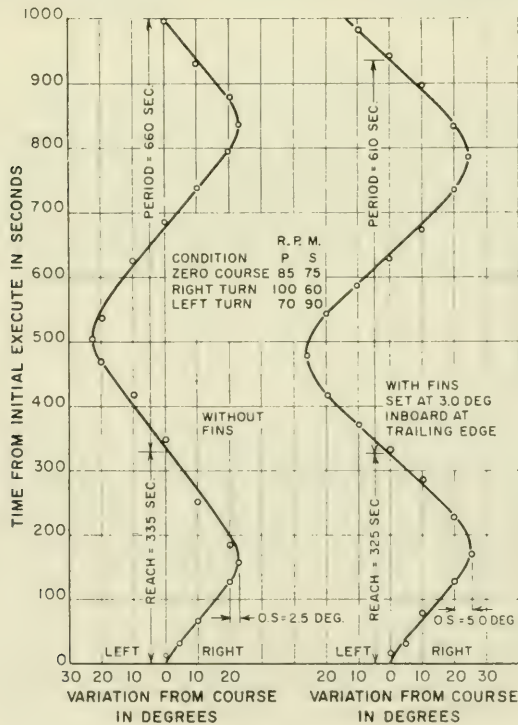


Fig. 12 - Time histories of zig-zag maneuvers at 8.0 knots

## Project Mohole Drilling Platform

trim tabs since in respect to other criteria they result in a slight improvement. In any event, the overshoot angles are somewhat less than those usually encountered in conventional ships and it may be concluded that controllability of the platform as designed is good. Other measures of controllability are the reach (time required for the vessel to return to its original base course at the end of the first half cycle), and the period (time to complete a full cycle). Small values of these parameters are usually associated with good controllability. Results of the tests show that reach is 335 seconds and the period 660 seconds. These values are in the same order as those encountered in ship hulls and are considered indicative of satisfactory maneuvering performance. Speed during these tests was nine to ten knots which is the range of speeds expected during transit from one site to another.

In addition to the self-propelled maneuvering tests, towing tests and wind tunnel tests at the University of Michigan Naval Tank and the David Taylor Model Basin provided information on resistance, side force, and moment due to wind, current and drift angle which must be resisted to maintain course. The installed main propulsion power of 7500 HP per shaft provides ample margin for headway and steering control during transit. Figure 13 is a curve of resistance versus speed at the 29-ft draft. It incorporates a 25 percent allowance over model test extrapolations for appendages and roughness. Superimposed on this is an approximate propeller thrust curve showing the speed which can be attained at full power, or 7500 HP per shaft. It will be noted that the positioning units will be retracted during long transit periods to reduce drag. Positioning unit propeller characteristics are such that they cannot be used effectively at the higher speeds required during transit conditions.

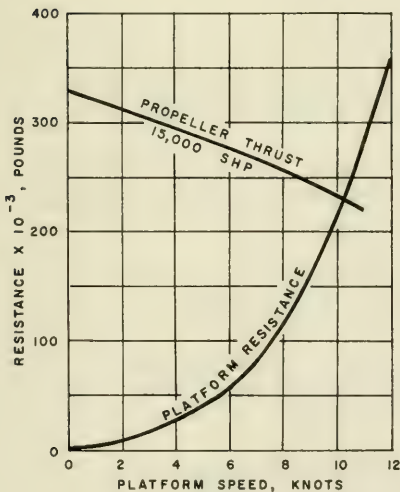


Fig. 13 - Platform resistance and thrust at 29 ft transit draft

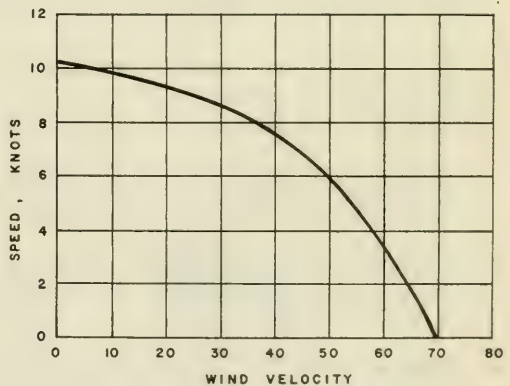


Fig. 14 - Transit speed vs head wind

The effect of wind on the platform may be expected to be greater than in the case of conventional ship forms. The large superstructure does result in appreciable wind force. Wind tunnel data have been utilized to develop the curve in Fig. 14 which indicates the effect on transit speed of a head wind. These were developed by adding the wind resistance at the relative wind speed to the water resistance and balancing that against the propeller thrust. It will be noted that under trade wind conditions the platform can still maintain a respectable 8 to 10 knots directly against the wind. Downwind, of course, the effect is reversed, but is of a smaller magnitude since the relative velocity is reduced.

The curve shows that headway can be maintained in winds up to 70 knots. This is not significant since the platform would not normally operate at shallow draft in winds of this intensity. Using positioning units to augment the main propellers the platform can maintain headway under stronger winds.

#### ACKNOWLEDGMENTS

The authors wish to express their appreciation to the many persons who participated in the hydrodynamic studies of the Mohole platform. Of special note are the contributions of Dr. H. K. Beckmann of Rice University in planning, analysis, and interpretation of tests, and Dr. E. C. Holt of Rice University in analysis and programming, including preparation of Appendix D.

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# Project Mohole Drilling Platform

## Appendix A

### PLATFORM MODEL

The platform model in the final configuration for towing and self-propulsion tests is shown in Fig. A1. The positioning units were not installed since the performance tests of these units determined the net thrust, that is the propeller-nozzle thrust less drag due to struts, housing, etc. Different model scales were used for different portions of the testing program. The largest model used was built to a scale of 1:27-1/2 and was used for towing and self-propulsion tests at the University of Michigan. Self-propulsion, maneuvering, towing and rotating arm tests were conducted at the David-Taylor Model Basin with a 1:45 scale model. Wind tunnel tests were conducted with above water and below water portions of the platform built to a scale of 1:60.

Tests were performed at various times during the period of development of the platform. Consequently, model configuration differed from the earlier to

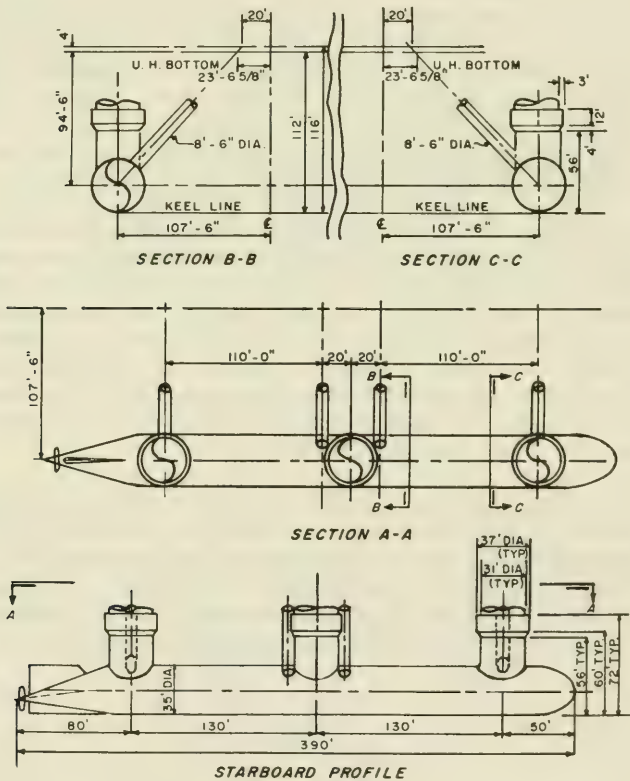


Fig. A1 - Mohole drilling platform propulsion test model



the later tests. For the resistance and moment tests the model was fitted with elliptical cross-section braces instead of circular members. Also damping tanks were not present. The data given in Figs. 2, 3, and 4 of the basic paper are not corrected for these additions, but the data used in the positioning maneuvers analysis have been appropriately increased. The shallow draft maneuvering tests were conducted earlier and in addition to the above changes the lower hulls were 20 feet shorter or 370 feet long and were 10 feet closer together.

## Appendix B

### POSITIONING UNIT, PROPELLER, AND NOZZLE

The drilling platform has six positioning units, one located in each of the platform's stability columns. Each unit consists of a 1000-HP DC motor installed in the lower hull gland, and an external assembly consisting of double right-angle gears, training assembly, housing, propeller, nozzle, and supporting struts (Fig. B1). The external assembly can be withdrawn or installed while the platform is in operation through the positioning unit trunk extending from the upper deck of the platform through the lower hulls. The flange of the positioning unit assembly rests on a mating flange and the trunk and inflatable seals enable the trunk to be kept dry during normal operation. The unit is steered by a reversible AC motor and worm gear drive. It can be rotated continuously in either direction at 2 rpm. The external assembly contains its own built-in lubricating system and is pressurized to minimize risk of leakage.

Installation of the units is accomplished by lowering the external unit into the trunk using one of the platform's deck cranes or a portable positioning unit handling fixture. The unit is seated on the seating flange in the trunk and the six hold-down latches are set. The inflatable seals are then actuated and dewatering begun, utilizing the platform's ballasting pumping system. With dewatering complete, a mechanic enters the trunk, removes the watertight covers over the drive shaft and the electric lead couplings and makes these up using short connecting pieces. The unit is then ready for check out and operation. Removal follows the same procedure in reverse.

The propellers and nozzles were designed in accordance with procedures given by van Manen.\* The propeller was based on van Manen's K 4-55 series. With the deep submergence of the propellers while operating, it has been possible to reduce the blade width without danger of cavitation. The nozzle is a modified van Manen No. 18 profile. The incidence angle was increased from 10.2 to 11.7 degrees to compensate for the tapered nacelle and propeller hub.

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\*J. D. van Manen and A. Superina, "The Design of Screw Propellers in Nozzles," International Shipbuilding Progress No. 55, March 1959.

## Project Mohole Drilling Platform

### Propeller Characteristics

|                                  |       |
|----------------------------------|-------|
| Number of blades                 | 4     |
| Diameter                         | 5'-9" |
| Design speed of advance          | 0     |
| Design shaft horsepower          | 720   |
| Design rpm                       | 300   |
| Pitch ratio at 0.7 R             | 0.92  |
| Expanded area ratio              | 0.412 |
| Design submergence to centerline | 64.5' |

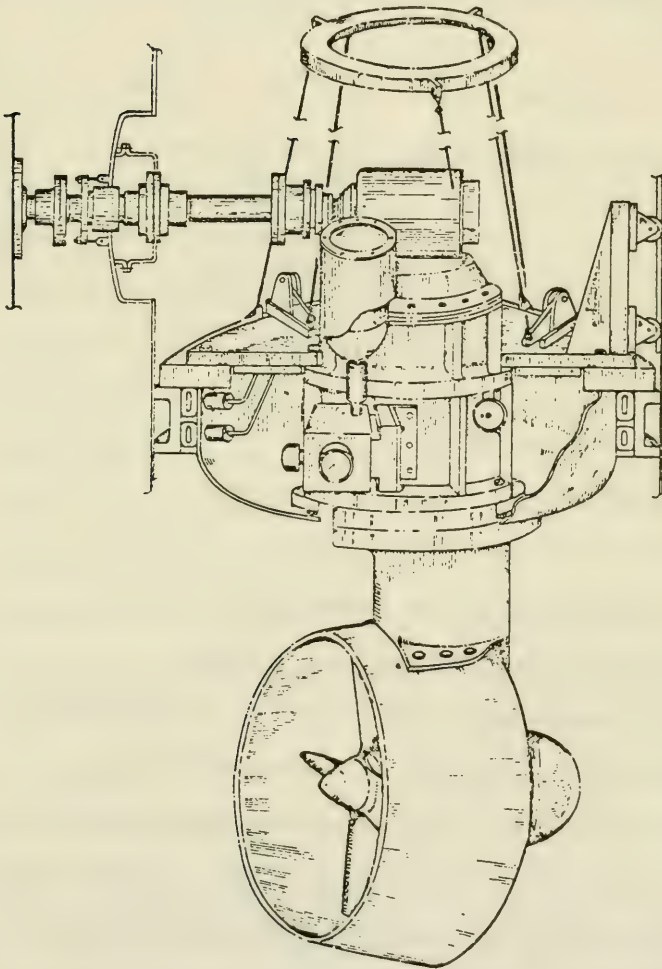


Fig. B1 - Positioning unit

## Appendix C

### POSITIONING MANEUVERS SIMULATION (Computer Program Description)

#### INTRODUCTION

This appendix presents the mathematical representation of the motion of the drilling platform moving over the surface of the sea under the control of the automatic positioning system. Starting with prescribed initial position and velocity and specified wind and current conditions, the equations of motion of the vessel are integrated to determine the position and heading of the vessel as a function of time.

#### EQUATIONS OF MOTION

The equations of motion of the vessel are as follows:

$$M\dot{U} = F_x(X, Y, \Psi, U, V, R, \dot{U})$$

$$M\dot{V} = F_y(X, Y, \Psi, U, V, R, \dot{V}, \dot{R})$$

$$I_z\dot{R} = M_z(X, Y, \Psi, U, V, R, \dot{V}, \dot{R})$$

where

$M$  = the mass of the vessel

$I_z$  = the mass moment of inertia of the vessel

$U$  = the forward velocity of the vessel

$V$  = the lateral velocity of the vessel, positive to port

$R$  = the angular velocity of yaw, positive to port

$X$  = the eastward displacement of the vessel from the hole

$Y$  = the northward displacement of the vessel from the hole

$\Psi$  = the heading of the vessel, measured counterclockwise from east

$F_x$  = the total longitudinal force on the vessel, positive forward

$F_y$  = the total lateral force on the vessel, positive to port

$M_z$  = the total yaw moment acting on the vessel, positive when producing yaw to port.

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The total forces in these equations are made up of the hydrodynamic forces, the aerodynamic forces, and the propulsive forces.

### HYDRODYNAMIC FORCES

#### Forces Due to Acceleration

The forces that the water exerts on the platform due to its acceleration are given by the formulas that follow:

$$F_x = -A_{11}\dot{U} + A_{22}RV + A_{26}R^2$$

$$F_y = -A_{22}\dot{V} - A_{11}RU - A_{26}\dot{R}$$

$$M_z = -A_{66}\dot{R} - A_{26}(\dot{V} + RU)$$

where

$F_x$  = longitudinal force, positive forward

$F_y$  = lateral force, positive to port

$M_z$  = yaw moment, positive when producing yaw to port

$U$  = longitudinal velocity of vessel relative to water, positive forward

$V$  = lateral velocity of vessel relative to water, positive to port

$R$  = angular velocity of yaw, positive to port

$A_{11}$  = added mass for longitudinal acceleration

$A_{22}$  = added mass for lateral acceleration

$A_{66}$  = added mass moment of inertia for yaw acceleration

$A_{26}$  = moment of added mass producing yaw moment due to lateral acceleration.

#### Forces Due to Velocity

##### General

These forces are produced by a combination of inertia and viscous effects. It is assumed at the outset that all forces vary as the square of velocity. In order to represent the total force and moment adequately for arbitrary motion of the vessel, it is necessary to consider separately the forces on the various parts of the platform. Where no formula for moment is given, it is calculated by multiplying the forces by the appropriate moment arms.



### Force on One Column

$$F_x = -C_{co1} |V_T| U$$

$$F_y = -C_{co1} |V_T| V$$

where

$C_{co1}$  = the projected area of the column times a drag coefficient

$V_T$  = the velocity of the column through the water

$U, V$  = components of  $V_T$ .

### Longitudinal Force on One Hull

$$F_x = -C_{long} |U| U$$

where

$C_{long}$  = the cross-sectional area of the hull times a drag coefficient

$U$  = the longitudinal velocity of the hull through the water.

### Lateral Force on One Hull

It is assumed that the hull is a segment of an infinitely long cylinder moving through the water and that the force intensity at a point depends only on the local cross velocity at that point. The resulting formulas are as follows:

Let  $X_0 = -V/R$

(1) For  $-L_2 \leq X_0 \leq L_1$

$$F_y = -\frac{1}{3} C_{lat} |R| R \left[ (L_1 - X_0)^3 - (L_2 + X_0)^3 \right]$$

$$M_z = -\frac{1}{12} C_{lat} |R| R \left[ (3L_1 + X_0) (L_1 - X_0)^3 + (3L_2 - X_0) (L_2 + X_0)^3 \right]$$

(2) For  $X_0$  outside above range

$$F_y = -C_{lat} |V| V \left[ L_1 + L_2 - \frac{L_1^2 - L_2^2}{X_0} + \frac{L_1^3 + L_2^3}{X_0^2} \right]$$

$$M_z = -C_{lat} |V| V \left[ \frac{1}{2} (L_1^2 - L_2^2) - \frac{2}{3X_0} (L_1^3 + L_2^3) + \frac{1}{4X_0^2} (L_1^4 - L_2^4) \right]$$

where

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$C_{lat}$  = diameter of hull times a drag coefficient

$V, R$  = lateral and angular velocities of the vessel through the water

$L_1$  = length of hull from vessel c.g. forward

$L_2$  = length of hull from vessel c.g. aft.

Inertia Moment on Both Hulls

$$M_z = -C_{inert} U V$$

where

$C_{inert}$  = added mass for steady velocity inertia moment

$U$  = longitudinal velocity of the vessel through the water

$V_n$  = lateral velocity of the "nose" of the vessel through the water; the "nose" being a point a distance  $L_3$  upstream from the vessel c.g.

Drag Force on One Fin

$$F_y = -C_{drag} |V_f| V_f$$

where

$C_{drag}$  = area of fin times a drag coefficient

$V_f$  = lateral velocity of fin through the water.

Lift Force on One Fin

$$F_y = -C_{lift} |U| V_f$$

where

$C_{lift}$  = area of fin times a lift coefficient

$U, V_f$  = components of velocity of fin through the water.

Keulegan Force on One Hull

This force due to wake formation drag is uniformly distributed over a part of the length of the hull from  $X_1$  to  $X_2$ .

(1) For  $U > 0$

$$X_1 = L_4$$

$$X_2 = X_1 - \left| \frac{2D}{V} \right| U \text{ or } -L_4,$$

whichever is algebraically greater.

(2) For  $U < 0$

$$X_2 = L_5$$

$$X_1 = X_2 - \left| \frac{2D}{V} \right| U \text{ or } L_5,$$

whichever is algebraically less.

$$F_y = -C_k |V| V (X_1 - X_2)$$

$$M_z = \frac{1}{2} F_y (X_1 + X_2)$$

where

$C_k$  = diameter of hull times a drag coefficient

$U$  = longitudinal velocity of hull through the water

$V$  = lateral velocity of vessel through the water

$X_1$  = coordinate of forward end of force

$X_2$  = coordinate of after end of force.

## AERODYNAMIC FORCES

The forces that the wind exerts on the vessel are given by the formulas below.

$$F_x = 1/2 \rho |V_T| (-C_D SU + C_C SV)$$

$$F_y = 1/2 \rho |V_T| (-C_D SV - C_C SU)$$

$$M_z = 1/2 \rho C_M SV_T^2$$

where

$$C_M S = 300,000 \sin [2(\alpha - 5^\circ)]$$

and

$F_x$  = longitudinal force, positive forward

$F_y$  = lateral force, positive to port

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$M_z$  = yaw moment, positive when producing yaw to port

$\rho$  = 0.00238 slugs/cu ft

$U$  = forward velocity of vessel relative to air

$V$  = lateral velocity of vessel relative to air, positive to port

$V_T$  = total velocity of vessel relative to air

$\alpha$  =  $\arctan V/U$

$C_D S$  = drag coefficient times reference area in square feet

$C_C S$  = cross-force coefficient times reference area

$C_M S$  = moment coefficient times reference area.

Values of  $C_D S$  and  $C_C S$  are tabulated in the program as a function of  $\alpha$ . The tabulated values were based on a DTMB Wind Tunnel Report.\* The formula for  $C_M S$  was also derived from the DTMB data for this item.

## PROPULSIVE FORCES

The propulsive forces produced by the main propellers and positioning units under the control of the positioning system computer are determined as described in the steps which follow. Where numerical values are given, these are the values typically used; they may be varied as desired from problem to problem.

### Apparent Vessel Position

The apparent position of the vessel with respect to the hole is determined as follows:

If the integrator is on,  $\Delta X$  and  $\Delta Y$  are incremented by a specified fraction of  $X$  and  $Y$ , respectively, at each sampling interval. If the integrator is off,  $\Delta X = \Delta Y = 0$ .

$$X_a = X + \Delta X$$

$$Y_a = Y + \Delta Y$$

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\*J. T. Matthews, "Low Speed Wind Tunnel Tests of a 1:60-Scale Mohole Drilling Platform Model," David Taylor Model Basin Report, Aero Tests A551, June 1964.



where

$X$  = the eastward displacement of the vessel from the hole

$Y$  = the northward displacement of the vessel from the hole

$X_a$  = the apparent eastward displacement

$Y_a$  = the apparent northward displacement.

The apparent bearing of the hole with respect to the vessel is given by

$$\theta = \arctan (Y_a/X_a) - \Psi + 180^\circ$$

where  $\Psi$  = the heading of the vessel, measured counterclockwise from east.

### Required Translational Thrust

The required translational thrust is computed as follows: If  $D > 50$  feet,

$$F_{reqd} = 1070 D - 37,500 ;$$

otherwise,  $F_{reqd} = 0$ .  $D$  is the apparent distance to the hole.

The longitudinal and lateral components are

$$F_{xreqd} = F_{reqd} \cos \theta = \text{longitudinal thrust, positive forward,}$$

$$F_{yreqd} = F_{reqd} \sin \theta = \text{lateral thrust, positive to port.}$$

There is no upper limit on  $F_{reqd}$  as such.

### Required Turning Thrust

The heading error is given by  $\Delta\Psi = \Psi_0 - \Psi$ , where  $\Psi_0$  is the ordered heading. Heading is controlled by thrusting ahead with the positioning units on one hull while thrusting astern the same amount with the positioning units on the other hull. The main propellers are not used to control heading. The required thrust per hull is computed as follows:

If  $|\Delta\Psi| > 3^\circ$ ,  $M_{reqd} = 40,000 \sin (\Delta\Psi)/\sin (30^\circ)$ , but not more than 40,000 pounds.

### Assignment of Thrust to Various Units

If  $F_{reqd} > 36,000$  pounds,  $F_m = 0.6 F_{xreqd}$  and  $F_p = 0.4 F_{xreqd}$ ; otherwise,  $F_m = 0$  and  $F_p = F_{xreqd}$ ; where  $F_m$  is the thrust assigned to the main propellers and  $F_p$  is the total longitudinal thrust assigned to the six positioning units.

## Project Mohole Drilling Platform

Each positioning unit is now assigned a longitudinal component of thrust  $F_x = F_p/6 \pm M_{reqd}/3$ , where the plus sign applies to the starboard hull and the minus sign to the port hull. Each unit is assigned a lateral component of thrust  $F_y = F_{yreqd}/6$ . The total thrust assigned for each unit is the vector sum of the components and the angle of train is determined from the components.

### Thrust Produced

The main propellers produce the thrust  $F_m$  up to the limit of 240,000 pounds forward or 120,000 pounds astern.

The thrust produced by each positioning unit is calculated separately as described below. The positioning computer actually controls rpm rather than thrust, whereas the thrust obtained depends not only on the rpm, but also on the local velocity of the water relative to the unit.

First the total thrust required from the unit is limited to 25,400 pounds. Then the thrust obtained is calculated as

$$F_{obt} = F_{ask} \times 1.075 (1 + 0.0313 V - 0.0696 V \cos \theta)$$

where

$F_{obt}$  = thrust obtained

$F_{ask}$  = thrust required

$V$  = absolute value of relative water velocity

$\theta$  = angle between water velocity vector and thrust vector.

### NUMERICAL VALUES

Numerical values of the constants can be varied as program input data. Typical values as derived from analysis of test data are given below. It will be noted that the dimensions  $L_1$  through  $L_5$  do not correspond to actual hull dimensions but to an idealized equivalent cylinder.

|                                       |                   |
|---------------------------------------|-------------------|
| Lateral spacing of hulls, feet        | 215               |
| Longitudinal spacing of columns, feet | 130               |
| Lower hull diameter, feet             | 35                |
| Column diameter, feet                 | 31                |
| Displacement at 65 ft draft, L tons   | 23,380            |
| Vessel mass, slugs                    | $1.63 \cdot 10^6$ |
| $A_{11}$ , slugs                      | $0.33 \cdot 10^6$ |
| $A_{22}$ , slugs                      | $1.78 \cdot 10^6$ |
| $A_{26}$ , slug-feet                  | $-29 \cdot 10^6$  |
| $A_{66}$ , slug-feet                  | $25 \cdot 10^9$   |

|                       |           |
|-----------------------|-----------|
| $L_1$ , feet          | 173       |
| $L_2$ , feet          | 178       |
| $L_3$ , feet          | 168       |
| $L_4$ , feet          | 75        |
| $L_5$ , feet          | 105       |
| Distance to fin, feet | 193       |
| $C_{col}$             | 930       |
| $C_{long}$            | 1400      |
| $C_{lat}$             | 17.5      |
| $C_{inert}$           | 1,300,000 |
| $C_{drag}$            | 970       |
| $C_{lift}$            | 2700      |
| $C_k$                 | 52.5      |

## Aerodynamic drag coefficients:

| Angle | $C_D S$ | $C_C S$ |
|-------|---------|---------|
| 0     | 14,500  | 0       |
| 45    | 22,500  | 7500    |
| 90    | 18,000  | 1000    |
| 135   | 22,500  | -7500   |
| 180   | 14,500  | 0       |

## Appendix D

## MOHOLE DYNAMIC POSITIONING SYSTEM

## GENERAL DESCRIPTION

The positioning system consists of three sonar reference subsystems, a two-channel analog positioning computer, display and motor control components, gyrocompass subsystems, and special purpose power supplies.

The subsystems include independent parallel channels capable of providing desired positioning information to the positioning computer or to the pilot, for manual control. Paralleling of all subsystems not only provides cross-checks on position information and control, but also makes possible trouble-free positioning when routine maintenance or component failure requires the subsystem to be shut down.

To meet the performance and maintainability requirements, each of the three reference subsystems is further paralleled or provided with standby elements to permit rapid changeover in the event of a failure.

In addition to the fully automatic mode of operation, several fall-back or emergency modes are provided which require various degrees of manual control.

These manual modes also provide smooth handling of the vessel for hole reentry. Further descriptive material on this dynamic positioning system has been published by H. W. Ragland.\*

#### LONG-BASELINE ACOUSTIC REFERENCE SUBSYSTEM

The Long-Baseline Sonar Subsystem is an acoustic ranging system that provides accurate ranges to each of four sea-floor mounted transponders positioned on a radius from the drill hole location approximately equal to the water depth.

The Interrogator generates an acoustic pulse which is transmitted to the water from the Transducer suspended below the drill vessel. This signal is received by each of the four Transponders and a signal returned. Each Transponder operates on a different frequency to permit positive identification. The return signals are received at the transducer at some later time. The interval between the original signal and the return signal is a function of the slant range between the vessel and the transponder.

The received signals are amplified and filtered and then passed to the Range-Measurement and Display Group, where the Range Counter generates and displays on the ship's bridge the true slant range in feet. At the same time, the Range-Deviation Counter subtracts the portion of the slant range outside the vessel operating limits and supplies the resultant to the Digital-to-Analog Converter, which provides an analog voltage to the positioning computer. Ancillary to the basic system is an Alarm circuit which provides a warning if a malfunction occurs or if widely divergent range information is being received. In addition, Precision Depth Recorders are provided to permit visual presentation of the signals both for monitoring and emergency use in the event of failure in all digital range-determining circuits.

In the Long-Baseline Sonar Subsystem, the positioning computer normally utilizes range differences between three of the four received ranges. One of the four channels is, therefore, always available as a standby.

An error analysis indicates that the maximum repeatability error of this subsystem will be less than 35 ft in 18,000 ft of water.

#### PHASE-COMPARISON ACOUSTIC REFERENCE SUBSYSTEM

The Phase-Comparison Subsystem is an acoustic system that provides accurate ranges from a beacon on the sea floor in proximity to the drill hole.

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\*H. W. Ragland, Honeywell, Inc., "A Dynamic Positioning System for the Mohole Drilling Platform," in "Ocean Science and Ocean Engineering, 1965," p. 1145, published by Marine Technology Society and the American Society of Limnology and Oceanography, Washington, D.C.



A Beacon transmits a continuous 10-kc signal, modulated by 100 cycles which is received by four Hydrophones located on the corners of the drill vessel. If the acoustic path length between the beacon and each hydrophone is different, a small signal phase (time) difference in the modulating frequency is apparent between each pair of hydrophones. The received signals are amplified and detected in the Receiver and sent to the Tilt Compensation Unit, where a correction is applied to compensate for vessel roll and pitch. This correction is derived from the Two-Axis Tilt Sensor. After correction, appropriate signals are selected by the Mode-Selection Relays and supplied to the Phase-Compare and Compute circuits, which generate an analog voltage to drive the positioning computer. The magnitude of this voltage is a function of the displacement of the vessel-oriented coordinate system from the drill hole. Displays on the bridge showing vessel displacement in feet from the drill hole are derived from the analog voltage by the use of an Analog-to-Digital Converter.

Only one bottom beacon is needed to provide a position signal. In normal operation, three hydrophones and receivers are used, and only two phase-compare and compute circuits are on line. The remaining elements are utilized to provide continuous system cross-checks, or to compare beacon position if a new alternate beacon is placed on the bottom.

Accuracy of this subsystem is about 40 ft in the depths of water anticipated.

#### SHORT-BASELINE ACOUSTIC REFERENCE SUBSYSTEM

Short-Baseline Subsystem is an acoustic system which provides accurate vessel displacement from bottom-mounted transponder located in the vicinity of the drill hole.

An acoustic interrogation pulse, with a center frequency of 16 kc and a predetermined width, is transmitted to the sea floor transponder.

The pulse is intercepted and decoded by the Transponder. If the pulse is of the proper duration, the Transponder transmits a 4-ms reply pulse at 16 kc. This return pulse is detected by four Hydrophones located on arms suspended below each corner column of the platform. Due to the Hydrophone separation, the return pulse does not arrive at each location at the same time.

Pulses detected by the four Hydrophones are routed to four independent signal-processing channels, where they are detected, amplified, and filtered. Timing marks are generated from these incoming signals and a differentiation technique is used which eliminates amplitude imbalance between channels.

The coordinate computation consists primarily of digital counters and storage registers.

The four coordinate values stored in the register are converted to an analog voltage, and corrected for platform inclination by the Tilt Compensator. The compensated coordinate values are sent to the computer, as well as for display purposes at the vessel bridge. Accuracy of this reference subsystem is expected to be approximately 34 ft under anticipated operating conditions.

## POSITIONING COMPUTER SUBSYSTEM

The Positioning Computer Subsystem is a dual-channel electromechanical analog computer which accepts range information from the three reference subsystems, computes vessel position with respect to the drill hole, and generates orders for the vessel power units.

Operation of a single channel may be explained as follows:

The Reference System Selector is used to select range data from one of the three reference systems for computation, and the Reference Point Selector is used to select two or three of the four reference points for computation. Using the selected range information, the X-Y Distance Computer computes the distance between the vessel and a coordinate center in X-Y coordinates. The Geographic to True Coordinate Converter then rotates these coordinates to a true north coordinate system. Parallax Offset between the Long Baseline Sonar transducer and the center of the vessel, and Coordinate Center Offset between the coordinate center and the drill hole are then added by Offset Addition. The resulting composite signal represents north and east distances from the vessel to the drill hole.

The composite signal is routed to total Range to Hole, North Range to Hole, and East Range to Hole indicators and to the Vessel Position Display. This display also accepts a gyrocompass input for displaying vessel heading as well as vessel position.

The composite signal is also sent through the Set Point Offset Integrator, which may be used to automatically generate set point offset signals. These signals are used to generate thrust requirements in the Fore-Aft Positioning Thrust Computer and the Port-Stbd. Positioning Thrust Computer.

Main propulsion motor thrust requirements are scaled from total fore-aft thrust requirements and subtracted from this signal. Ordered Heading, gyrocompass information and fore-aft thrust requirements are combined in Vessel Heading addition circuits to generate required vessel turning force signals. Positioning unit orders are generated in the Port Pos. Unit Thrust and Azimuth Computer and in the Stbd. Pos. Unit Thrust and Azimuth Computer.

Computed quantities from computer channel A and channel B are compared at several points. Alarms are associated with each comparison to indicate that a difference exists. Only one channel is used for vessel control and position display at one time.

## MOTOR CONTROL SUBSYSTEM

The motor control subsystem consists of servo modules and electronic circuitry necessary to control the main propulsion motors and the positioning units.

Propulsion unit control orders are available from either computer channel. In addition, these orders are routed through Mode Select circuits for selection

of the operating mode. If the system is operated in the automatic mode, orders are derived from the positioning computer. If the manual mode is in use, propulsion unit orders are manually introduced.

Positioning unit azimuth orders are routed to the Port Pos. Unit Azimuth Indicator-Control and the Stbd. Pos. Unit Azimuth Indicator-Control, on the positioning control console, where this information is displayed and compared to actual positioning unit azimuth. If a discrepancy exists, signals are sent to the port and starboard Pos. Unit Azimuth Drive motor controllers which drive the positioning units to the desired azimuth. Signals from Pos. Unit Azimuth Synchros geared to the units are used to insure that ordered and actual azimuth angles are the same.

Positioning unit thrust rpm orders are sent to port and starboard Pos. Unit Thrust Indicator-Control modules where they are compared to the actual positioning unit propeller shaft rpm. If a discrepancy exists, signals are sent to port and starboard Pos. Unit Gen. Field Control modules in the engine room. These modules control appropriate generator fields which in turn control the speed of the Pos. Unit Drive Motors. Port and Starboard Pos. Unit Tachometers generate shaft rpm signals for comparison to rpm orders.

Main propulsion motor thrust orders are routed in a manner similar to the positioning unit thrust orders. In this case, functions travel to Main Propulsion Indicator-Control modules, Main Prop. Gen. Field Control modules and Main Propulsion Motors. Main Shaft Tachometers generate signals which are used for ordered and actual shaft rpm comparison.

Alarms associated with each indicator-control module indicate when a sustained discrepancy exists between actual and ordered quantities.

Based on component tolerances, error analysis of the computer circuits indicates an error of about 20 ft when the platform is over the hole. Because of simplifying assumptions in the mathematical model, this error increases to approximately 40 ft at the maximum allowable distance off.

\* \* \*



# TOWING, MOTIONS, AND STABILITY CHARACTERISTICS OF OCEAN PLATFORMS

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## GENERAL

For the purposes of this paper an ocean platform is defined as a floating unit which can be towed or in some cases self-propelled to a location where it is anchored or dynamically positioned to fulfill its function. In special cases platforms are propelled slowly while working. However, in general, an ocean platform is considered to be a unit which is designed for performing some type of work in a more or less fixed location rather than one that is engaged in transporting commodities. While single hulls such as ships or barges may be positioned for a working assignment, these are not considered as true platforms, although they are mentioned in the paper. A platform is envisioned as a unit that has a relatively large working deck of considerable width in respect to its length, where the length to width ratio of the platform itself may be anything from 1.0 to 3.0.

Such platforms have a variety of uses, and with modern technology new uses are being developed. Some of these are as follows: oil well drilling; oil well producing; oil well workover; oil storage; undersea mining; heavy lifting; pipe laying; oceanographic research; weather stations; missile launching; beacons - visual, radar, loran, etc.; landing platforms - air or water vehicles.

Ocean platforms fall principally into the following categories: fixed pile structures having built-in buoyancy for towing only; single hulls - ship or barge; catamarans - floating or submerged; fixed height noncolumn stabilized structures; jack-up platforms with or without bottom hull; column stabilized platforms; and special types: (a) deadweight anchor type platforms and (b) spar vessels.

Many of the types of platforms listed above are shown in Figs. 1 through 15. Fixed pile structures such as that shown in Fig. 1 are not properly a part of this paper as they are ordinarily transported to location on the deck of barges. However, a few have been towed either with the use of attached pontoons to provide stability or by having at least two of the corner tubular members made sufficiently large to provide stability and buoyancy. Such pontoons are lowered into position by selective flooding. Once in position on the bottom, they are ordinarily regarded as being permanently placed, although it is not inconceivable



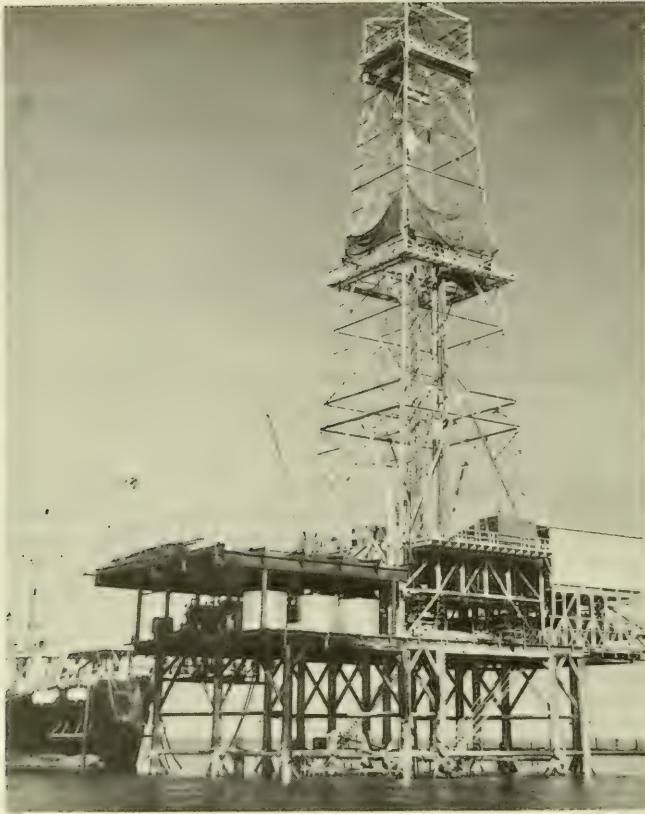


Fig. 1 - Fixed pile supported platform

that some of these could later be removed by the same method by which they were first placed.

Single-hull drilling rigs of the ship and barge type are shown in Figs. 2 and 3.

The platform size for a single-hull vessel is limited by stability and structural support considerations. A catamaran offers the possibility of having a very large working platform with normal proportions for the supporting hulls which is beneficial to mobility and maneuverability. Figure 4 shows the *E. W. Thornton*, one of the few existing catamarans.

Fixed-height noncolumn stabilized structures are limited to relatively shallow water, and therefore, cannot be considered as true ocean platforms. To submerge such a platform it is necessary to either lower one end to the bottom first to provide a stable base while the other end is going down or else provide auxiliary pontoons to maintain stability during sinking. This was the



Fig. 2 - Floating ship with center well (Global Marine, Inc., Glomar III)

common type of offshore drilling platform in the early days but is no longer being built except for relatively shallow water oil storage platforms such as that shown in Fig. 11, a tank-battery platform.

Jack-up platforms are basically a buoyant hull which is at the same time the working platform. Attached to this hull are legs which are lowered to the bottom until they encounter soil resistance after which jacks on the columns raise the hull out of the water to the desired clearance. The bottoms of the legs may penetrate the soil independently like those of Figs. 5 and 7 or they may be attached to a buoyant or a nonbuoyant hull as shown in Fig. 6. With a lower hull much better soil bearing is obtained, but this is at the expense of additional steel structure. Although a number of accidents have occurred with jack-up platforms, the type is very successful and economical and there are perhaps more platforms of this type in service than any other type.

Column stabilized barges consist of a hull having several large-diameter columns supporting a fixed platform. These may be submerged by flooding, and the platform will sink vertically either all the way to the bottom or to some intermediate draft. Once the hull has submerged below the surface, the stability is afforded by the water plane moment of inertia of the stability columns. Such platforms are shown in Figs. 8, 9, 10, and 12.

Figures 13 and 14 show a small column stabilized platform for oil well servicing having hinged stability columns. When the columns are each rotated 90

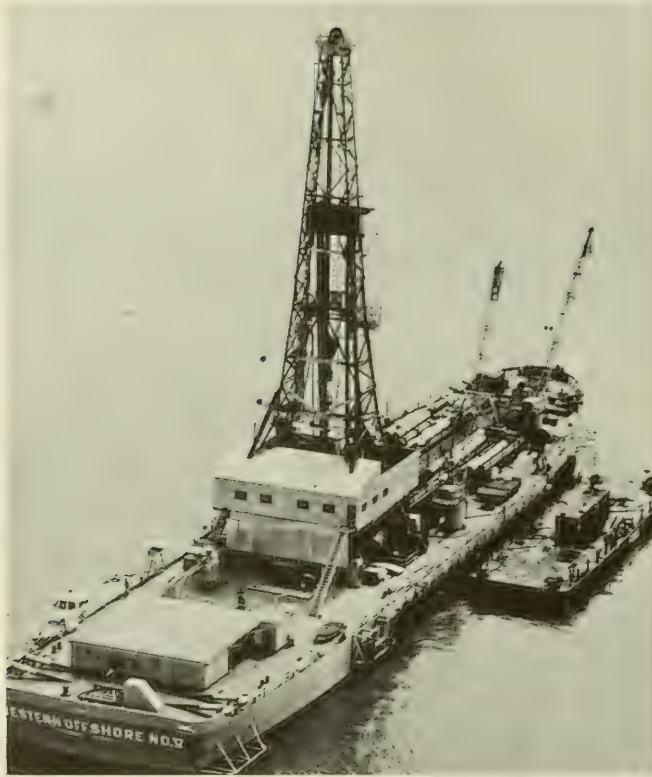


Fig. 3 - Floating single barge (Western Offshore Drilling and Exploration Co., Wodeco V)

degrees in a vertical plane, the columns become catamaran hulls, and in this mode the unit is moved by self-propulsion.

There are several special types which are not in common use. One of these is the deadweight anchor type (Fig. 15), consisting of a hull with small columns extending upward to support the platform. When the platform has arrived at location, very heavy anchors suspended directly below the structure are lowered to the bottom. Then with the use of winches, the unit is pulled under the water so that the hull is below the maximum wave forces, with the working platform remaining above the surface. The proportions of the anchor weights and buoyancy members need to meet the following criteria:

1. The anchors must be heavy enough so that the upward components of the wave forces will not lift the hulls and raise the anchors off the bottom.
2. The reserve buoyancy of the hulls must be enough so that the downward components of the wave forces will not cause slack in the anchor cables, allowing the platform to drop.

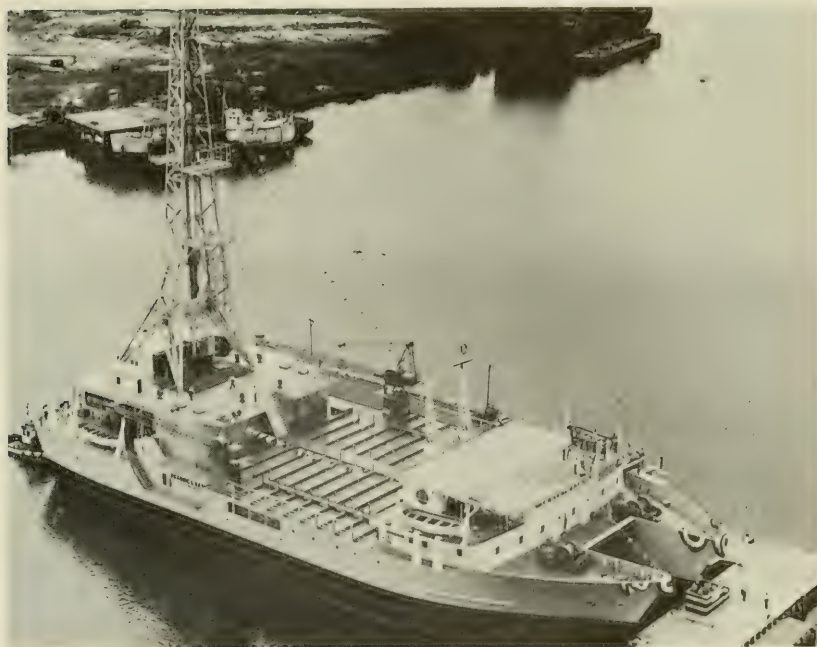


Fig. 4 - Floating catamaran (Reading and Bates,  
E. W. Thornton)

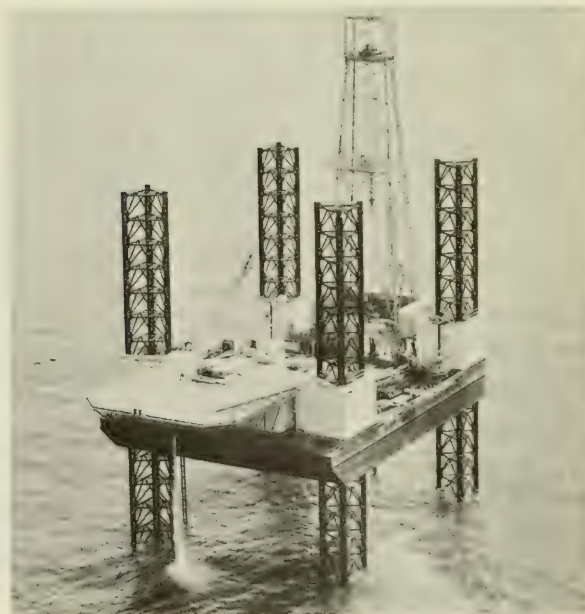


Fig. 5 - Jack-up platform for  
a 200-foot water depth (The  
Offshore Co., Jubilee)



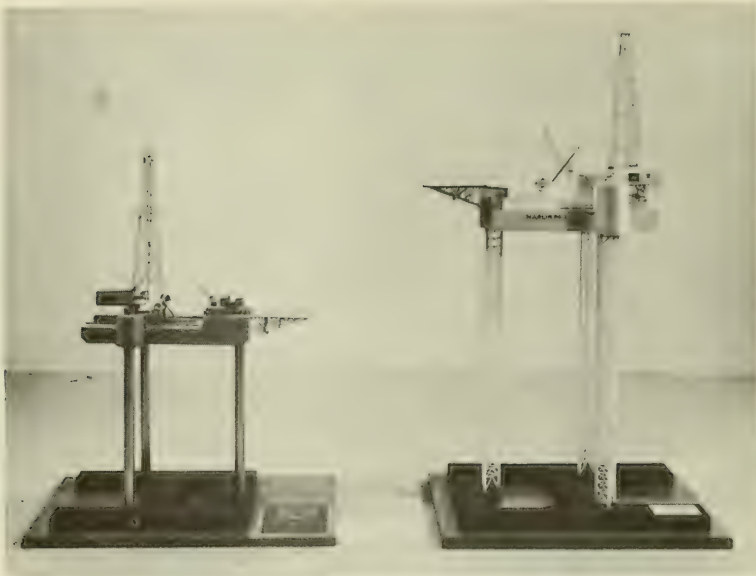


Fig. 6 - Comparative models of jack-up platforms with bottom mats (Bethlehem, Stormdrill III for a 140-foot depth and Marlin No. 3 for a 250-foot depth)



Fig. 7 - Jack-up platform without a bottom hull for a 300-foot water depth (Loffland Brothers Ocean Master I)

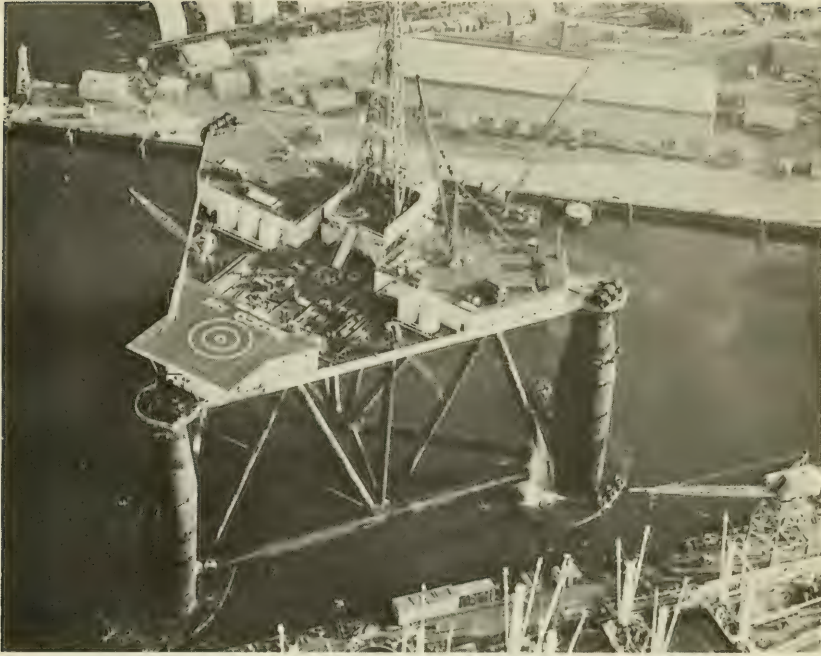


Fig. 8 - Platform on columns stabilized for a 135-foot water depth or for floating semisubmerged in greater depth (South-eastern Drilling Co., Sedco 135)

If these two criteria are met the platform will be virtually motionless in a vertical direction but must be restrained laterally by an anchoring system.

A spar platform is simply a long cylinder which is towed horizontally but is erected vertically for working purposes. Its use has been limited to oceanographic research, but it could be used for much heavier work. The great vertical dimension when upright results in a very long rolling period with good damping. The small water-plane area results in a long heaving period.

Another special type of rig is the new Production Platform No. 3 for the Freeport Sulphur Company. This is actually a column stabilized platform; however, it is unique in that it rests on the bottom at the beginning of its assignment in 50 feet of water, but it will eventually, due to subsidence of the bottom as the sulphur is mined, find itself no longer on bottom but in water approaching twice this depth. Eight spuds are provided which, once the bottom has started to subside, will assist the stability columns in maintaining stability and, at the same time, will prevent lateral shifting.

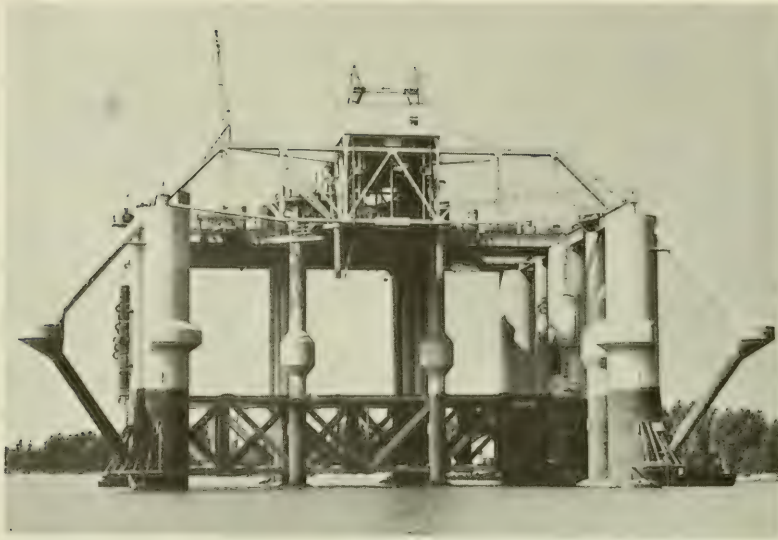


Fig. 9 - Platform on columns (note column enlargements) stabilized for a 90-foot water depth or for floating semi-submerged (Odeco, Ocean Queen)

## TOWING RESISTANCE

Ocean platforms were originally used almost entirely in the Gulf of Mexico; distances between locations were not great, and towing speed was not an important factor. Today, however, platforms are towed all over the world, and the importance of speed is demonstrated by a recent performance where a 9000-horsepower tug towed a platform having a hexagonal shape hull from Orange, Texas, to Rotterdam, a distance of 5500 nautical miles with a towing time of 58 days. The average speed was only 4 knots. It is entirely possible that on certain days the rig lost mileage rather than gained. Many of the hulls of offshore platforms are nothing but rectangular blocks without any shape. Some of these hulls have a slot in them to permit the platform to get away from a finished oil well. Such slots are usually placed aft when towing.

Figure 16 illustrates most of the hull forms used or proposed. Single hull vessels having the proportions of ships or normal cargo barges have much greater towing speeds, but their deck space, stability, and motion characteristics are not ideal for use as ocean platforms. Floating catamarans are capable of high towing speeds and several of them have been built, one of which is self-propelled, the *E. W. Thornton*, shown in Fig. 4.

The type of high load and space capacity platform that has the best working and motion characteristics is the column stabilized platform, which is usually a square, or nearly a square, or a triangular working platform. The first hulls for such column stabilized platforms were of a grid pattern, which was practically

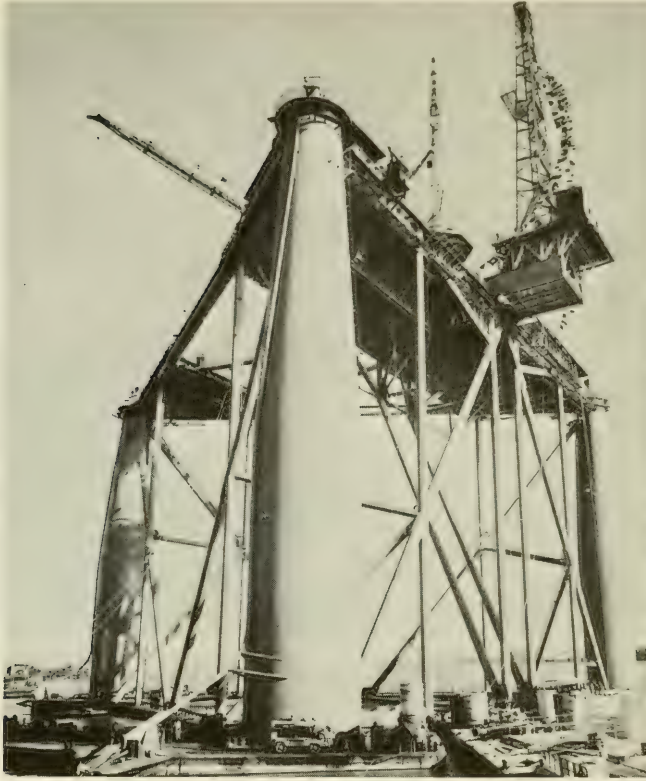


Fig. 10 - Platform on columns stabilized for a 175-foot water depth (Kerr-McGee, Rig 54)

square in plan. These hulls have extremely poor resistance and maneuvering characteristics, and it is usually necessary to have one or two tugs ahead as well as a tug astern to control yaw.

Recent developments in column stabilized platforms have used catamaran hulls, such as the Mohole platform described in the preceding paper. It is possible also to use three or four parallel hulls without any cross connection in the water, all connections being made at a considerable elevation above the surface. An example of this is the *Ocean Queen* shown in Fig. 9. Such parallel hulls permit very high towing speeds and yet permit the platform to be of optimum proportions. A nonparallel catamaran has some structural advantage, and the increase in resistance (Fig. 17) is tolerable.

The *Sedco 135*, Fig. 8, is a column stabilized platform with elliptical footings at the bottom of each column. However, the speed-to-length ratio is very unfavorable due to the short length of each footing. On the contrary the





Fig. 11 - Platform comprising oil storage tanks with a submersible bottom mat (Chevron Oil Co., S-92)

speed-to-length ratio of long catamaran hulls is very low and results in a low resistance per ton as compared with that of individual footings.

Figure 17 speaks better than words in describing the vast variation in resistance of platform types. It is difficult to obtain a sufficient quantity of good data on towing resistance on some of the unusual forms that are used. This is due to the fact that most model tests are made by the individual companies for their own use, and at the present time such data is regarded as confidential.

Extreme refinement in hull forms for elevated deck ocean platforms is not justified, because the large platform structure raised on columns high above the hull has wind resistance which is a large component of the total resistance. Furthermore, high freeboard in the hulls of column stabilized platforms is not necessary from a stability standpoint, as the columns provide stability anyway when the hull is being submerged. In addition, the presence of an excess hull volume compared with the column volume results in the center of buoyancy being unfavorably low for stability. Therefore, there is a tendency to keep not only the displacement but the freeboard of the hulls to a minimum. This has a disadvantage in towing in that even moderate waves will wash over the hull and strike the columns, causing a considerable augment in resistance. While most of this is due to the large stability columns, there is considerable effects on the smaller support columns as well.

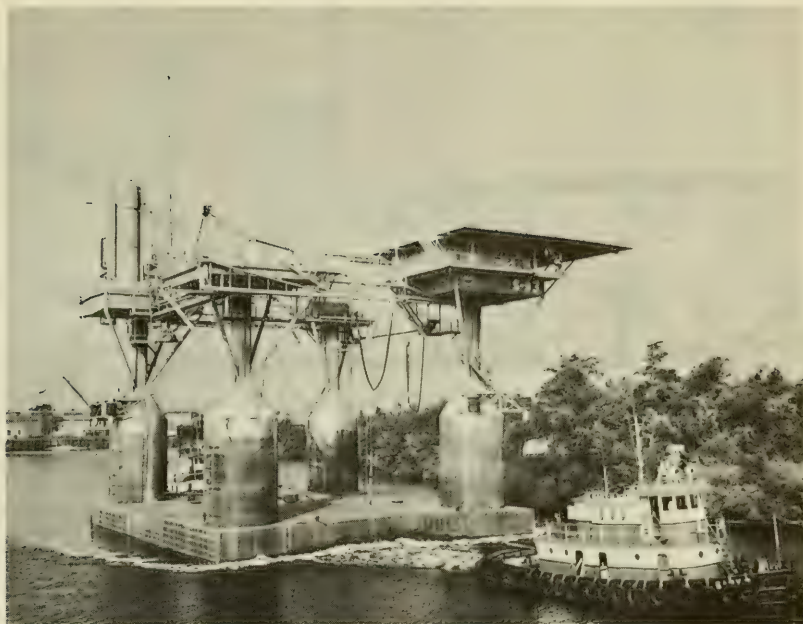


Fig. 12 - Oil storage and production platform stabilized by 20,000-bbl columns (Chevron Oil Co., S-93)

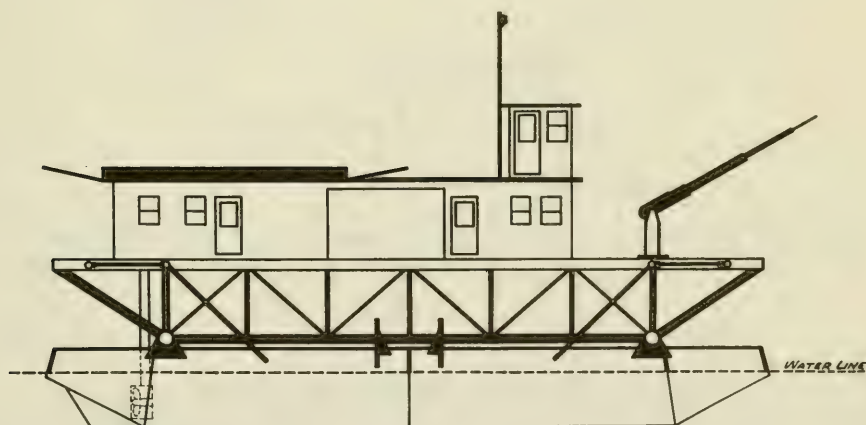


Fig. 13 - Otis Engineering Corporation column stabilized catamaran wire line unit in the transit position

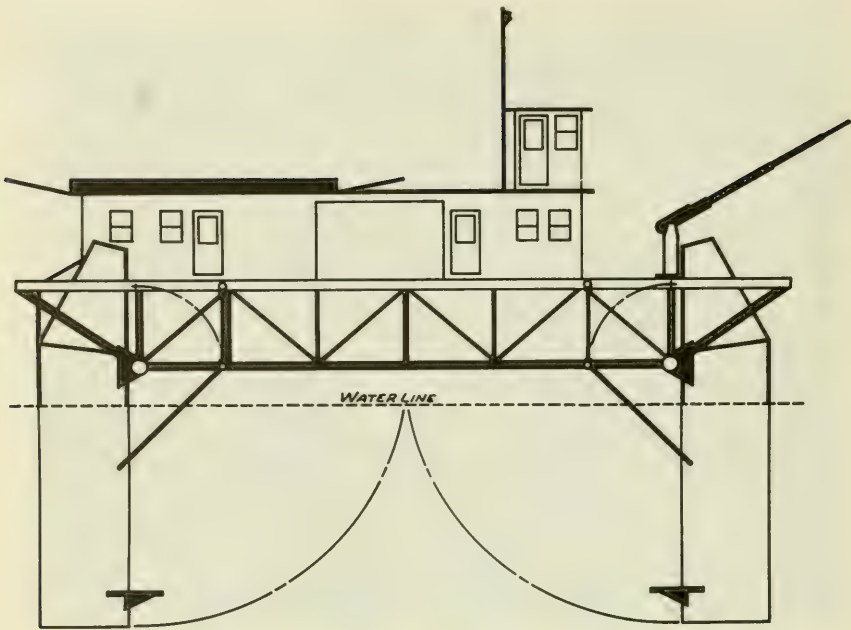


Fig. 14 - Otis Engineering Corporation column stabilized catamaran wire line unit in the working position

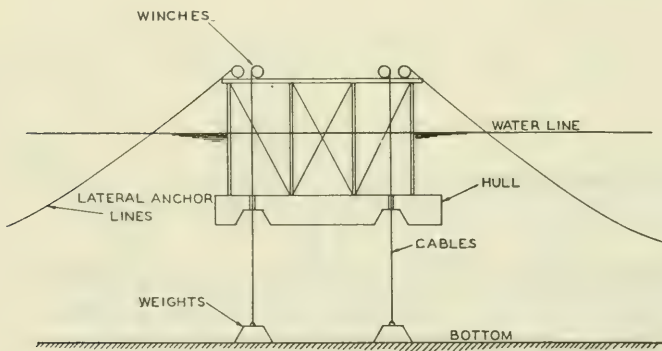


Fig. 15 - Deadweight anchor type platform

## Towing, Motion and Stability of Platforms

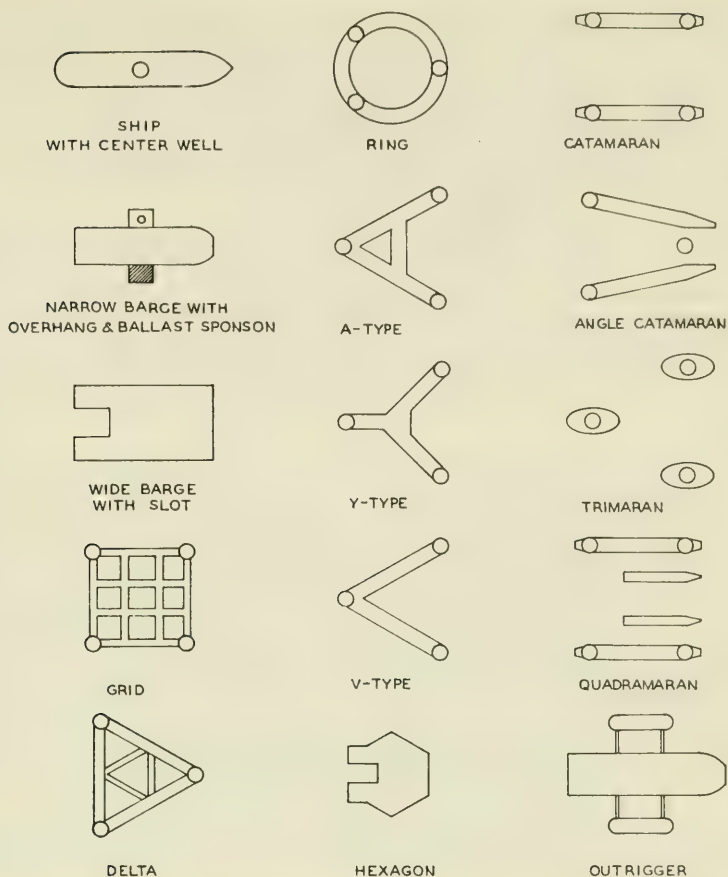


Fig. 16 - Platform hull types

Nearly all of the forms shown in Fig. 16 have a tendency to yaw, with the exception of the forms which have single or multiple parallel hulls.

### STABILITY

It is necessary to discuss stability in this paper due to the fact that it is very closely associated with considerations of motion. The stability of floating vessels or barges is in all respects similar to that of ordinary ships and requires no special comment here. The only problems of a special nature are the substantial wind heel moment due to the high structures required on the decks including the drilling derrick itself. Also if drilling is done on a cantilever extension over one side, compensating water ballast is usually needed on the opposite side with pumps of very high capacity to control the compensating moment. The load hanging from the drilling derrick can be as high as one million pounds,



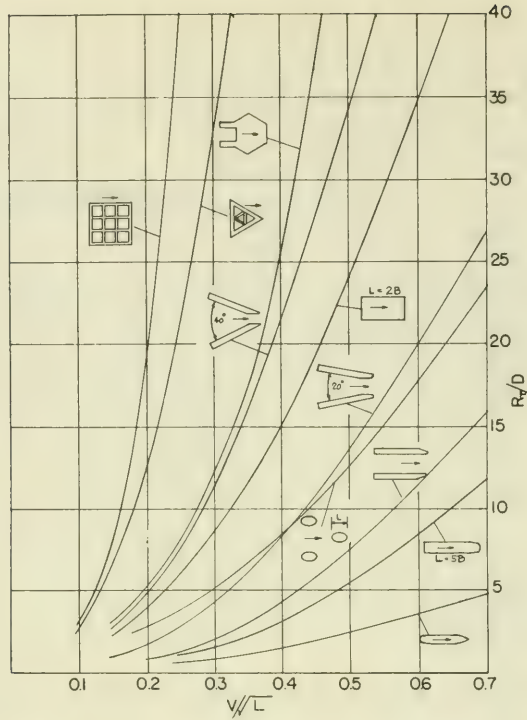


Fig. 17 - Towing resistance of mobile platforms in smooth water. The data for these composite curves were obtained from actual towing performances or from model tests.

and this load can be fully effective as soon as the bit is raised off the bottom when pulling it out of the hole. For this reason, high-capacity single-hull drilling rigs ordinarily drill in a well on the center line.

Catamaran hulls or the outrigger type vessel have tremendous metacentric height, and stability is no problem at all, though motion and mooring are.

Likewise, jack-up platforms usually have a hull of substantial length and width and static stability presents no difficulty at least in the upright condition. The range of stability however is sometimes a problem, as the great width of such hulls means that at a relatively small angle the deck is immersed or the bilge is emerged, resulting in a loss of range of positive righting lever. Therefore, it is best to have the freeboard equal to the draft so that both occur simultaneously.

The stability of jack-up platforms may be a problem if there is a common buoyant lower hull at the bottom of the legs or if there are individual buoyant

pads at the bottom of each leg. At least one platform has been lost due to the presence of buoyancy in such footings. If these spaces are insufficiently filled with water ballast, during raising and lowering, the center of buoyancy of the entire assembly may be too low and the platform may capsize. Where it is necessary to have ballast in the lower hull, arrangements should be made so that this ballast is added automatically by an open stand pipe extending up to a predetermined height. This insures that such tanks will be filled at an early stage.

The stability of jack-up platforms when actually resting on the bottom is a matter of overturning moment due to wind and wave versus the resisting moment provided by the bottom of the sea. This is a matter of soil mechanics and is outside of the scope of this paper. However, platforms have capsized in this manner and have been completely lost. The provision of a buoyant upper hull would have in several cases resulted in the platform at least not disappearing out of sight, and salvage would have been possible.

Column stabilized platforms are one of the most successful and safest types of platforms; yet their stability presents some very unusual problems. The stability of such platforms when floating on the surface is tremendous but as soon as they are submerged to the point where the hull is awash, the moment of inertia of the water plane drops so that the metacentric height of hundreds of feet becomes only 5 to 25 feet (Fig. 18). Exceptional care has to be taken in the sequence of ballasting in order to provide proper trim as well as to restrict free surface. The trim problem is aggravated with certain forms in that the center of flotation may abruptly jump a considerable distance when the hull becomes awash. For example, a configuration having a catamaran hull would have its center of flotation at midships when surfaced but if this platform has three stability columns, which is entirely feasible, the center of flotation, once the hull were submerged, would be at one-third the distance from one end.

It can be shown (4) that for a symmetrical arrangement of stability columns, the moment of inertia of the water plane is the same about any axis.

The diameter and spacing of stability columns are usually a compromise. Economical construction calls for relatively large columns closely spaced so that the connecting structures at the hull and at the platform are of minimum proportions. However, considerations of motion and wave transparency dictate small diameter columns very widely spaced. On the other hand, to prevent heaving motions from being too short, small columns are desirable, but this results in a very low value of tons per foot of immersion, so that considerable changes in draft and trim occur when adding or removing loads. This particularly applies to the drilling load, which in some arrangements is at one end.

In general drilling should be performed at the center of flotation in order to provide minimum motion and minimum change in trim. One of the drilling platforms of the Ocean Drilling and Exploration Co., the *Ocean Queen* (Fig. 9), has some interesting features — one of which is enlargements on some of the secondary structural support columns which provide an increase in water plane area at the normal drilling draft. In the event of severe motions and synchronism the barge could be ballasted or deballasted to bring these deeper in the water or out of water and therefore change the heaving period. Also, presumably

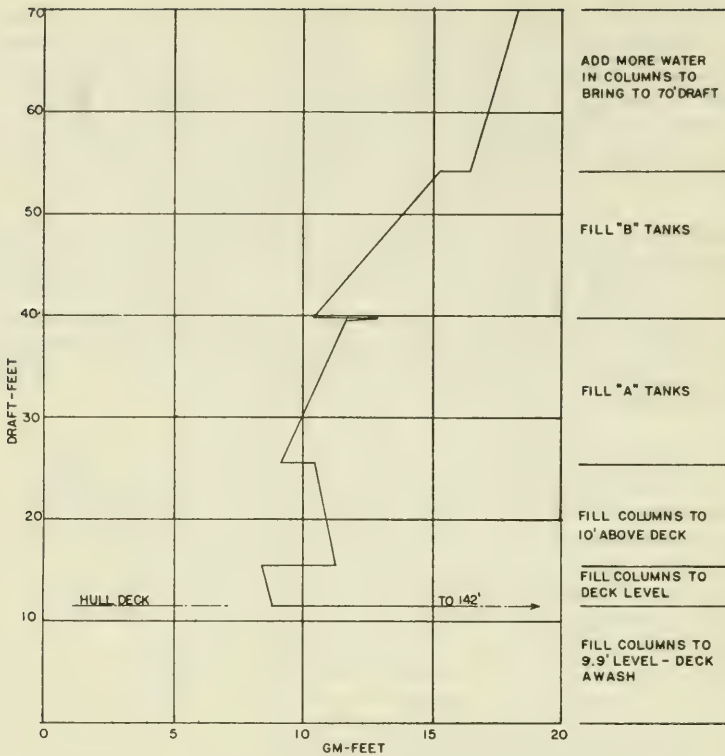


Fig. 18 - Variation in the metacentric height (GM) of a column stabilized platform during submerging

to effect a movement of the center of flotation in a more favorable direction, there are two outriggers at one end which contain cylinders to provide additional water plane area at a considerable lever arm in both directions.

Drilling platforms which are designed for resting on the bottom in a particular water depth do not need as thorough a stability investigation as those which are intended for drilling semisubmerged. The reason for this is that the tremendous length and width of these platforms is such that in a fixed water depth for which a platform is specially designed, it would heel over to the point where one edge of it was aground long before the rig could capsize. On the other hand platforms designed for drilling semisubmerged could conceivably be in water much deeper than the width of the platform. Fortunately, the range of righting levers of column stabilized platforms is very large — 50 to 70 degrees being usual.

As a safety factor, intact buoyancy above the water line is desirable. One form of this buoyancy is horizontal subdivision in the columns. Some of the compartments are occasionally filled with foamed-in-place plastic or empty oil



drums to reduce the permeability. It is considered good practice to provide in the stability columns a safety deck so located that in the event of inadvertent total flooding the platform could not completely sink. An alternative procedure is to make the upper platform structure of a watertight box configuration which has the dual advantage of restricting the angle of heel in case of unsymmetrical flooding or preventing foundering in case of total flooding of the lower hull. In the latter alternative, however, each and every opening of every kind in the upper platform must be made watertight, which means that drains must all be connected into a centralized drain tank, and that watertight doors must be avoided in the upper platform enclosure.

The calculation of righting levers for complex platforms which not only have stability columns but also secondary structural support columns which may or may not be vertical is rather complex, and Ref. 5 describes a method of facilitating such calculations by the use of optical projection in which a beam of light is projected through a scale model of the platform onto a screen.

In order to obtain the maximum metacentric height with a minimum of structure, it is obvious that the center of buoyancy should be as high as possible. An optimum situation is to have a column stabilized platform which has no lower hull of any kind and the entire displacement is afforded by the columns themselves. In this case the center of buoyancy is at half the draft. This arrangement has been proposed several times but not actually constructed due to the fact that the draft would be too high for construction of a large platform in most shipyards. Therefore column stabilized platforms usually have a buoyant lower hull, and in keeping with this principle the displacement of this hull should be at an absolute minimum. It is not even strictly necessary that there be any positive freeboard at all in the surfaced condition. However, from the practical consideration of accessibility to tank manholes, a freeboard of several feet is usually provided. As discussed in the preceding section, however, a greater freeboard insures that the columns supported on the hull will be high enough above the water so that the effect of waves on towing resistance is reduced.

Unless the stability columns contain some watertight subdivisions, the flooding of one column will usually result in the loss of the platform. This is especially true of a three-column platform. A four column platform may survive actual loss when one column is damaged but will be inclined at a very great angle. A platform having six columns can survive at a moderate angle the damage to any one column; but for commercial purposes six columns are not used due to economic considerations. The platform proposed for the Mohole project, however, will have six columns.

Ordinarily, the tops of stability columns are tapered in diameter somewhat above the maximum water line. This reduces the area exposed to waves, although it has a disadvantage of taking away what would otherwise be very desirable reserve buoyancy in the event of damage. The savings in structural cost is very minor. Platforms having lower hulls or mats which rest on the bottom of the sea are subject to scouring due to current or, in fairly shallow water, effects from storm waves. There are three principal methods of reducing the effect of scouring. One is to slope the sides of the hull into a flat wedge. This is no longer popular in that it results in increased dimensions and construction



cost. The second method is to provide scour coamings of the "cookie cutter" type which extend 4 or 5 feet below the bottom of the hull. These are also of benefit in preventing lateral shifting on bottom during storms. Care must be taken to have vent pipes through the hulls to prevent trapping air behind these coamings. The coamings are somewhat vulnerable to damage, and in very soft bottoms such as encountered in the Gulf of Mexico off Louisiana they would have to be uneconomically deep to do any real good. A third solution is to provide the platform with a number of spuds which penetrate deep into the bottom of the sea. While these do not actually prevent scouring, they will prevent lateral shifting, and if the spuds are of sufficient number and capacity and are wedged or locked into their guides, they will help to prevent undue settling of one corner or one edge as a result of scouring.

In soft bottoms such as in Louisiana it is not at all unusual for the lower hulls of platforms to be completely buried in the mud after a long period on station. To remove an imbedded hull from the bottom requires considerable precautions. Water jet piping is a necessity, but care has to be taken in pumping out water ballast that the platform does not suddenly break loose from the bottom and emerge at an excess velocity. It is ordinarily preferable to try to get one end free before the other end. With a column stabilized platform disaster could hardly occur, because as the platform starts to come up, the displacement of the stability columns falls off rapidly, functioning as a natural hydrostatic brake. Old type platforms which had large buoyant lower hulls and only small-diameter columns did not have this advantage of rapid change in displacement on emerging and could conceivably come up to the surface of a terrific rate.

In cold areas it is necessary to consider the effect of ice on stability. One platform had a loss of metacentric height of 1.5 feet as a result of 3 inches of ice on the platform deck, not including ice elsewhere. To make matters worse, ice formation is greater on the windward side resulting in a heeling moment. Ice is not likely to form on the columns, as the waves would keep it washed off, but it can form under the platform. As a result of snow, ice may develop on high-center-of-gravity structures such as drilling derricks. Reference 2 contains an extensive bibliography on icing of ships.

Ocean platforms exceeding 100 gross tons, other than those which rest on the bottom, are subject to U.S. Coast Guard regulations, and rather complete stability calculations are required.

## MOTION

Aside from the usual considerations of comfort and structural design, minimum motion is a necessity to facilitate the work assignment of a platform. Excessive angular motion or surge in an oil drilling platform will cause bending stress in the rotating drill pipe, while heave will result in tension or compression in the drill pipe. Telescopic sleeve joints just above the bit prevent cyclic lifting of the bit off the bottom.

Experience has shown that if a flared structure like a curved inverted funnel is fitted below the derrick substructure, that a half angle of as much as 20

degrees of roll can be tolerated. The sleeve joints in the drill string consists of multiple sections, and enough of them can be used to permit tolerating a heave of around 20 feet. The platform must be reasonably restrained laterally, and a rough rule of thumb is that a drift from the vertical of 10 percent of the water depth is acceptable. Even so, all these motions must be kept to a minimum, and platforms of various types have greatly differing motion properties.

Broadly speaking, the usual types of platforms may be separated into three categories as shown in Table 1.

Table 1  
Platform Categories

| Platform Type                                                                                                                                                 | Roll Angle     | Pitch Angle | Roll Period | Pitch Period | Heave Period |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------|-------------|-------------|--------------|--------------|
| 1. Single hulls on the surface                                                                                                                                | Medium to high | Small       | Medium      | Short        | Short        |
| 2. Twin hulls or wide single hulls on the surface.<br>Column stabilized platforms on the surface.<br>Jack-up platforms on the surface or during leg lowering. | Small          | Small       | Short       | Short        | Short        |
| 3. Column stabilized platforms submerged                                                                                                                      | Small          | Small       | Long        | Long         | Long         |

Jack-up platforms have motion problems when raising and lowering. Since the jacking speed is rather slow, when lowering the legs at a stage when the legs are close to the bottom, there is an appreciable interval of time when the leg bottoms are touching or nearly touching the bottom. This also applies when the legs are being raised and have just come off the bottom. The leg bottoms are then subject to pounding on the bottom of the sea even in a very moderate sea condition, due to the heaving motion of the floating platform. This effect is very much less if there is a bottom mat, due to its inertia and damping effect. A jack or motor failure just at this time would be serious.

Another critical condition exists from leg bending stress when the legs are hanging almost to the bottom, due to rolling acceleration. Even in moderate sea conditions, the combination of a short rolling period and a long radius to the leg bottom results in a considerable length of arc of movement of the leg. The acceleration is sufficient to cause very large stresses in the legs unless they are designed with this in mind. The use of a bottom hull instead of individual legs tends to dampen this acceleration.

Similarly with the platform on the surface, legs may extend several hundred feet in the air and during bad weather, which might be encountered during towing, the top of the legs will be moving through a considerable arc.

Many of the earlier drilling platforms were equipped for drilling off one end or over the side. Aside from stability and trim control problems, this is undesirable because the vertical components due to rolling or pitching are added to the heaving motion. If the drilling is done at the center of flotation, the vertical component of roll and pitch is zero, and there remains only the small horizontal components. But unless the platform contains some type of slot in one end, or has a Y, A, or catamaran configuration, the platform could not be withdrawn from a permanent pile supported production platform. This has led to the development of sub-sea completions, of which many forms have been proposed and some tried with success. However, it is still far from a routine operation.

The natural period for unresisted rolling or pitching in still water is

$$T = \frac{1.108 K}{\sqrt{GM}}$$

where  $T$  is period for a complete roll,  $K$  is radius of gyration, and  $GM$  is the metacentric height.

The heaving period is

$$T = \frac{2\pi}{\sqrt{\frac{g t}{D}}}$$

where  $t$  is tons per foot of immersion and  $D$  is displacement in tons.

In the case of a wall-sided vessel, where the tons per foot of immersion is constant,  $D/t$  may be replaced by the draft  $d$ , and the formula becomes

$$T = \frac{2\pi}{\sqrt{\frac{g}{d}}}$$

There are two basic philosophies in adopting motion criteria. One is to maintain all of the natural frequencies below the range of periods of the larger waves expected. This will not avoid the possibility of synchronism, because wave periods will be encountered all the way from zero to the maximum contemplated, and at least one period will give synchronism for each of the roll, pitch, and heave conditions. There may also be secondary peaks in the response curves. However, if the platform is a type with short roll, pitch, and heave periods, the amplitudes even at synchronous motion may not be too great.

The second approach is to design all of the natural frequencies to be greater than the period of the maximum or the nearly maximum expected wave. In this case, synchronism will never be reached. The column stabilized platform and the less common spar platform are the only ones which can attain this criterion. The determination of the design wave period for a particular location is an oceanographic matter, but generally in the Gulf of Mexico a 16-second wave is about the maximum to be expected, and in the Pacific 20-second waves



can be expected. Column stabilized platforms can be designed to have natural frequencies beyond these periods.

Figure 19 shows typical heave response curves with an assumed maximum wave period superimposed. This shows that a column stabilized platform can be designed to have a natural frequency beyond the range of most anticipated waves. The situation for roll and pitch is similar.

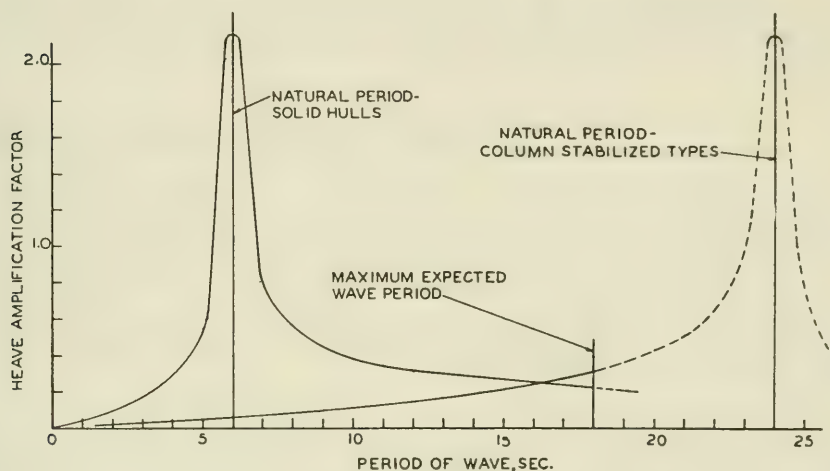


Fig. 19 - Typical heave response curves

Normal single hulls cannot have a large radius of gyration due to their limited width. However, they have the benefit of reasonably low GM so that the roll period may not be too short. Due to the large tons per foot the heave period is short. Surface catamarans, very wide floating barges, and jack-up platforms have large radii of gyration and high metacentric heights at the same time, and even larger tons per foot, so that all motions are of a short period.

Column stabilized platforms when submerged combine large radius of gyration with medium GM to obtain long roll and pitch periods. The low tons per foot of the columns coupled with a high displacement results in a long heaving period.

Platform design selection should include studies of its motion characteristics in association with data on the wave periods expected at the contemplated sites. For preliminary purposes, the roll and pitch formulas given seem to agree well enough with actual results. The heave periods seem to be always longer, by as much as 25 percent due to the damping of the hull or footings. It is recommended that preliminary designs be later confirmed by model tests. However, it should be pointed out that rolling and pitching tests are no better than the precision of the calculation of the mass moment of inertia or radius of gyration. This is a laborious calculation if done thoroughly.



To some extent motion periods or amplitudes may be adjusted. The obvious means of controlling roll and pitch motions is by changing the radius of gyration. This can be changed in wide hulls or catamarans by shifting ballast between outboard and inboard locations. The shift must be done horizontally to prevent modifying the metacentric height. Merely adding water ballast does little good as the gain in radius of gyration tends to be offset by the increase in the metacentric height.

The metacentric height may be adjusted in several ways to change the rolling period. Figure 18 is the ballasting curve of a four-column grid hull platform, and shows the wide choice of GM available with only moderate draft adjustments.

The metacentric height may also be changed by increasing or decreasing the free surface in the ballast tanks. The free surface can be increased only up to the limits imposed by stability requirements. The metacentric height may also be adjusted on column stabilized platforms by providing enlargements or bulges on some or all of the columns near the working water line (Fig. 9) so that the draft may be set on the enlargements, or above or below them. Also, if stability permits, the water plane area of the columns may be decreased by reducing the stability column diameter over a portion of the length of the columns. Usually holes are provided in the columns and a smaller inner core is provided to maintain watertight integrity.

The devices outlined in the preceding paragraph also change the tons per foot and are therefore effective in changing the heave period. Another obvious way to modify the heave period is to change the displacement, but increasing the draft to do so results in an increase of surge force and mooring line tension. Heave damping plates have been tried and are sometimes effective if placed deeply enough. Otherwise, if they are in the larger wave particle orbits near the surface, they could actually augment heave.

It seems appropriate to introduce a word of caution about these motion control devices; a patent study should always be made before adopting any of them. Many devices used or proposed on ocean platforms for stability or motion control based on principles known for many years now seem to be covered by recent patents.

## MOORING

Motion and mooring are closely related in that minimum motions may be obtained from high tension in mooring lines, literally "tying down" the rig. This is at the expense of greatly increased mooring equipment, which on a large platform can easily reach \$1 million in cost.

Three-column stabilized columns are usually moored at each column with a total of 3, 6, or 9 anchor lines on the platform. Four-column platforms as well as jack-up platforms in a catamaran have four to eight lines. The anchor lines should have a moderate degree of pretension, and while constant tension winches are desirable, their high cost is usually prohibitive. However, in any case line tension recorders should be installed.

Mooring lines are commonly of the following types: all wire rope, wire rope with a nylon section, wire rope with a chain section, all chain, or wire rope from the winch to the buoy and chain from the buoy to the anchor. The resiliency afforded by the second and the last types are attractive but complicate the winding of lines on drums or wildcats. This is less serious than would appear as most platforms are not maneuverable enough to set their own anchors, and this has to be done by auxiliary supply vessels. Anchors are usually of the Navy LWT or Danforth types.

Model tests show that for minimum motion the mooring lines must not be placed too low. A test of a platform 140 feet high at 70 feet draft showed that raising the point of attachment of mooring cables from 30 feet above the bottom to 100 feet resulted in a reduction of heel due to aerodynamic lift from wind under the platform deck from 8 degrees to 1 degree. This was accompanied by a reduction of mooring line tension of 40 percent.

This aerodynamic effect is important, and various types of "spoilers" have been tried with little or no gain and should be the subject of further research. Care should be taken to avoid overhanging platforms such as cantilevered pipe racks or helicopter platforms which might be subject to such lift.

A disadvantage of the multiple anchor system is the difficulty in changing heading in order to improve the motion or kill synchronism. The anchor lines originate at fixed points on the platform and only a few degrees of swing can ordinarily be made. The Offshore Company's platform *Discoverer* is a single-hull platform with a rotatable annular ring in the center of the hull, having the anchor lines attached to the bottom of the ring. By the use of thruster propellers, the hull can be rotated 360 degrees. There are, however, some formidable problems of maintenance and servicing the anchor lines under the hull.

Position can be maintained by rotatable propellers instrumented for automatic position sensing and control. Conventional mooring arrangements become too expensive in water more than about 600 feet deep, and dynamic positioning will likely be a necessity in very deep water.

## PLATFORM SELECTION

Table 2 gives an indication of the suitability of several common platform types. Water depth is a relative term; water deep enough for a platform 50 by 50 feet is shallow for one 250 by 250 feet. However the table is based on relatively large platforms. While spar platforms are shown, their primary field seems to be in oceanographic research, where a great load capacity is not required. The principal conclusions drawn from this table are (a) that below 50 feet platforms should rest on bottom, (b) between 50 and 300 feet platform types are available which can float, rest on the bottom, or be semisubmerged, and (c) above 300 feet the floating or semisubmerged platforms seem the best.

New developments in details of existing types as well as new types of platforms are being developed. Unfortunately model test and actual performance data does not exist to nearly the degree as in the case of ship data. Furthermore, the majority of platforms in use are for the oil industry, and data is

Table 2  
Relative Characteristics of Several Common Platform Types

| Platform Type        | A, B, C, or D Grading, Where A Indicates the Best With<br>Regard to a Given Characteristic |      |       |       |                   |                 |                  |                  |                |
|----------------------|--------------------------------------------------------------------------------------------|------|-------|-------|-------------------|-----------------|------------------|------------------|----------------|
|                      | Towing<br>Speed                                                                            | Roll | Pitch | Heave | Water Depth Range |                 |                  |                  |                |
|                      |                                                                                            |      |       |       | Below<br>50 ft    | 50 to<br>150 ft | 150 to<br>300 ft | 300 to<br>600 ft | Over<br>600 ft |
| Floating:            |                                                                                            |      |       |       |                   |                 |                  |                  |                |
| Single hull          | A                                                                                          | C    | A     | D     | D                 | B               | B                | B                | B              |
| Catamaran            | A                                                                                          | A    | A     | D     | D                 | B               | B                | B                | B              |
| Spar                 | A                                                                                          | A    | A     | A     | D                 | D               | C                | B                | A              |
| Semisubmerged:       |                                                                                            |      |       |       |                   |                 |                  |                  |                |
| Column<br>stabilized | A to D                                                                                     | A    | A     | A     | D                 | B-C             | A                | A                | A              |
| On Bottom:           |                                                                                            |      |       |       |                   |                 |                  |                  |                |
| Fixed pile           | -                                                                                          | -    | -     | -     | A                 | B               | C                | D                | D              |
| Fixed height         | C                                                                                          | C    | C     | D     | A                 | D               | D                | D                | D              |
| Jack-up              | C                                                                                          | C    | C     | D     | B                 | B               | C                | D                | D              |
| Column<br>stabilized | A to D                                                                                     | A    | A     | A     | A                 | A               | C                | D                | D              |

regarded as proprietary and is not exchanged as freely as in the more orthodox branches of naval architecture. It is hoped that in time the industry will come to realize the mutual benefits of a free exchange of data.

#### ACKNOWLEDGEMENTS

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Monday, October 3, 1966

Morning Session

## OCEAN WAVES

Chairman: G. Neumann

New York University  
New York, New York

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# LOW-FREQUENCY TAIL OF THE OCEAN WAVE SPECTRUM

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University of California, San Diego  
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I will review briefly four relatively new observations concerning ocean waves whose frequency is below that of ordinary wind waves. The topics are vaguely related and they emphasize the importance of nonlinear processes.

At 2 to 3 cycles per minute one occasionally finds well-developed spectral peaks associated with swell from distant storms. These peaks shift from day to day toward higher frequency, and the rate of shift gives information concerning source time and distance.

One new development concerns an attempt by Snodgrass et al. (1) to measure waves simultaneously at widely separated points. These measurements confirm nicely what had already been fairly clear since the days of Barber and Ursell (2): each wave component moves at a group velocity appropriate to its frequency. The purpose of the experiment was to measure the *attenuation* of the ocean waves. They diminish rapidly in the near zone (within one storm diameter) and very slowly thereafter. The behavior can be accounted for in terms of "Stokes scattering": interacting wave groups from the storm area scatter energy into high frequencies, where they are ultimately removed by white capping. From what we know of scattering cross sections, there is no insuperable quantitative discrepancy. The super-exponent behavior follows from the fact that the interaction depends on the products of the interacting amplitudes, on the beamwidth, and on the bandwidth in the frequency domain; all these quantities diminish with distance from the storm area.

So far we have been concerned with nonlinear interactions leading to higher (sum) frequencies. Insofar as these interactions lead to difference frequencies, one can explain to some extent oscillations at around 1 cycle per minute, which have been observed with appropriate instrumentation (3).

Further down the frequency scale I would like to refer to one of the few clear-cut results in some long-time efforts to make sense of the low-frequency oscillations (4). We used an array of instruments separated by various distances along shore, ranging from 1/2 mile to 20 miles. By a suitable analysis one can find the description of wave energy in  $\omega, k$ -space, where  $\omega$  is the temporal frequency and  $k$  is the spacial frequency. Suitable units are cycles per



hour and cycles per kilometer, respectively. It turns out that nearly all the energy is concentrated along two or three narrow bands, thus defining empirically the corresponding  $\omega(k)$  relations. We have computed the corresponding relations for trapped edge waves, using the observed off-shore topography, and these agree very satisfactorily with the empirical relations. There can be no doubt that most of the energy is in the form of low-order trapped edge waves. Approximately equal energy travels up and down the coast. At the frequency under consideration (cycles per hour), the effect of the Coriolis force leads to a slight splitting of the  $\omega(k)$  curves. As one approaches daily frequencies, the situation changes radically, and present remarks are not applicable.

Finally, I wish to comment on an unexplained observation in the analysis of tide records (Munk and Cartwright, 1966): the continuum spectrum peaks sharply in the vicinity of the strong tidal lines. The explanation is not yet known, but it appears that there are a number of processes that could be responsible for this peaking, and each of these might be important under some circumstances. One is the nonlinear interaction of the low-frequency sea-level oscillation with the tidal "carrier" frequencies, giving rise to modulated side bands. Another is the relatively small surface effects associated with internal tides, these being demonstrably incoherent with tide producing forces. And finally, there is the effect of multiple line interaction under highly nonlinear circumstances, such as a near bore situation. For this latter case, the peaked spectrum is discrete, not continuous.

This is an outline of an interesting set of problems. As the central frequencies of the ocean wave spectrum have become populated, I find myself moving monotonically towards lower frequencies.

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# CONTINENTAL SHELF WAVES

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## INTRODUCTION

When a static, normal stress distribution acts upon the sea surface, it reacts as an inverse barometer. In the literature such a response is often termed "isostatic"; however, in this paper we shall call such a response "barometric." In the past it has also been assumed that the sea surface responds barometrically when the distribution is time-dependent, such as in ordinary, moving weather systems, provided the fluctuations are of sufficiently low frequency (considerably less than the Coriolis parameter). Recently, however, Hamon (1-3) and Hamon and Hannan (4) have observed that the daily mean sea level fluctuations at the Australian coast are not barometric: At four stations on the east coast the behavior is appreciably less than barometric (that is, the magnitude of the "barometric factor," the ratio of daily mean sea level changes to atmospheric pressure changes, is appreciably less than the expected value of 1.0 cm/mb), and at two stations on the west coast the behavior is appreciably greater than barometric. An analysis of a simultaneous record of the daily mean sea level fluctuations at Lord Howe Island, on the other hand, indicates a barometric behavior.

Another unexpected phenomenon Hamon (2,3) observed is the existence of adjusted sea level phase lags between neighboring coastal stations whose distance separation is of the order 100 km. (The adjusted sea level is defined as the observed sea level with the atmospheric pressure eliminated according to the hydrostatic relation 1-cm decrease in sea level for 1-mb increase in pressure.) More specifically, for a series of six east coast stations between Eden (37°S) and Urangan (25°S), the lags between adjacent stations are related in a manner which suggests the presence of a northward-traveling wave with a speed of 400 cm/sec (3). The lagged correlations between adjusted sea levels at (west coast stations) Geraldton, Fremantle, and Bunbury, on the other hand, suggest the presence of a southward traveling wave. In this case the wave speed has not been determined accurately, but it is estimated to lie in the range 300 to 600 cm/sec (3).

Robinson (5) attributed the nonbarometric behavior at the east coast to a resonant response of the sea surface to a plane-wave pressure distribution which travels northward along a sloping shelf which sharply drops off to water at constant depth. His solution also predicts the existence of a northward-traveling shelf wave with a speed of 250 cm/sec, which is of the order of the

observed wave speed at the east coast. However, the solution does not yield any information about the variation of sea level behavior with latitude. Also, Robinson's mathematical representation of the forcing atmosphere pressure variations is questionable in view of the fact that at midlatitudes ordinary weather systems tend to progress eastward rather than northward.

To overcome these two shortcomings of the theory, we consider in this paper the following problem: the response of the sea surface to a plane-wave pressure distribution which progresses eastward across a circular continent with a sloping shelf which sharply drops off to water at constant depth.

The solution to this problem (given in the next section), when applied to the Australian stations where an anomalous sea level behavior has been observed, does in general predict the observed behavior, provided the forcing frequency lies in a small neighborhood of an eigenfrequency. However, the behavior is indeterminate when the forcing frequency is equal to an eigenfrequency, since both viscosity and nonlinear effects have been neglected. In view of this shortcoming, we also obtain the solution (given in the third section) for the same geometry and driving force as used in the next section but with bottom friction incorporated into the equations of motion. The solution given in these next two sections also predicts the existence of circularly traveling shelf waves which in the case of the southern hemisphere, move in the observed direction, namely, counterclockwise. However, the theoretical wave speeds are somewhat less than the observed, especially for the values of the parameters which apply to the east coast. In the fourth section the geometry of the model is modified so as to include a finite-slope continental slope region. The unforced solution for this geometry does give a west coast wave speed which lies well within the error bounds of the observed speed; however, there is still a significant discrepancy between the theoretical and observed speeds for the case of the east coast. In view of the fact that the theory does not take into account the intense current and associated stratification which is present off the east Australian coast (6,7), this discrepancy is, perhaps, not too surprising. In the last section the change in wave speeds due to deep-sea stratification (idealized by a two-layer model) and a uniform, upper-layer deep-sea current is determined.

We mention here that this paper is primarily intended to serve as an introduction to the theory of continental shelf waves; a more complete treatment will be published elsewhere.

## FORMULATION OF PROBLEM AND ITS SOLUTION

### Governing Equations

In view of the nature of the phenomena we are considering, we shall use the equations of linear shallow-water theory with the addition of the Coriolis parameter. Therefore, in terms of a cylindrical polar coordinate system  $(r, \psi, z)$  centered at the origin of the continent with  $z$  measured vertically upward from the undistorted sea surface, the adjusted sea level  $\bar{\eta}(r, \psi, t)$  and velocity components  $\bar{u}_r(r, \psi, t)$  and  $\bar{u}_\psi(r, \psi, t)$  satisfy the equations



# Continental Shelf Waves

$$\frac{\partial \bar{u}_r}{\partial t} - f \bar{u}_\psi + g \frac{\partial \bar{\eta}}{\partial r} = 0, \quad (1)$$

$$\frac{\partial \bar{u}_\psi}{\partial t} + f \bar{u}_r + \frac{g}{r} \frac{\partial \bar{\eta}}{\partial \psi} = 0, \quad (2)$$

$$\frac{\partial}{\partial r} (r h \bar{u}_r) + \frac{\partial}{\partial \psi} (h \bar{u}_\psi) + r \frac{\partial}{\partial t} (\bar{\eta} + \varphi) = 0. \quad (3)$$

Here  $f = 2\Omega \sin \theta_0$  ( $-0.59 \times 10^{-1} \text{ sec}^{-1}$  for Australia),  $g$  is the acceleration of gravity,  $z = -h(r, \psi)$  the equation of the sea bottom, and

$$\bar{\eta} = \bar{\zeta} - \varphi, \quad (4)$$

where  $z = \bar{\zeta}(r, \psi, t)$  is the sea-level distortion and  $\varphi(r, \psi, t)$  is the negative of the atmospheric pressure fluctuations measured in centimeters of water. From Eq. (4) we note that  $\bar{\eta}$  is a measure of the deviation of the sea-level behavior from exact barometric behavior. It is thus our purpose to derive an equation for  $\bar{\eta}$  and examine its magnitude relative to  $\varphi$ .

We assume that the driving force  $\varphi$  is of the form of an eastward moving plane wave, viz.,

$$\varphi = \varphi_0 \exp[i(kr \cos \psi - \omega t)], \quad (5)$$

where  $\varphi_0$  is a constant of order 10 cm. The choice of Eq. (5) as a model for the ordinary weather systems which progress across mid-Australia is motivated by two facts: (a) the weather systems essentially progress from west to east, and (b) for a specified season and at any given station the power spectrum of the daily mean atmospheric pressure fluctuations, which have an amplitude of about 10 cm, has a single, albeit broad, maximum (1). In particular, the spectrums for "winter" (April-September) and "summer" (October-March) are peaked at 9 days ( $\omega_w = 0.81 \times 10^{-5} \text{ sec}^{-1}$ ) and 5 days ( $\omega_s = 1.45 \times 10^{-5} \text{ sec}^{-1}$ ) respectively. As a typical example, the winter spectrum of the atmospheric pressure at Sydney is given in Fig. 1. In this figure the product of frequency and spectral energy density has been plotted against the logarithm of frequency.

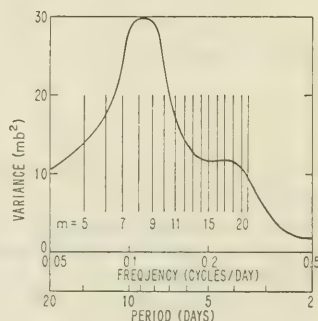


Fig. 1 - Winter spectrum of the daily mean atmospheric pressure at Sydney. The vertical lines represent the lowest mode eigenfrequencies  $\omega_{1,m}$  ( $m = 5, 6, \dots, 20$ ) for  $\ell = \ell_E$ .



We do not present here the response of  $\eta$  to a driving force  $\varphi$  which has a continuous frequency distribution of the form shown in Fig. 1; this case will be dealt with in a future paper.

In view of Eq. (5) we assume a simple harmonic time factor  $\exp(-i\omega t)$  for all the unknown quantities:

$$\left\{ \begin{array}{l} \bar{\eta} \\ \bar{u}_r \\ \bar{u}_\psi \end{array} \right\} = \left\{ \begin{array}{l} \eta(r, \psi) \\ u_r(r, \psi) \\ u_\psi(r, \psi) \end{array} \right\} \exp(-i\omega t). \quad (6)$$

Employing Eq. (6), Eqs. (1) and (2) yield, for  $\omega^2 \neq f^2$ ,

$$u_r = \frac{g}{f^2 - \omega^2} \left( i\omega \frac{\partial \eta}{\partial r} - \frac{f}{r} \frac{\partial \eta}{\partial \psi} \right), \quad (7)$$

$$u_\psi = \frac{S}{f^2 - \omega^2} \left( f \frac{\partial \eta}{\partial r} + \frac{i\omega}{r} \frac{\partial \eta}{\partial \psi} \right). \quad (8)$$

Hence Eq. (3) becomes, for  $h = h(r)$ ,

$$rh \nabla^2 \eta + h'(r\eta, r + if\omega^{-1}\eta, \psi) - r(f^2 - \omega^2)g^{-1}\eta = r(f^2 - \omega^2)g^{-1}\varphi_0 \exp(ikr \cos \psi). \quad (9)$$

For the circular geometry described in the Introduction,  $h(r)$  takes the form

$$h = \begin{cases} D, & r > R + \ell & (\text{deep sea region}) \\ d(r - R)/\ell, & R < r < R + \ell & (\text{shelf region}), \end{cases}$$

where  $D$  is the depth in the deep-sea region,  $d$  is the depth at the edge of the shelf,  $\ell$  is the shelf width, and  $R$  is the continental radius. For Australia,  $D = 5 \times 10^5$  cm,  $d = 2 \times 10^4$  cm,  $\ell_{\text{east coast}} = \ell_E = 5 \times 10^6$  cm,  $\ell_{\text{west coast}} = \ell_W = 7.5 \times 10^6$  cm, and  $R = 2.05 \times 10^8$  cm. Equation (9) is to be solved separately in the deep-sea and shelf regions, subject to the boundary conditions  $|\eta(R, \psi)| < M$  (a constant) and  $\eta(\infty, \psi) = 0$ . At the edge of the shelf we stipulate that  $\eta$  and  $u_r$  (radial transport component) be continuous.

In Eqs. (7), (8), and (9), we assume that  $\omega^2 \ll f^2$ . Hence in the deep sea region, Eq. (9) reduces to

$$(\nabla^2 - \beta^2)\eta = \beta^2\varphi_0 \exp(ikr \cos \psi), \quad (10)$$

where  $\beta^2 = f^2/gD$ . Upon examining the forcing term in Eq. (10), we note that for  $r \sim k^{-1}$  and  $\beta^2 \ll k^2$ ,  $\eta = O(\beta^2\varphi_0/k^2)$ , which implies that in the deep-sea region the amplitude of  $\eta$  is much smaller than that of  $\varphi$ . For the case of Australia, the sea-level behavior in the deep-sea region is barometric to within  $O(10^{-2})$  if we set  $k = \pi \times 10^{-8} \text{ cm}^{-1}$ . In the shelf region, Eq. (9) reduces to

$$\xi\eta, \xi\xi + \eta, \xi - i\gamma\eta, \psi = \epsilon a\varphi_0 \exp(ikR \cos \psi), \quad 0 < \xi < 1, \quad (11)$$

for  $\epsilon = \ell/R \ll 1$  and  $\gamma = -f\ell/\omega R$ ,  $\alpha = f^2 R \ell / g d$ ,  $\partial/\partial \xi$ , and  $\partial/\partial \psi$  each of order unity; the dimensionless variable  $\xi$  is defined by  $\xi = (r - R)/\ell$ . Hence  $\eta = O(\epsilon \alpha \varphi_0 / \gamma)$ , which implies that the shelf sea-level behavior is also essentially barometric unless resonance occurs. The unforced equations obtained from Eqs. (10) and (11) together with the above stated boundary and continuity conditions constitute an eigenvalue problem with eigenvalue  $\gamma$ . Since  $\gamma \propto \omega^{-1}$ , it follows that when the forcing frequency is in a small neighborhood of an eigenfrequency,  $\eta$  on the shelf is amplified, thereby leading to a nonbarometric behavior. For a nonresonant response however, the Australian shelf sea level behavior is also barometric to within  $O(10^{-2})$ .

### Unforced Solution: Continental Shelf Waves

The appropriate homogeneous solutions to Eqs. (10) and (11) which are single-valued in  $\psi$  are given by

$$\eta = \begin{cases} \sum_m A_m K_m [\beta(\ell\xi + R)] \exp(im\psi), & \xi > 1, \\ \sum_m B_m J_0(2\sqrt{\gamma m}\xi) \exp(im\psi), & 0 < \xi < 1, \end{cases} \quad (12)$$

where  $m = 0, \pm 1, \dots$ ,  $K_m$  is the  $m$ th order modified Bessel function of the second kind, and  $J_0$  the zeroth order Bessel function of the first kind. Application of the continuity conditions to Eq. (12) implies that  $A_0 = B_0 = 0$  and implies the eigenvalue equation

$$J_0(2\sqrt{\gamma m}) [1 - \Delta + \epsilon(1 + \epsilon) \beta R K'_m / \gamma m K_m] - [\Delta(1 + \epsilon) / \sqrt{\gamma m}] J'_0(2\sqrt{\gamma m}) = O(m \neq 0), \quad (13)$$

where  $K_m$  and  $K'_m$  are evaluated at  $\xi = 1$  and  $\Delta = d/D$ . For  $\beta R K'_m / K_m = O(1)$  and  $\epsilon/\Delta = O(1)$ , Eq. (13) implies that to be consistent with our earlier approximations we must take the eigenvalues to be given by the zeros of  $J_0$ . Let  $\lambda_{0j}$  ( $j = 1, 2, \dots$ ) denote the roots of the equation  $J_0(\lambda) = 0$ ; then the eigenfrequencies are given by

$$\omega_{j,m} = -4f\ell m / R \lambda_{0j}^2 \quad (m \neq 0). \quad (14)$$

Finally, in view of Eq. (14), the adjusted sea-level eigenfunctions can be written in the form

$$\bar{\eta}_{j,m}(r, \psi, t) = \begin{cases} O(\Delta), & \xi > 1, \\ J_0(2\sqrt{\gamma_{0j} m} \xi) \exp[i(m\psi - \omega_{j,m} t)] + O(\Delta), & 0 < \xi < 1, \end{cases} \quad (15)$$

where  $\gamma_{0j}$  is defined by  $2\sqrt{\gamma_{0j} m} = \lambda_{0j}$ .

The following properties of the unforced solution should be noted:

1. Each eigenfunction  $\bar{\eta}_{j,m}$  is in the form of a circularly traveling wave almost entirely confined to the shelf; hence we adopt the term "continental shelf

waves" for the  $\bar{\eta}_{j,m}$ 's. These waves correspond to Robinson's plane-traveling shelf waves (5).

2. The eigenfrequencies  $\omega_{j,m}$  form a doubly discrete set. In the Cartesian geometry the eigenfrequencies are given by  $\omega_{j,k}^2 = -4f^2 k^2 \lambda_{0j}^2$ , where  $k$  is a N-S wave number which lies in the range  $k^2 \lambda_{0j}^2 \ll 1$  (5). It is clear that in the case of the circular geometry, the effect of the periodicity requirement is to quantize the wave number  $m$ .

3. In both geometries the waves are nondispersive and the phase velocity (wave speed) of the  $j$ th mode is given by

$$c_j = -4f\lambda_{0j}^2. \quad (16)$$

From Eq. (16) we note that the wave speed is proportional to the shelf width and is independent of the depth and acceleration of gravity, which is quite unlike the case for gravity waves and edgewaves. In the circular geometry the waves travel counterclockwise in the southern hemisphere and clockwise in the northern hemisphere.

### Forced Solution

To solve the forced problem, we expand  $\bar{\eta}$  on the shelf in terms of the eigenfunctions given in Eq. (15) with  $\omega_{j,m}$  in the exponent replaced by  $\omega$ . The unknown coefficients are then determined from Eq. (11). The complete solution takes the form

$$\begin{aligned} \bar{\eta}(\xi, \psi, t) = & \epsilon \alpha \varphi_0 \sum_{j,m} \frac{i^m J_m(kR)}{\sqrt{\gamma_{0j} m} J_1(2\sqrt{\gamma_{0j} m})} \frac{m(\gamma - \gamma_{0j})}{m(\gamma - \gamma_{0j})} \\ & \times J_0(2\sqrt{\gamma_{0j} m} \xi) \exp[i(m\psi - \omega t)] \quad (m \neq 0). \end{aligned} \quad (17)$$

Equation (17) explicitly shows that unless the forcing frequency  $\omega$  lies within a "resonant" neighborhood of  $\omega_{j,m}$  defined by

$$|m(\gamma - \gamma_{0j})| < \epsilon \alpha \varphi_0 |J_m(kR)/\sqrt{\gamma_{0j} m} J_1(2\sqrt{\gamma_{0j} m})|, \quad (18)$$

the sea level behavior on the shelf is very nearly barometric. However, if  $\omega$  lies within a resonant neighborhood of  $\omega_{j,m}$ , then  $\bar{\eta}$  has an amplitude of  $O(10 \text{ cm})$ , the leading contribution being  $N_{j,m}$ , the  $j, m$  term of Eq. (17). In such a case the theoretical sea-level amplitude on the shelf is given by  $N_{j,m}$  with  $\omega$  in the neighborhood specified above. However, since the eigenfrequencies corresponding to  $j = 2, 3, \dots$  in the neighborhood of  $\omega_w$  or  $\omega_s$  have large values of  $m$  ( $m = O(40), O(10^2), \dots$ ) and since  $J_m(kR)$  rapidly decreases with increasing  $m$ , the resonant neighborhood (Eq. (18)) will be largest for the eigenfrequencies corresponding to  $j = 1$ . Hence we conclude that an anomalous sea-level behavior on the shelf is essentially determined by  $N_{1,m}$  with  $\omega$  within the resonant neighborhood of  $\omega_{1,m}$  given by Eq. (18) with  $j = 1$ . The following question



now arises. For a specified  $\ell$  ( $\ell_E$  or  $\ell_w$ ) and season (summer or winter), which  $\omega_{1,m}$ 's are most likely to be excited? (It should be made clear here that although the theoretical shelf wave  $N_{1,m}$  propagates around the whole continental shelf (of width  $\ell$ ) with a fixed frequency given by Eq. (14), we now wish to apply the theory locally to a small region of the coast in view of the two scales of  $\ell$  involved, viz.,  $\ell_E$  and  $\ell_w$ . Hence Eq. (14) implies that the eigenfrequencies corresponding to each region will differ.) Figure 1 shows the winter spectrum of the atmospheric pressure at Sydney as a function of frequency along with the eigenfrequencies  $\omega_{1,m}$  as given by Eq. (14) with  $\ell = \ell_E$ . For this case it is evident that  $\omega_{1,8}$  and  $\omega_{1,9}$  are most apt to be excited since only these eigenfrequencies lie at the peak of the spectrum. For  $\ell = \ell_w$ , a similar analysis reveals that  $\omega_{1,5}$  and  $\omega_{1,6}$  are most likely to be excited. During summer, the eigenfrequencies most likely to be excited are  $\omega_{1,14}$  and  $\omega_{1,15}$  for  $\ell = \ell_E$ , and  $\omega_{1,9}$  and  $\omega_{1,10}$  for  $\ell = \ell_w$ .

To illustrate the coastal sea-level behavior in the neighborhood of the above specified eigenfrequencies, we use the normalized amplitude

$$S_m(\psi, \omega; \ell) = |\varphi + N_{1,m}|_{\xi=0} / \varphi_0$$

$$= \left\{ 1 + A_m^2 + 2A_m \cos \left[ m \left( \psi + \frac{\pi}{2} \right) - kR \cos \psi \right] \right\}^{1/2}, \quad (19)$$

where

$$A_m = \epsilon \alpha J_m(kR) / 0.62(m\gamma - 1.44).$$

It is clear that when  $S_m > 1$  or  $< 1$  the behavior is greater or less than barometric. From Eq. (19) we note that when  $A_m = O(1)$ , the sea-level behavior does depend on  $\psi$  (as well as on  $m$ ,  $\omega$ , and  $\ell$ ). Hence it is not too surprising that the observed Australian sea level at the east coast is quite different from that at the west coast. In Figs. 2 through 5 the amplitude  $S_m$  is plotted as a function of frequency for those east and west coast stations where an anomalous sea-level behavior has been observed during winter and/or summer. These stations are listed in Table 1 along with their observed behavior ( $<$  and  $>$  represents respectively a less or greater than barometric behavior) and position  $\psi$ . (We have taken the origin of our coordinate system to lie at Alice Springs, which is the geographic center of Australia.)

The following were observed from Figs. 2 through 5:

1. For both  $\ell = \ell_E$  and  $\ell = \ell_w$ , the amplitude  $S_m \rightarrow +\infty$  as  $\omega \rightarrow \omega_{1,m^+}$  or  $\omega \rightarrow \omega_{1,m^-}$ . This phenomenon arises because both viscosity and nonlinear effects have been neglected. To determine the amplitude when the forcing frequency is equal to an eigenfrequency (within the framework of a quasi-steady model), at least one of these effects must be incorporated into equations. In the next section we obtain the solution for the amplitude which is damped by bottom friction.

2. Behavior at the east coast ( $\ell = \ell_E$ ). The behavior at all stations during winter is distinctly less than barometric only when the forcing frequency lies in the neighborhood  $5 \times 10^{-9} \lesssim \omega_{1,m} - \omega \lesssim 2 \times 10^{-8} \text{ sec}^{-1}$  ( $m = 8, 9$ ) (Fig. 2).



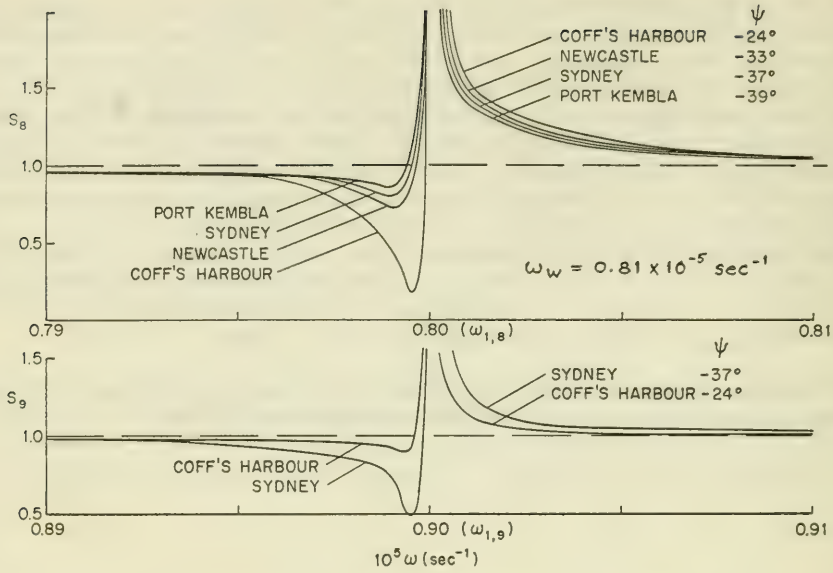


Fig. 2 - Undamped sea-level amplitude vs frequency in the neighborhood of  $\omega_{1,8}$  and  $\omega_{1,9}$  for  $\ell = \ell_E$ . The amplitudes  $S_9$  for Port Kembla and Newcastle have been omitted as they are nearly identical to those for Sydney.

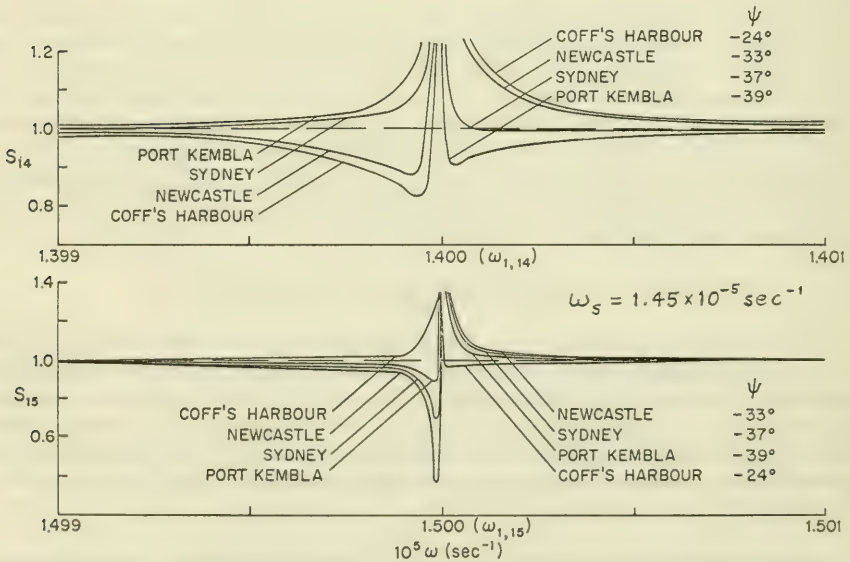


Fig. 3 - Undamped sea-level amplitude vs frequency in the neighborhood of  $\omega_{1,14}$  and  $\omega_{1,15}$  for  $\ell = \ell_E$

# Continental Shelf Waves

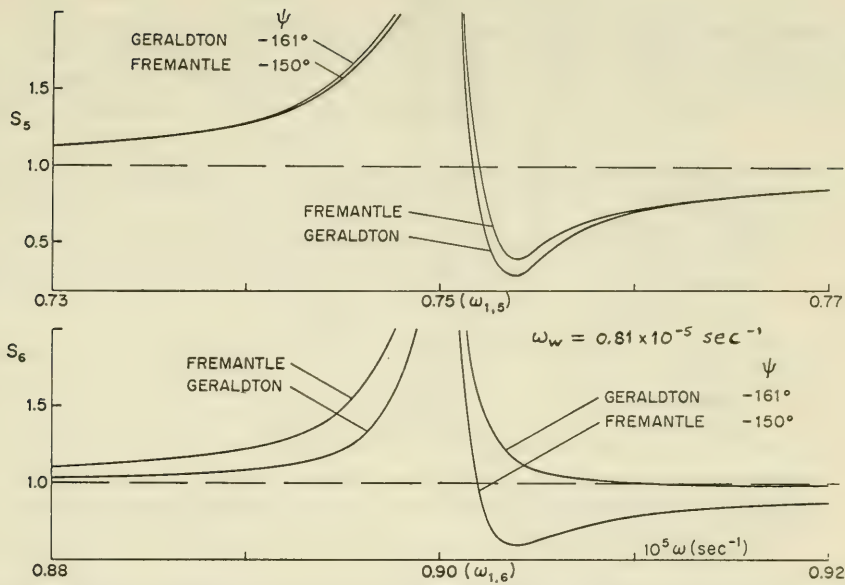


Fig. 4 - Undamped sea-level amplitude vs frequency in the neighborhood of  $\omega_{1,5}$  and  $\omega_{1,6}$  for  $l = l_w$

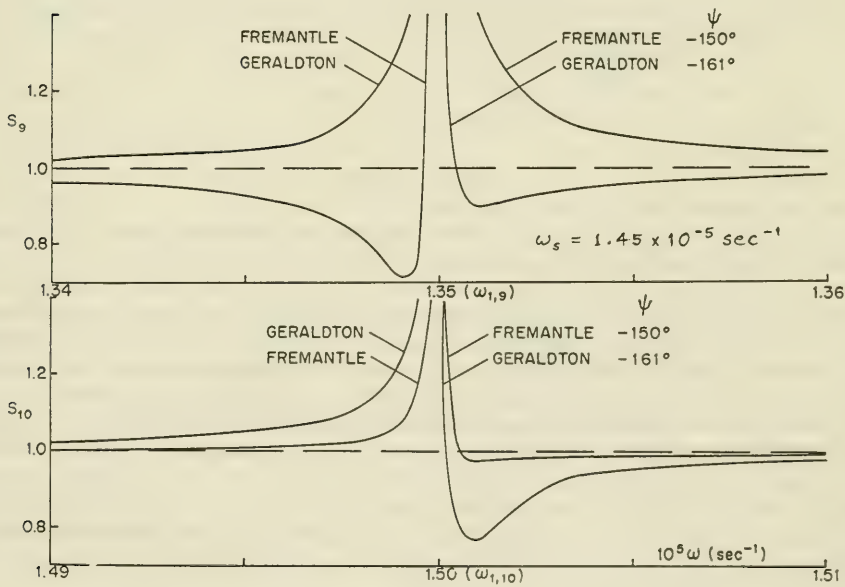


Fig. 5 - Undamped sea-level amplitude vs frequency in the neighborhood of  $\omega_{1,9}$  and  $\omega_{1,10}$  for  $l = l_w$

Table 1  
Australian Stations with Observed Anomalous Sea-Level Behavior

| Station           | Position $\psi$ | Observed Behavior |        |
|-------------------|-----------------|-------------------|--------|
|                   |                 | Winter            | Summer |
| <u>East Coast</u> |                 |                   |        |
| Coff's Harbour    | $-24^{\circ}$   | $<$               | $<$    |
| Newcastle         | $-33^{\circ}$   | $-*$              | $<$    |
| Sydney            | $-37^{\circ}$   | $<$               | $<$    |
| Port Kembla       | $-39^{\circ}$   | $-*$              | $<$    |
| <u>West Coast</u> |                 |                   |        |
| Fremantle         | $-150^{\circ}$  | $>$               | $>$    |
| Geraldton         | $-161^{\circ}$  | $-*$              | $>$    |

\*No winter sea level behaviors are available for Newcastle, Port Kembla, and Geraldton; hence in the winter column these stations are marked with dashes.

During summer, on the other hand, the behavior never becomes distinctly less than barometric at some stations (see the Sydney curve in the top part of Fig. 3 and the Coff's Harbour curve in lower part of Fig. 3). However, the behavior is less than barometric at Newcastle and Coff's Harbour; Port Kembla; and Newcastle, Sydney, and Port Kembla when  $\omega$  lies in the neighborhoods  $5 \times 10^{-10} \lesssim \omega_{1,14} - \omega \lesssim 3 \times 10^{-9} \text{ sec}^{-1}$ ;  $2 \times 10^{-10} \lesssim \omega - \omega_{1,14} \lesssim 2 \times 10^{-9} \text{ sec}^{-1}$ ; and  $10^{-10} \lesssim \omega_{1,15} - \omega \lesssim 10^{-9} \text{ sec}^{-1}$  respectively.

3. Behavior at the west coast ( $\ell = \ell_w$ ). Figure 4 is essentially the mirror image of Fig. 2. During winter the behavior at both stations is greater than barometric when  $2 \times 10^{-8} \lesssim \omega_{1,m} - \omega \lesssim 10^{-7} \text{ sec}^{-1}$  ( $m = 5, 6$ ) and when  $10^{-8} \lesssim \omega - \omega_{1,m} \lesssim 2 \times 10^{-8} \text{ sec}^{-1}$  ( $m = 5, 6$ ). During summer (Fig. 5) the behavior at both stations is greater than barometric when  $5 \times 10^{-9} \lesssim \omega_{1,10} - \omega \lesssim 10^{-8} \text{ sec}^{-1}$  and  $10^{-9} \lesssim \omega - \omega_{1,10} \lesssim 2 \times 10^{-9} \text{ sec}^{-1}$ . For  $\omega$  in the neighborhood of  $\omega_{1,9}$  however, no general statement can be made regarding the anomalous behavior.

In summary we note that during winter the behavior is less than barometric at all east coast stations and greater than barometric at all west coast stations (as observed) when the forcing frequency lies in a neighborhood of  $\omega_{1,m}$  of the form  $\delta\omega_2 < \omega_{1,m} - \omega < \delta\omega_1$ , where  $\delta\omega_1/\delta\omega_2 = O(10)$ . With the exception of a few cases, the same statement also applies to the behavior during summer. At the east coast  $(\delta\omega_1)_{\text{winter}} = O(10^{-8} \text{ sec}^{-1})$  and  $(\delta\omega_1)_{\text{summer}} = O(10^{-9} \text{ sec}^{-1})$ , whereas at the west coast  $(\delta\omega_1)_{\text{winter}} = O(10^{-7} \text{ sec}^{-1})$  and  $(\delta\omega_1)_{\text{summer}} = O(10^{-8} \text{ sec}^{-1})$ . This is to say, for a given season, the behavior at the east coast is more finely "tuned" than that at the west coast, and for a given coast the behavior is more finely tuned during winter than during summer. The reasons for these results are that  $\ell_E < \ell_w$  and  $\omega_w < \omega_s$  respectively.

We conclude this section by noting that the lowest mode ( $j = 1$ ) shelf wave travels counterclockwise with a speed given by  $c_1 = -f/1.44$  (see Eq. (16)). In

the neighborhood of the east coast ( $\ell = \ell_E$ ), therefore, this wave travels northward with a speed  $c_1 = 200$  cm/sec, which is half the observed speed of 400 cm/sec. In the neighborhood of the west coast ( $\ell = \ell_w$ ), the wave travels southward with a speed  $c_1 = 310$  cm/sec, which just lies within the error bounds for the observed speed, viz., 300 to 600 cm/sec.

## EFFECT OF BOTTOM FRICTION ON SEA LEVEL BEHAVIOR

If we vertically integrate the horizontal momentum equations, retain only linear terms, and neglect lateral eddy viscosity, we obtain

$$\frac{\partial U_r}{\partial t} - fU_\psi + gh \frac{\partial \bar{\eta}}{\partial r} = \frac{1}{\rho} (F_s - F_B), \quad (20)$$

$$\frac{\partial U_\psi}{\partial t} + fU_r + \frac{gh}{r} \frac{\partial \bar{\eta}}{\partial \psi} = \frac{1}{\rho} (G_s - G_B), \quad (21)$$

where

$F_s, G_s = r, \psi$  components of the wind stress on the sea surface,

$F_B, G_B = r, \psi$  components of the frictional stress of water on the ocean floor,

$U_r, U_\psi = r, \psi$  components of transport.

Upon combining (20) and (21) with the continuity equation, it can be shown that for typical values of the parameters, the forcing terms due to  $F_s$  and  $G_s$  are an order of magnitude smaller than the term due to the atmospheric pressure fluctuations; henceforth we take  $F_s = G_s = 0$ . We assume (8)

$$\left. \begin{aligned} F_B &= K_\rho U_r, \\ G_B &= K_\rho U_\psi, \end{aligned} \right\} \quad (22)$$

where  $K$  is a small positive constant of dimensions  $\text{sec}^{-1}$ . For our problem we take  $K$  to lie in the range  $10^{-7}$  to  $10^{-8} \text{ sec}^{-1}$ , which is obtained by equating the bottom stress as given by Eq. (22) to the well-known expression for the bottom-layer Ekman stress in which the vertical eddy viscosity (for the shelf region) is taken to lie in the range  $1$  to  $10^{-2} \text{ cm}^2/\text{sec}$  (see page 482 of Ref. 9). The same estimate for  $K$  is also obtained if we equate  $F_B, G_B$  (with velocities  $u_r = O(10^{-1} \text{ cm/sec})$ , and  $u_\psi = O(1 \text{ cm/sec})$ ) to the well-known bottom-frictional force expression  $k\rho u^2$ , where  $k$  is a drag coefficient usually taken to have the value  $2.5 \times 10^{-3}$  (see page 136 of Ref. 10). From Eq. (20), Eq. (21), and the continuity equation, we obtain, for  $h = h(r)$ , the following equation for  $\bar{\eta}$ :

$$hL \nabla^2 \bar{\eta} + h'(L \bar{\eta}_r + f r^{-1} \bar{\eta}_\psi) + (L^2 + f^2) g^{-1} \bar{\eta}_t = (L^2 + f^2) g^{-1} \varphi_t, \quad (23)$$

where  $L = \partial/\partial t + K$ .

In the unforced case we anticipate a solution of the form



$$\bar{\eta} = \eta(r, \psi) \exp[-(i\omega + K)t], \quad t > 0. \quad (24)$$

Substituting Eq. (24) into Eq. (23), and assuming that  $\omega^2 \ll f^2$ ,  $K \ll \omega$ ,  $\epsilon \ll 1$ , and  $f^2 \ell^2 \ll g d$ , and that  $\partial/\partial \xi$ ,  $\partial/\partial \psi$ , and  $\gamma$  are each of order unity, we find that the solution for  $\eta$  is the same as that given in the preceding section. However, in view of Eq. (24), the unforced shelf waves now decay with time, in contrast to the inviscid case. But since  $K \ll \omega$ , the effolding time of these waves is of the order of several years.

In the forced case, the solution is given by

$$\begin{aligned} \bar{\eta}(\xi, \psi, t) = \epsilon \alpha \varphi_0 \sum_{j,m} \frac{i^m J_m(kR)}{\sqrt{\gamma_{0j} m} J_1(2\sqrt{\gamma_{0j} m}) m (\gamma - \gamma_{0j} - i\gamma_{0j} K/\omega)} \\ \times J_0(2\sqrt{\gamma_{0j} m} \xi) \exp[i(m\psi - \omega t)] \quad (m \neq 0). \end{aligned} \quad (25)$$

Hence, to compare the theoretical sea-level behavior at the coast with the observed behavior, we consider

$$\bar{S}_m(\psi, \omega; \ell) = \left\{ 1 + \bar{A}_m^2 + 2\bar{A}_m \cos \left[ m \left( \psi + \frac{\pi}{2} \right) + \alpha_m - kR \cos \psi \right] \right\}^{1/2}, \quad (26)$$

where

$$\bar{A}_m = \epsilon \alpha J_m(kR) / 0.62 [(\gamma m - 1.44)^2 + (1.44 K/\omega)^2]^{1/2},$$

and

$$\tan \alpha_m = K/\omega (\gamma m / 1.44 - 1), \quad 0 < \alpha_m < \pi.$$

Equation (26) is the generalization of Eq. (19), so that  $\bar{S}_m$  has the same physical interpretation as  $S_m$ . But Eq. (26) differs from Eq. (19) in one important way: when  $\omega = \omega_{1,m}$  (a resonant response), Eq. (26) implies that  $\bar{S}_m$  is finite because of the nonzero term  $(1.44 K/\omega)^2$  in the denominator of  $\bar{A}_m$ . Further, for the values of  $K$  quoted above,  $\bar{S}_m = O(1)$  when  $\omega = \omega_{1,m}$ , so that the hypernonbarometric behavior which occurs in the inviscid model when  $\omega = \omega_{1,m}$  is now eliminated. Finally, we note that for  $K \ll \omega$  the behavior as given by  $\bar{S}_m$  will be essentially the same as that given by  $S_m$  except for  $\omega$  in the small neighborhood  $|\omega - \omega_{1,m}| < \delta\omega$  where  $\delta\omega = O(10^{-8} \text{ to } 10^{-9} \text{ sec}^{-1})$ .

In Table 2 the value of  $\bar{S}_m$  at each station listed in Table 1 is given when  $\omega = \omega_{1,m}$ ,  $\omega = \omega_{1,m}^-$  (that is, when  $\omega_{1,m} - \omega = \delta\omega > 0$ ), and  $\omega = \omega_{1,m}^+$ . We interpret these data as follows: when  $(\bar{S}_m)_{\text{maximum}} - 1 > \text{or} < 1 - (\bar{S}_m)_{\text{minimum}}$  at a given station and in the neighborhood of a particular  $\omega_{1,m}$ , we regard the behavior in this case as distinctly greater or less than barometric respectively. With this interpretation we note that at the east coast the behavior is distinctly less than barometric (the observed behavior) in 50 percent of the cases. In particular, at Newcastle, Sydney, and Port Kembla during winter, the behavior in the neighborhood of  $\omega_{1,8}$  is distinctly less than barometric for an "off-resonant" response to the left. (That is,  $\bar{S}_8$  at these stations has a minimum when

Table 2  
Damped Sea-Level Amplitude  $\bar{S}_m$  for  $\omega = \omega_{1,m-}, \omega_{1,m},$  and  $\omega_{1,m+}$   
(The values of  $k$  used in Eq. (26) have been chosen so that  
 $\bar{S}_m/b = O(1)$ , where  $b$  is the observed barometric factor)

| Station        | $\bar{S}_m$ During Winter (with $k = 10^{-7} \text{ sec}^{-1}$ ) |                |                 |                 |                |                 | $\bar{S}_m$ During Summer (with $k = 10^{-8} \text{ sec}^{-1}$ ) |                 |                  |                  |                 |                  |
|----------------|------------------------------------------------------------------|----------------|-----------------|-----------------|----------------|-----------------|------------------------------------------------------------------|-----------------|------------------|------------------|-----------------|------------------|
|                | $\omega_{1,8-}$                                                  | $\omega_{1,8}$ | $\omega_{1,8+}$ | $\omega_{1,9-}$ | $\omega_{1,9}$ | $\omega_{1,9+}$ | $\omega_{1,14-}$                                                 | $\omega_{1,14}$ | $\omega_{1,14+}$ | $\omega_{1,15-}$ | $\omega_{1,15}$ | $\omega_{1,15+}$ |
| Coff's Harbour | 0.8                                                              | 1.2            | 1.4             | 1.1             | 1.3            | 1.2             | 1.0                                                              | 1.2             | 1.1              | 1.0              | 1.1             | 1.0              |
| Newcastle      | 0.6                                                              | 0.8            | 1.1             | 0.9             | 1.1            | 1.2             | 0.9                                                              | 0.8             | 1.0              | 0.9              | 1.0             | 1.1              |
| Sydney         | 0.5                                                              | 0.6            | 1.0             | 0.9             | 1.0            | 1.1             | 0.9                                                              | 0.8             | 0.9              | 0.9              | 0.9             | 1.0              |
| Port Kembla    | 0.5                                                              | 0.6            | 1.0             | 0.9             | 1.0            | 1.1             | 1.0                                                              | 0.8             | 0.9              | 1.0              | 0.9             | 1.0              |
| Fremantle      | $\omega_{1,5-}$                                                  | $\omega_{1,5}$ | $\omega_{1,5+}$ | $\omega_{1,6-}$ | $\omega_{1,6}$ | $\omega_{1,6+}$ | $\omega_{1,9-}$                                                  | $\omega_{1,9}$  | $\omega_{1,9+}$  | $\omega_{1,10-}$ | $\omega_{1,10}$ | $\omega_{1,10+}$ |
|                | 1.1                                                              | 0.9            | 0.8             | 1.2             | 1.2            | 1.0             | 0.6                                                              | 0.7             | 1.1              | 0.9              | 0.8             | 0.9              |
| Geraldton      | 1.2                                                              | 1.1            | 0.9             | 1.2             | 1.3            | 1.1             | 1.0                                                              | 0.8             | 0.7              | 1.1              | 1.0             | 0.9              |

$\omega = \omega_{1,8-}$ .) This is not unexpected if we note that the very low minimums of  $S_8$  are well to the left of  $\omega_{1,8}$  in Fig. 2. During summer at these stations, however, the behavior in the neighborhood of  $\omega_{1,14}$  is distinctly less than barometric for a "true" resonant response. (That is,  $S_{14}$  at these stations has a minimum when  $\omega = \omega_{1,14}$ .) Again, this is not too surprising in view of the closeness of the  $S_{14}$  minimums to  $\omega_{1,14}$  (Fig. 3). At the west coast, the behavior is distinctly greater than barometric (the observed behavior) only during winter, and in two of the three cases in which this is true the response is also off-resonant to the left, which is also not surprising in view of Fig. 4. While this theory does predict, in many of the cases, the observed behavior when the forcing frequency is in a small neighborhood of an eigenfrequency, the discrepancies which do exist perhaps suggest that bottom friction is not the only amplitude-limiting mechanism involved. In fact it can be shown that the terms due to bottom friction (for the values of  $k$  given above) are the same order of magnitude as the nonlinear, inertial acceleration terms which have been hitherto neglected. It would thus be of value to determine the theoretical sea-level response which is damped by both bottom friction and inertial acceleration terms.

## INFLUENCE OF CONTINENTAL SLOPE

If a finite-slope continental slope region is included in the theory of edge-waves, which are similar in form to continental shelf waves, the wave speed is increased by about 2 percent (11). Since the wavelengths and periods associated with continental shelf waves are much greater, it is quite plausible that a similar geometrical modification would give rise to an even larger increase in the shelf wave speed. In fact it has been suggested (3) that the discrepancy between the observed and theoretical wave speeds may be due to the oversimplified bottom topography used by Robinson (5). Of course the same criticism also applies to the circular model used above.

To answer this question we assume that between the deep-sea and shelf regions there exists a uniformly sloping continental slope of width  $\ell'$  with depth  $d$  at  $r = R + \ell$  and depth  $D$  at  $r = R + \ell + \ell'$ . Then it can be shown that to within our order of approximations the eigenfrequencies are given by

$$2\sqrt{\gamma m} = \lambda_{0j} + q_j \sqrt{\Delta'} \quad (m \neq 0; j = 1, 2, \dots), \quad (27)$$

where  $q_j$  is a constant of order unity and  $\Delta' = d\ell'/(D-d)\ell$ . For the case of Australia we take  $\ell'_E = 7.5 \times 10^6$  cm and  $\ell'_W = 12.5 \times 10^6$  cm. For these values of  $\ell'$  we find that  $\Delta'_E = 6 \times 10^{-2}$  and  $\Delta'_W = 7 \times 10^{-2}$ . Equation (27) therefore implies that a continental slope region produces a significant shift in the eigenfrequencies. For  $j = 1$  the eigenvalue equation implies that  $q_1 = -0.9$  and  $-1.1$  for the east and west coasts respectively.

From Eq. (27) the new wave speeds are now readily determined; the results are given in Table 3, along with the theoretical values computed for the previous model and the observed values. From Table 3 we first note that with a continental slope the wave speed at either coast is increased by about 30 percent, which, as conjectured earlier, is much greater than the increase for edgewaves. The theoretical wave speed for the west coast now lies well within the error bounds

Table 3  
Observed and Theoretical Lowest-Mode Wave Speeds

|            | Wave Speed (cm/sec)             |                              |          |
|------------|---------------------------------|------------------------------|----------|
|            | Theoretical                     |                              | Observed |
|            | Without<br>Continental<br>Slope | With<br>Continental<br>Slope |          |
| East coast | 200                             | 270                          | 400      |
| West coast | 310                             | 410                          | 300-600  |

of the observed speed. For the east coast, however, a considerable discrepancy still exists between the observed and theoretical wave speeds.

#### EFFECT OF DEEP-SEA CURRENT AND STRATIFICATION

The two outstanding phenomena which have not been considered in this theory are the East Australian Current (6) and the associated deep-sea stratification (7). To gain a rudimentary understanding of the interaction processes which exist between the shelf waves and these two phenomena, we obtain the unforced solution for the simple two-layer model depicted in Fig. 6. If we assume the basic current to be geostrophic and baroclinic, then the interface must slope downward as we proceed away from the coast. However, in the equations for the wave heights  $\eta$  and  $\zeta'$ , we make the simplifying approximation that the depths of the upper and lower layers are constant.

In terms of the notation shown in Fig. 5 we assume the following forms for  $(\eta, u, v)$  and  $(\zeta', u', v')$ :

$$\begin{Bmatrix} \eta \\ u \\ v \\ \zeta' \\ u' \\ v' \end{Bmatrix} = \begin{Bmatrix} N(x) \\ X(x) \\ Y(x) \\ Z'(x) \\ X'(x) \\ Y'(x) \end{Bmatrix} \exp[i(ky - \omega t)] . \quad (28)$$

Then, for  $\omega^2 \ll f^2$ ,  $\omega_0^2 \ll f^2$  ( $\omega_0 = \omega + kV_0$ ),  $k^2 \ell^2 \ll 1$ ,  $f^2 \ell^2 \ll g d$ ,  $\mu^2 = (\rho' - \rho)/\rho \ll 1$ ,  $\alpha^2 = f^2/k^2 g d_0 = O(1)$ , and  $\alpha'^2 = f^2/k^2 g' D_0 = O(1)$  ( $g' = g(1 - \rho/\rho')$ ), the conservation equations give



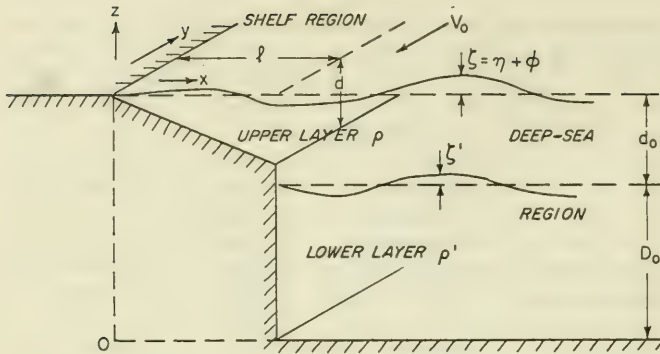


Fig. 6 - Two-layer model. The uniform, basic flow  $V_0$  is confined to the deep-sea region and to the upper layer. For the east Australian coast region, typical values of the parameters shown are  $V_0 = 100$  cm/sec,  $d_0 = 2.5 \times 10^4$  cm,  $D_0 = 5 \times 10^5$  cm,  $\ell = 5 \times 10^6$  cm,  $d = 2 \times 10^4$  cm,  $\rho = 1.025$  g/cm<sup>3</sup>, and  $\rho' - \rho = 2.5 \times 10^{-3}$  g/cm<sup>3</sup>.

$$N = \begin{cases} A_s J_0(2\sqrt{\nu}\xi), & 0 < x < \ell, \\ A_1 \exp[-(\alpha/\mu)(\chi - k\ell)] + A_2 \exp[-(\chi - k\ell)], & x > \ell, \end{cases} \quad (29)$$

$$Z' = -(\mu^2) A_1 \exp[-(\alpha/\mu)(\chi - k\ell)] + A_2 \exp[-(\chi - k\ell)], \quad x > \ell, \quad (30)$$

where  $\xi = x/\ell$ ,  $\chi = kx$ , and  $\nu = -fk\ell/\omega$ . In obtaining Eqs. (29) and (30), we have imposed the boundary conditions  $|N(0)| < M$ ,  $N(\infty) = 0$ , and  $|Z'(\infty)| < M$ . Application of the boundary condition  $X'(\ell) = 0$  yields  $A_2 = 0$ ; application of the continuity conditions to the upper layer solutions implies the eigenvalue equation

$$J_0(2\sqrt{\nu}) [\beta^* \nu (1 - \Delta_0) - (1 + \nu R_0) \sqrt{\Delta_0}] - \beta^* \Delta_0 \sqrt{\nu} J'_0(2\sqrt{\nu}) = 0, \quad (31)$$

where  $\beta^* = \mu(gd)^{1/2} (\ell|f|)^{-1}$ ,  $\Delta_0 = d/d_0$ , and  $R_0 = -V_0/f\ell$ . We first note that in the limit  $\beta^* \rightarrow 0$  ( $\rho' - \rho \rightarrow 0$ ) and  $R_0 \rightarrow 0$  ( $V_0 \rightarrow 0$ ), Eq. (31) implies that the eigenvalues are given by the zeros of  $J_0$ , in agreement with the unstratified case with no basic deep-sea current. However, with stratification and a deep-sea current, Eq. (31) implies that the eigenvalues are quite different, since  $\beta^*$ ,  $\Delta_0$ , and  $R_0$  are each of order unity. The physical reason for this change in the eigenvalues is that the motion of the surface wave  $\eta$  in the deep-sea region is now coupled to that of the interfacial wave  $\zeta'$  through the conservation equations, so that new modes of oscillation result. Finally, from Eqs. (28) and (31) we observe that the waves are still nondispersive and progress northward at the east coast of a continent which lies in the southern hemisphere.

## Continental Shelf Waves

In Fig. 7 the lowest-mode wave speed  $c_1$  ( $= -4f\ell/\nu_1$ , where  $\nu_1$  is the lowest eigenvalue as determined by Eq. (31)) is plotted as a function of  $V_0$  for the case of the east Australian coast in the neighborhood of Sydney to Coff's Harbour ( $f = -0.73 \times 10^{-4} \text{ sec}^{-1}$ ). From Fig. 7 we observe that for  $V_0$  in the range 50 to 100 cm/sec (typical East Australian Current speeds) and for all values of  $\Delta_0$  considered, there is now excellent agreement between the theoretical and observed wave speeds. Even for  $V_0 = 0$ , it is interesting to note,  $c_1$  is only about 10 percent greater than the observed speed; this is to say, the main shortcoming of the earlier models was the absence of deep-sea stratification. However, the wave speeds are relatively insensitive to changes in stratification (Fig. 7). Finally, we conclude this paper by noting that in each  $\Delta_0$  curve,  $c_1$  decreases (essentially linearly) with increasing  $V_0$ . This result suggests that we can theoretically determine the speed of an essentially surface deep-sea current which flows alongside a continental shelf by analyzing the sea level at the coast. However, since the wave speed in any small coastal section (distance of the order

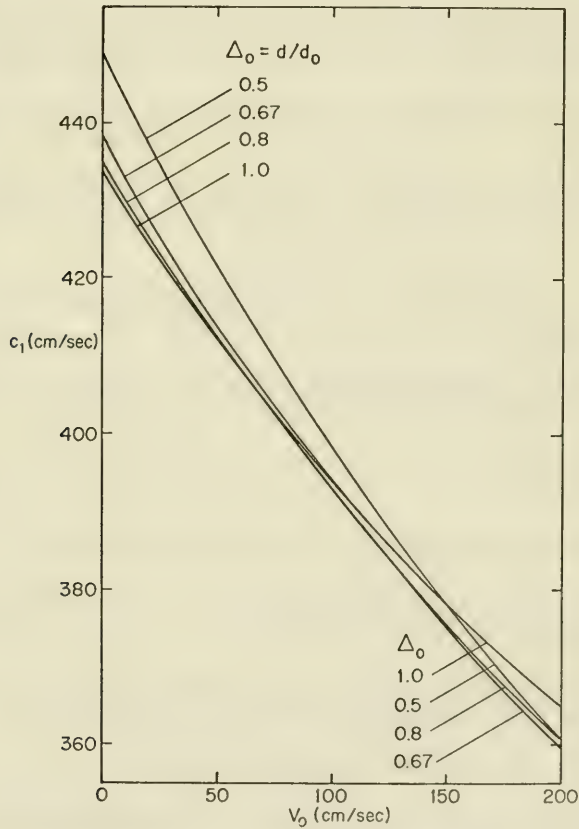


Fig. 7 - East Australian coast shelf wave speed of lowest eigenvalue vs deep-sea current speed for varied deep-sea stratification

100 km) is determined by computing the cross spectrum of daily mean sea level data from two stations bounding this section, the theory can at most predict seasonal current speeds, or the seasonal difference in the current speed between two widely separated regions (distance separation of the order 500 km).

## ACKNOWLEDGMENTS

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## DISCUSSION

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With reference to the history of the ratio of sea level changes to atmospheric changes varying from a value of 1, Ted Saur reported on this in AGU several years ago and the Coast and Geodetic Survey published on this over 100 years ago, using a four-year record of tide and pressure at Boston.

In considering additional forces related to the problem, I note that you set the wind stress on the surface as equal to zero. I suggest that although it is an order of magnitude less than the term due to atmospheric pressure fluctuation and thus not large enough to account for the variations noted, it still can be a significant disturbing parameter.

\* \* \*





# NONLINEAR PROCESSES IN LONG-CRESTED WAVE TRAINS

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## ABSTRACT

This paper is a commentary on some recent findings in water-wave theory which have been reported in previous papers by the present authors, and which relate closely to ideas developed recently by Whitham and by Lighthill. The principal aim here is to clarify the practical significance of the new results, with regard both to ocean waves and to waves formed in model basins.

The original object of the work to be reviewed was to explain an impressive, clearly nonlinear phenomenon that was discovered during experiments with a wave tank. To demonstrate this phenomenon, a long-crested progressive wave train of fairly large amplitude needs to be generated on deep water by a wavemaker whose reciprocating motion is slightly modulated either in amplitude or frequency. It may then be observed that the modulations imposed on the basic wave train undergo enormous amplifications with distance from the origin, the final result being a complete disintegration of the wave train and redistribution of its energy over a broad spectrum.

A theory agreeing very satisfactorily with the experimental observations has been worked out for the initial phase of these events, during which the modulations can be represented adequately by two infinitesimal wave components at side-band frequencies separated by a fraction  $\delta$  from the fundamental frequency. The theory shows in effect that, on sufficiently deep water, the waves of permanent form described by the well-known Stokes approximation are unstable to perturbations of this type over a certain range of  $\delta$ . Nonlinear interaction between the side-band components and the second harmonic of the basic wave train is crucial to the mechanism of instability. If  $\delta$  is not too large, a balance can gradually be attained between the nonlinear effect of the phase velocities of the side-band components and the effect of frequency dispersion on them, so that thereafter the interaction is resonant; each side-band component is then subject to a synchronous forcing action proportional to the amplitude of the other, and thus the two grow mutually at an exponential rate.

In an application of the theory to wave trains on water of arbitrary depth  $h$ , it has been found that instability is possible only if  $kh > 1.363$ , where  $k$  is the fundamental wavenumber ( $= 2\pi/\text{wavelength}$ ). This critical value of  $kh$  has also been discovered by Whitham. By means of a variational principle using an averaged form of Lagrangian function, he

obtained equations governing extremely gradual -- but not necessarily small -- variations in wave properties, and showed that these equations are of elliptic type if  $kh > 1.363$  (a result also obtained by Lighthill in the case of deep-water waves,  $kh \rightarrow \infty$ ) but are hyperbolic if  $kh < 1.363$ . On the whole there is a nice accord among the results of Whitham's, Lighthill's, and the present authors' theories, and the available experimental results fit well into the established picture.

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The complete text of this paper was not available for publication.

# COMPUTER-BASED PROCEDURES FOR PREPARING GLOBAL WAVE FORECASTS AND WIND FIELD ANALYSES CAPABLE OF USING WAVE DATA OBTAINED BY A SPACECRAFT

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and

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## NEED

Techniques are needed for improved descriptions of waves on the surface of the sea. Improvements are needed both in accuracy and in detail of the descriptions. With such descriptions, ships can be routed across the ocean more economically, and search and rescue operations at sea can be carried out more efficiently and safely. Warnings of conditions which could produce hazardous events such as the recent damage to the Michelangelo could be given. An adequate knowledge of the prevailing wave conditions on the ocean would permit the design of better merchant ships and military vessels. A concept under investigation by the Maritime Administration involving surface effect ships would require very accurate wave forecasts. Coastal recreation areas need wave forecasts.

Moreover, the state of the sea is an integrated indication of the weather over the oceans and should play an important role in numerical weather prediction calculations. Many air-sea interaction processes are a function of the state of the sea such as the amount of salt nuclei that enters the atmosphere. A better understanding of the wave conditions over the oceans will be needed in order to improve upon numerical weather prediction techniques.

## PRESENT DATA

Present knowledge of conditions at sea is gained from ship reports, from ships with shipborne wave recorders, and from special scientific installations



and investigations such as those made at Argus Island and from FLIP. The estimates of waves reported by ships are of dubious quality. The World Meteorological Organization's instructions for observers are so nebulous in their statement of what is to be done that the reported wave heights can, and do, differ from the significant wave heights that are observed instrumentally by a factor of two, either too high or too low. A great contribution to the problem of observing and forecasting ocean waves at sea could be made by improving the instructions for the visual observation of waves.

The winds reported by ships are also of dubious quality, except for those cases where the ships are equipped with calibrated anemometers and the height of the anemometer is known. Finally, the ships report a quantity called the air-sea temperature difference which ought to be quite useful in determining the turbulent structure of the wind over the water. However, the method of recording the sea temperature leaves much to be desired, and there is considerable debate as to the accuracy and usefulness of the values currently reported.

In addition to the conventional transient merchant ship reports and the visual estimates by weather ships of wave heights, waves have been recorded since about 1952 by the British weather ships using the Tucker shipborne wave recorder. These data are perhaps the most valuable, because they cover the widest range of conditions and because they are in a part of the ocean where a great deal of variation in wave height occurs from day to day.

The Argus Island installation is also an extremely valuable one for recording waves. It has been used as a primary standard of calibration for many other kinds of wave recorders which are under development at the Naval Oceanographic Office, and the amount of data, though less in quantity to that available from the shipborne wave recorder, is sufficient to permit useful interpretation and analysis.

FLIP has been used for one month at a station halfway between Hawaii and Alaska by Snodgrass et al. (1) to measure waves in the North Pacific. It is evident, however, that new techniques for observing waves on a much broader scale over the oceans are needed and that the installation of shipborne wave recorders, or other wave recording techniques, on many ships at sea would provide much valuable data for problems connected with both the analysis of the wind field and the waves over the ocean. In a later section of this paper, the potentials of a special kind of radar on a spacecraft will be discussed in the context of a procedure for enhancing the quality of data obtained over the oceans.

## THE NORTH ATLANTIC PROJECT

The work to be described here on the problem of developing computer-based features for wave hindcasts and forecasts reached a certain stage in 1964 upon the completion of a climatology of wave conditions for the North Atlantic ocean. Those results have been described in numerous reports to the Navy and to the scientific community. References of interest are those by Baer (2), Thomasell and Welsh (3), Moskowitz, Pierson, and Mehr (4), Moskowitz (5), Pierson and Moskowitz (6), Pierson (7), and Pierson and Tick (8). This entire project has

been summarized by Bunting (9). These results have been used by Wachnik and Zarnick (10) in the study of aircraft carrier motions. This file of climatological wave conditions for the year 1959 is proving to be of great value in a number of other investigations that have not yet reached the stage of publication.

The numerical technique that was developed at that time involved the analysis of the surface wind field over the Atlantic Ocean on the basis of reports from weather ships at sea and from conventional ships, the description of the wave spectra at 15 frequencies and 12 directions at each of 519 grid points, the propagation of the various spectral components at their group velocities, certain dissipation mechanisms essentially based on gross Austausch turbulence, and a wave growth theory obtained from a study of the data tabulated in the reports by Moskowitz, Pierson, and Mehr.

## AVAILABLE THEORIES

At the time that the work described above was completed in 1964, theories for the growth of waves due to the wind were available but they had not been checked, and indications were that they did not account for the observed growth of waves at sea. Since that time, some field work has led us to the conclusion that perhaps these theories can be used in a modified way to describe wave growth, and the purpose of this section is to review the available theoretical results on wind-generated gravity waves in the context of our computer-based procedures.

The shape of the spectrum of a fully developed wind sea was studied in the work completed prior to December 1964. This analysis by Moskowitz (5), Pierson and Moskowitz (6), and Pierson (7) seems to have resolved a number of the important discrepancies in this part of wind-generated-wave theory. The major source of the discrepancies seems to have been a combination of calibration effects, which explain the very low values obtained by Darbyshire (11), and differences in the elevation above the surface of the sea at which the wind was measured. A correction for these differences places many of the theories within reasonable range of each other.

The propagation of waves is also quite well understood, based on the work of Barber and Ursell (12), Pierson (13), and the more recent study of Snodgrass et al. (1) in the Pacific. There can be no doubt that each spectral component in a wind sea propagates at group velocity in an appropriate direction, according to where it lies in the directional spectrum, and that such components can be tracked on a great-circle route for a distance comparable to half the circumference of the earth.

The problem of the growth of the spectrum of a wind-generated sea has received considerable attention. Recent work, starting with that of Eckart (14), and continuing through the work of Miles (15) and Phillips (16) laid the theoretical foundations for this problem. It has been found that the growth that can be calculated from these theories is apparently too low to explain the observed growth of the spectral components in a wind sea. The fascinating field study by Snyder and Cox (17), however, does show that the growth of a particular spectral



component has the behavior predicted by these theories in that the growth is at first quite slow and linear and then growing to the full amplitude rather quickly once a certain minimum is reached. The evaluation by Snyder and Cox, for example, led to the conclusion that the Miles term was about eight times smaller than is actually needed to explain the growth of the spectral component they investigated.

Inoue (18) in some recent work has attempted to use the functional forms postulated in the theories of Miles and Phillips with constants picked in such a way that the growth will match the growth which was observed in the data provided by the British shipborne wave recorder. The first effort employed a flat growth independent of frequency for the Phillips term, and the functional form of the Miles term increased by a factor of about eight to model the Miles term. This effort indicated a growth of significant wave height versus fetch (and duration) very much like that obtained by Sverdrup and Munk many years ago. Current investigation involves the study of a more realistic Phillips term in which the spectral growth will be a function of both frequency and wind speed. The spectral growth from a flat, calm ocean with a 40-knot wind according to Inoue (18) is shown in Fig. 1.

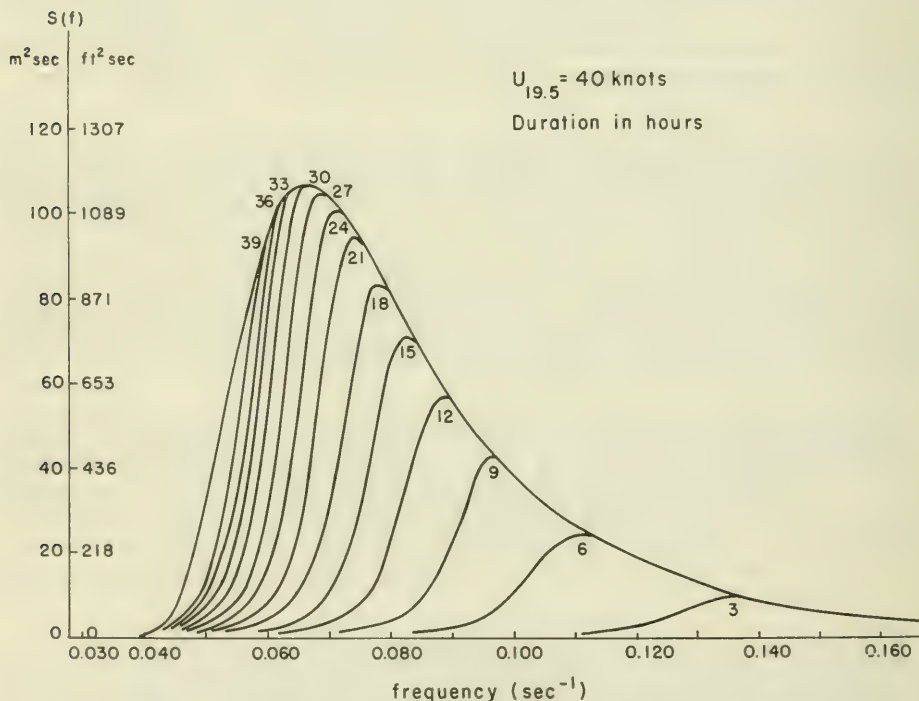


Fig. 1 - Spectral growth from a flat calm ocean with a wind of 40 knots 19.5 meters above the sea surface

The problem of the growth of the spectrum of a wind-generated sea is a complicated one in nature for a number of reasons. The winds over the ocean, either blowing offshore, or on the open ocean, are not constant in velocity for the length of time required in most of the theories for full development. They shift in direction and speed over time periods of the order of 6 to 12 hours. They can build from 20 to 25 knots to 40 to 60 knots in a half a day, remain at 60 knots over an area 400 or 500 nautical miles on a side for 12 to 18 hours, and then drop in speed to 15 knots in another 18 hours. The models used by oceanographers in attempting to develop wave forecasting methods in the past, that assumed a constant wind blowing for a certain duration, do not fit conditions at sea very well. It has been our experience in the study of the data that are available that the spectra very nearly track the fully developed spectra as the winds build with time, except for very high winds. The area wherein the difficulties lie is in the forward face of the spectra and in getting the contribution at these frequencies correct as a storm builds up and dies down. The high-frequency portion of the spectrum is more or less always saturated at or near the value predicted from the spectral form given by Pierson and Moskowitz (6).

An important aspect of the problem of wave generation is the widely different rates of growth reported by various investigators. The point raised by Phillips concerning the presence of a background spectrum in the ocean and its absence in the lee of land seems to explain these different growth rates. A very low background can reduce the time to reach full development to less than half that required if the waves grow from a flat calm. The spectral growth for a 40-knot wind with a background present is shown in Fig. 2 according to Inoue.

There remains the problem of how waves are dissipated at sea. If there was no dissipation mechanism, the spectrum of the waves on the open ocean would soon become isotropic, with spectral components traveling in all directions at nearly every point. This is observed not to be the case, and there is some process by which waves traveling against strong wind-generated seas are rapidly attenuated. The study of Snodgrass et al. (1) on the propagation of waves from the South Pacific to the North Pacific found that the greatest amount of attenuation was right in, or near, the generating area and that the swell could then travel through the subtropical highs and the trade wind regions without very much additional loss.

It seems to us that the primary reason for the dissipation of the waves is the turbulence generated by the breaking waves in a wind-generated sea. An attempt has been made to model this effect by taking the simple theories proposed by Lamb for viscous attenuation and using an Austausch coefficient in its place that enhances this effect. This method very strongly attenuates waves that travel against the wind. This dissipation, as discussed in a later section, seems to be an important contribution to the decrease of waves after a storm passes and the wind shifts. There is certainly plenty of room for improvement in this particular aspect of our computer-based procedures. However, it does appear to yield consistent results when tested against available sequences of wave spectra computed from wave records obtained from both Argus Island and from the British weather ships. As an example, for the square North Atlantic grid system, Figs. 3, 4, and 5 show the results near the Weather Reporter for the original



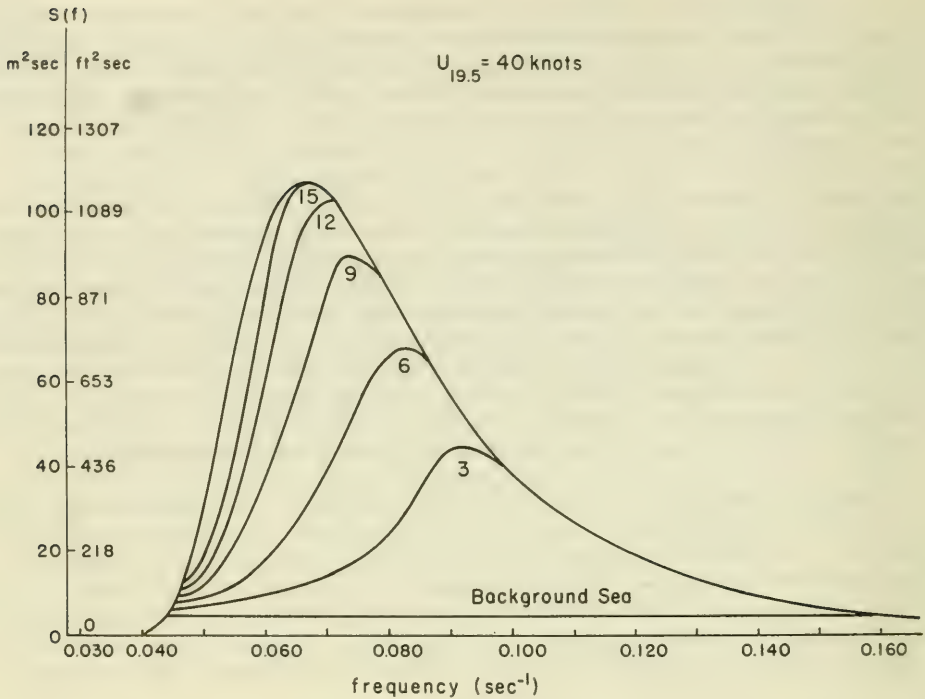


Fig. 2 - Spectral growth for 40 knots with a white noise background present

growth scheme, the original scheme without dissipation, and the growth scheme of Inoue (18), to be described later.

There is another aspect of wave theory that might possibly play a role in the modification of the spectrum of a wind sea. It consists of the theory given by Phillips (19) on third-order interactions for intersecting trains of gravity waves and the extension by Hasselman (20). Pierson (21) has prepared a technical report that attempts to show that these theories are not fully substantiated. It is the opinion of Pierson that these third-order nonlinear interactions are not a feature of nature and that they are not necessary to forecast the correct shape of wave spectra on the open sea. We understand that certain computer techniques are being employed in an attempt to find out whether or not changes in the spectrum due to such interactions can explain what is observed. A number of laboratory experiments are also being planned that may answer some crucial questions. However, it is important to point out that energy cannot be destroyed in such a process if it exists; it must reappear at some other frequency. Unless after it reappears at some other frequency, it is then removed by the process of wave breaking, there is the tendency to make the directional spectrum nearly isotropic, and this is not an observed condition on the open ocean.

# Global Wave Forecasts Using Spacecraft Data

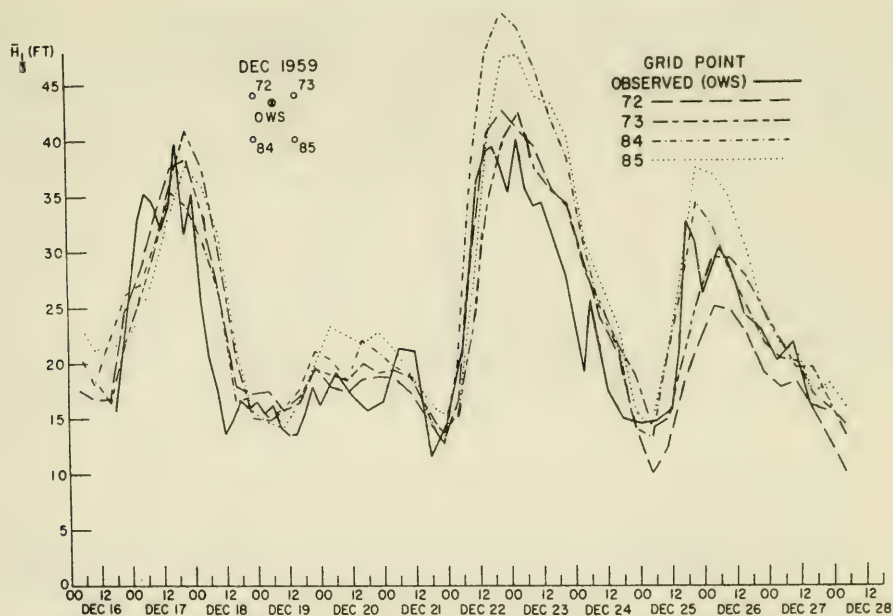


Fig. 3 - Observed and computed significant wave heights based on a spectral growth theory developed in 1964

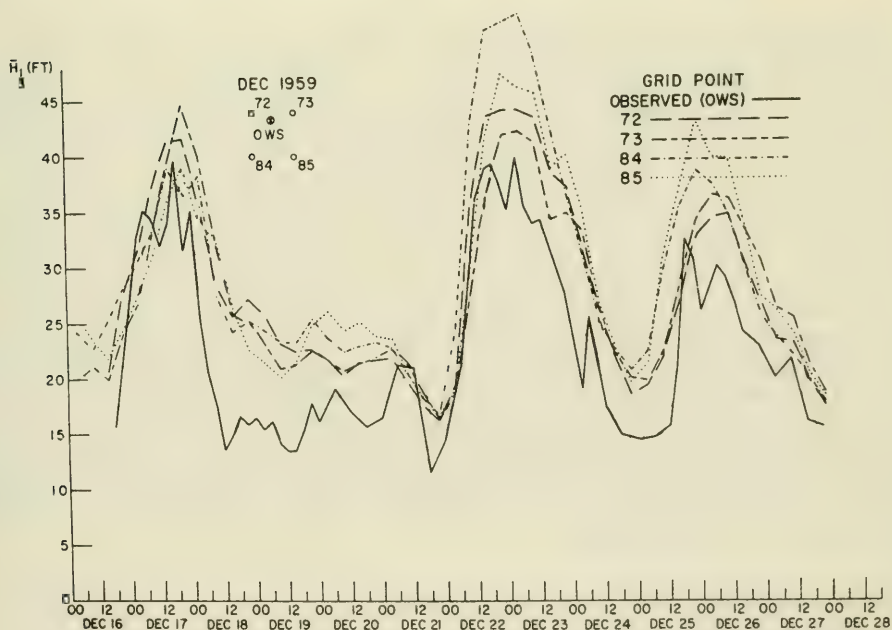


Fig. 4 - Observed and computed significant wave heights based on the same growth and propagation as in Fig. 3 but without any dissipation

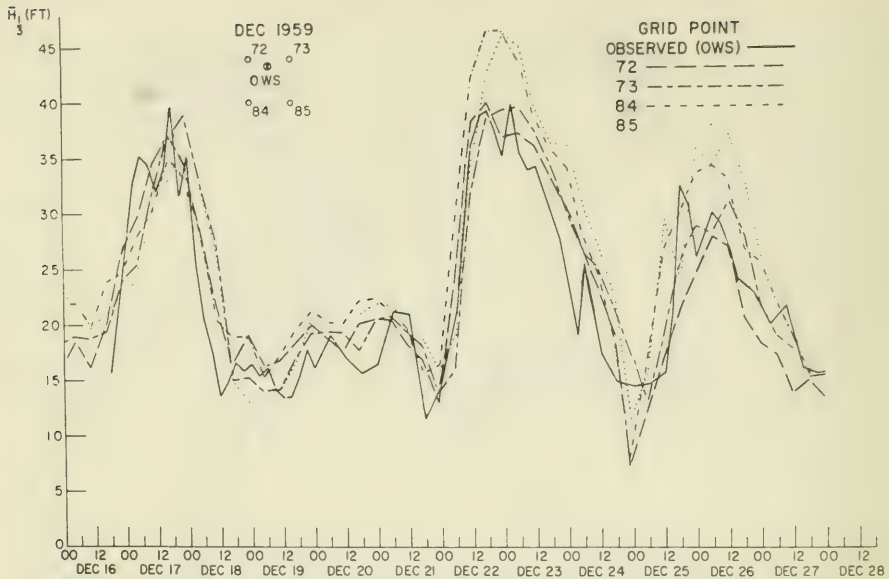


Fig. 5 - Observed and computed significant wave heights based on the same wind fields, dissipation and propagation as Fig. 3 but with Eqs. (7), (8), and (9) describing the wave growth

Finally, the character of the turbulence in the wind over the ocean should play an important part in determining the rate of growth of the waves. A great deal of new theoretical work on turbulence is becoming available in the literature. These results suggest that the drag coefficient, and the roughness length, depend on atmospheric stability and on the height of the sea present in that particular area. It would be highly desirable to develop a better way to modify the growth of a wave spectrum on the basis of how high the waves are at the time of the observation and on the basis of the stability of the air over the water. Miles' theory states that the rate of growth of a particular spectral component is determined by the ratio of the curvature of the wind profile to the slope of the wind profiled at that elevation above the sea surface where the phase speed of the waves equals the wind speed. Since the phase speeds observed in fully developed seas are considerably in excess of the winds measured 19.5 meters above the surface, this means that knowledge of the wind profile over the water must extend both theoretically and observationally to elevations well over 70 meters. Also, since the wind profile changes its character as a function of stability, the rate of growth of different spectral components will be quite different depending upon how the particular ratio mentioned above compares to the ratio that would exist under neutral conditions.

#### A COMPUTER-BASED GLOBAL PROCEDURE

A procedure for hindcasting and forecasting wave spectra over the world ocean is under development. It will include many of the results described above

plus a wave propagation procedure that has new and desirable features for large oceans. It will include procedures for specifying the wind field over the oceans, and the turbulent structure of the winds. The wave spectra will be computed at each grid point by three steps for each 2-hour time step. The three steps are called: growth, dissipation, and propagation. The procedures under development for specifying the wind field, and for the three steps of the wave spectra computation will now be described.

## Wind Fields

Presently, the wind field is described by a direction and speed at each grid point. The speed is that which would supposedly be observed at an elevation of 19.5 meters over the sea surface under conditions of neutral stability, under the assumption that the drag coefficient given by Sheppard (22) as a function of the wind at 10 meters is correct.

The surface pressure field is used to obtain a first estimate of the wind field by means of a regression equation given by Thomasell and Welsh (3). The weather ships are used next to correct the winds at the points on the grid closest to the ships. The anemometer heights of the weather ships are used to compute the wind at 19.5 meters under the above assumptions.

All other ships with anemometers are then used. If a grid point is corrected, subsequent ship reports are not allowed to change the wind at that grid point.

Finally, the transient merchant ships are used as if the reported wind had been observed at 19.5 meters above the sea surface. These winds were also first treated as if they had been observed at 10 meters above the sea surface and increased according to the logarithmic wind profile to 19.5 meters above the sea surface. This led to waves that were much too high.

The wind fields are obtained on a JNWP grid (23) and interpolated to the wave forecasting grid.

In the system under development, the number of points on the JNWP grid will be quadrupled by halving the distance between grid points. This will provide better resolution in areas of strong winds and strong wind shear. Also more ships will be used in correcting the wind field from the regression analysis. Recent Russian papers (and unpublished work by P. S. DeLeonibus) suggest that air-sea temperature differences and wave height can be used to modify the wind profile over the waves in a way that will influence the curvature and slope of the profile so as to permit a modification of the growth rate for different spectral components.

## Growth

The growth portion of the computer program is based on the concept that if the wind blows uniformly in direction and speed over a large enough area for a



long enough time, and if swell does not enter the area from some other region, the spectrum in the area will settle down eventually to the fully developed spectrum represented by

$$S(\omega, \theta, v) = \frac{\alpha g^2}{\omega^5} e^{-\beta(\omega_0/\omega)^4} [F(\omega, \theta^*, v)] , \quad (1)$$

where  $\alpha = 8.1 \times 10^{-3}$ ,  $\beta = 0.74$ , and  $\omega_0 = g/v_{19.5}$  and where

$$F(\omega, \theta^*, v) = \frac{1}{\pi} \left[ 1 + \left( 0.50 + 0.82 e^{-(\omega v_{19.5}/g)^4/2} \right) \cos 2\theta^* + 0.32 e^{-(\omega v_{19.5}/g)^4/2} \cos 4\theta^* \right] \quad (2)$$

for  $-\pi/2 < \theta^* < \pi/2$  and is zero otherwise ( $\theta^*$  being measured with reference to the direction of the wind).

The fully developed sea develops as the waves originally present propagate out of the area, which may take a long time, and as the effect of fetch and duration become established.

This fully developed sea, designated by  $S_\omega(f_i, \theta_j)$  is approximated over a band of frequencies and directions  $f_i$ ,  $i = 1, 2, \dots, 15$ , and  $\theta_j$ ,  $j = 0, 1, \dots, 23$ , by

$$S_\omega(f_i, \theta_j) = \int_{f_i - \Delta f_i}^{f_i + \Delta f_i} \int_{\theta_j - \Delta \theta_j}^{\theta_j + \Delta \theta_j} S(2\pi f, \theta, v_{19.5}) d\theta \, 2\pi df , \quad (3)$$

where Eqs. (1) and (2) define  $S(2\pi f, \theta, v_{19.5})$ , the  $\theta_i$  are  $0, 15^\circ, 30^\circ$ , etc., and all  $\Delta\theta$ 's are  $7.5^\circ$ .

In general, the spectrum at a particular grid point differs from the 360 numbers defined by Eq. (3) (half of them are zero from Eq. (2)) because the sea is not necessarily a fully developed sea, because components outside the range of  $\pm 90^\circ$  to the wind may be present, and because higher components from other winds can remain. The spectrum actually present is also described by 360 numbers according to the formula

$$S(f_i, \theta_j) = \int_{f_i - \Delta f_i}^{f_i + \Delta f_i} \int_{\theta_j - \Delta \theta_j}^{\theta_j + \Delta \theta_j} S(2\pi f, \theta) d\theta \, 2\pi df . \quad (4)$$

After the procedure has gone through a number of time steps, there will in general be 360 values of  $S(f_i, \theta_j)$  at a particular grid point. For the particular time step a wind speed  $v$  and a wind direction  $\theta_w$  will be given for that grid point.

For appropriately defined angles certain  $\theta_j$  will be within  $\pm 90^\circ$  to the wind direction:

$$\theta_w - 90^\circ < \theta_j < \theta_w + 90^\circ \quad (5)$$

The sum for  $\theta_j$  satisfying the above inequality is defined as

$$S_w(f_i) = \sum_j S(f_i, \theta_j) \quad (6)$$

and represents the sum of all contributions to the spectrum for a given  $f_i$  traveling within  $90^\circ$  to the wind. If  $S_w(f_i) > S_\infty(f_i)$ , there is no change in the values of  $S(f_i, \theta_j)$ , but if  $S_w(f_i) < S_\infty(f_i)$ , there will be growth during the next time step. The growth of each frequency band  $f_i$  is then given by the solution to the ordinary differential equation

$$\begin{aligned} \frac{d}{dt} S_w(f_i) = & A(v) \left\{ 1 - \left[ \frac{S_w(f_i)}{S_\infty(f_i)} \right]^2 \right\}^{1/2} + B(v, f_i) S_w(f_i) \\ & - \left[ \frac{S_w(f_i)}{S_\infty(f_i)} \right]^2 \left( A(v) \left\{ 1 - \left[ \frac{S_w(f_i)}{S_\infty(f_i)} \right]^2 \right\}^{1/2} + B(v, f_i) S_w(f_i) \right) \end{aligned} \quad (7)$$

where

$$A(v) = 1.4 \times 10^{-8} v^3 \quad (8)$$

and

$$B(v, f) = 6.27 (v/c)^2 e^{-0.017(c/v)^4} 2f\Delta f \quad (9)$$

for those frequency bands where  $2\Delta f = 1/180$  and where Eq. (7) has units of feet<sup>2</sup>/hour and  $v$  and  $c$  are in knots (18). Equation (7) can be integrated for the initial value  $S_w(f_i) = 0$  to yield  $S_w(f_i; t_e)$  so that one can solve for the effective time  $t_e$  corresponding to  $S_w(f_i)$  at the start of the time step. From this,  $S_w(f_i; t_e + \Delta t)$  for the end of the 2-hour time step can be found. The incremental growth

$$S_w(f_i) = S_w(f_i; t_e + \Delta t) - S_w(f_i; t_e)$$

is then spread out in direction according to Eq. (2) with  $\theta_w$  corresponding to  $\theta^* = 0$ . The increments are not allowed to exceed the values given by Eq. (3) for each  $\theta_j$ , although the shape of  $F(\omega, \theta, v)$  is not always preserved.

It would be possible to treat each  $S_w(f_i, \theta_j)$  separately as an initial-value problem. As the wind direction changed, such a procedure would tend to produce spectra in excess of Eq. (1), whereas this procedure avoids this, at least in part.

A modification based on the work of Snyder and Cox (17) and Barnett (24) for the term  $A(v)$  in Eq. (8) is under investigation. The Phillips term would then be a function of both wind speed and frequency (but still not direction according to

$$A(v, f) = \int_{-\pi/2}^{\pi/2} \frac{A^* v^{3.25} (2\pi f)^{0.25} d\theta}{\left[ (0.52)^2 + \left( \frac{v}{c} \sin \theta \right)^2 \right] \left[ \frac{(0.33)^2 \sqrt{k}}{\sqrt{v/c}} + \left( \frac{v}{c} \cos \theta - 1 \right)^2 \right]}, \quad (10)$$

where  $v$  is in meters/second,  $k$  is in meters<sup>-1</sup>, and  $c = g/2\pi f$ . The growth of the wave spectrum for a 40-knot wind with Eq. (10) replacing Eq. (8) is shown in Fig. 6.

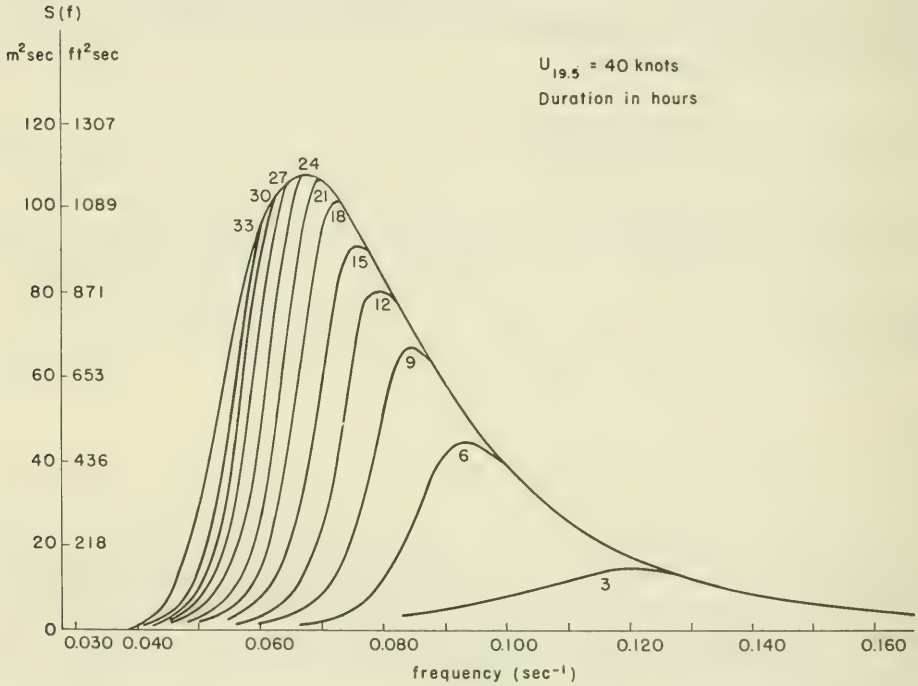


Fig. 6 - The growth of a spectrum for a 40-knot wind with Eq. (10) replacing Eq. (8) in Eq. (7)

There are several possibilities for further modification of Eq. (9). The low frequencies do not seem to grow fast enough, so that a change in the two constants is still possible. Also, from the analysis of the stability and roughness length, it may be possible to represent the wind profile at each grid point by an equation of the form

$$v = v [z, T_a - T_w, S(\omega, \theta)], \quad (11)$$

and for the condition  $v = c = g/2\pi f_i$ , the height  $z_{fi}$  can be found:

$$z_{fi} = z [g/2\pi f_i, T_a - T_w, S(2\pi f_i)]. \quad (12)$$

Since a neutral profile assuming Sheppard's drag coefficient  $c_{10}$  was used to construct the original wind field as in

$$v_N(z) = v(z; c_{10}), \quad (13)$$

the ratio

$$R = \frac{d^2v(z)/dz^2}{dv(z)/dz} \frac{dv_N(z)/dz}{d^2v_N(z)/dz^2} \quad (14)$$

can be evaluated at  $z_{fi}$  to get  $RB(v, f)$ , which will replace  $B(v, f)$  in Eq. (7).

Stable conditions will probably make  $R$  near one for high frequencies and less than one for low frequencies because the greatest variation in the wind profile is near the surface. Unstable conditions will probably make  $R$  greater than one over the whole frequency range.

The effect will be to produce more rapidly developing seas in unstable air, but the fully developed spectrum will not be changed. Whether or not this corresponds to all aspects of reality is open to question. Present data do not seem to show any dependence of the fully developed spectrum on air sea temperature difference (5).

The papers at the International Symposium on Turbulence in the Boundary Layer held at Kyoto, Japan, in September 1966 (IUGG-IUTAM) provided many new concepts for the integration of the pressure field with the boundary layer. In particular, we hope to be able to use the ideas of Dr. Swinbank, Dr. Priestly, and Prof. Lettau in the further development of the preceding concepts.

## Dissipation

The spectral components traveling against the wind will not receive energy from the wind and must undergo some form of dissipation. This dissipation in our model is governed by the height of the wind sea. With  $S_w = \sum_i S_w(f_i)$ ,

$$\frac{S_D(f_i, \theta_j)}{S_0(f_i, \theta_j)} = \left( e^{-78 \sqrt{S_w} f_i^4} \right)^N, \quad (15)$$

where  $S_0(f_i, \theta_j)$  is a component traveling against the wind,  $S_D(f_i, \theta_j)$  is the component after 2 hours of dissipation, and  $N = 4$  if  $\theta_j$  is opposite to the wind,  $N = 3.5$  if  $\theta_j$  is  $\pm 15^\circ$  to the wind,  $N = 3$  if  $\theta_j$  is  $\pm 30^\circ$  to the wind, and so on.

For a spectral component of  $0.10 \text{ sec}^{-1}$  traveling opposite to a fully developed 30-knot-wind sea,

$$\frac{S_D(0.10, \pi)}{S_0(0.10, \pi)} = e^{-0.1248}$$



for a 2-hour time step, so that in 16 hours the component will decay to  $1/e$  of its original value. A frequency of  $0.20 \text{ sec}^{-1}$  is attenuated 16 times faster, so that the local chop traveling against the wind rapidly disappears, whereas a frequency of 0.05 takes more than 10 days to be attenuated to  $1/e$  of its original value.

The effect of this attenuation is shown in Figs. 3 and 4, where it has not been included in one computation and it has been included in a second computation. The effects of propagation account for a major part of the variation at the weather ship, but attenuation appears to be needed to explain the full sequence of observations. This attenuation is only effective if the spectral component is traveling against the wind. In Figures 3, 4, and 5, for example, the wind shifted after the peak of the December 17 cyclone, and this kind of dissipation provides for closer agreement between theory and observation. For the peak of December 22-23, the wind speed decreased, but the winds did not change direction, so that the dissipation term did not operate. Although, without dissipation, the peak waves are too high, the decrease after the peak is at the same rate and is due to the effects of propagation in all of the figures. The correct determination of the effect of dissipation will be extremely important, because Eq. (7) indicates a quite different behavior, depending on the initial value of  $S_w(f_i)$ . It would be highly desirable to replace the above dissipation computation by some measure of breaking waves and white caps in the wind sea, but this problem is not yet clearly enough understood for this to be possible.

### Propagation on a Global Scale

To forecast both local sea and swell, the winds from essentially the entire ocean area are needed at some time. Thus if the spectral component is traveling at 33 knots, the effects of the winds 12 hours ago and 400 miles away from the forecast point will be needed. Similarly, day-old information is needed from 800 nautical miles away and 2-day-old information from 1600 nautical miles. Other components travel at a different velocity, so that data over most of the ocean is needed to make a single forecast. By judicious scanning of all of these wind fields, one might conceivably drop many of the areas and reduce the problem to manageable size for a single forecast point. However, for many requirements, such as ship routing, wave forecasts are needed for such large areas that it is worthwhile to consider forecasting for at least a large region of the ocean and perhaps for all the oceans.

One of the problems is how to represent the wave conditions over such a large area. From the previous discussion it is obvious that the spectrum can have an infinite number of different shapes, so that no simple system can represent the condition at a particular location. We have therefore searched for a simple function which could represent the spectrum and be remembered in the computer by a reasonable number of parameters. Many such functions are available, but they are not amenable to the components being propagated independently. We have therefore been forced to use individual discrete frequency and directional components to represent the spectrum. After rather extensive testing, we found that 15 frequencies and 12 directions did well for the North Atlantic. An increase to 24 directions is planned in the present development

and should be fairly optimum for the time and space scales and quality of data expected. The very high frequencies are always treated as fully developed and can therefore be handled quite simply.

The geographical variations are represented by remembering the spectrum at discrete grid points. As can be seen from Figs. 3, 4, and 5 the variation of significant wave height at adjacent grid points 120 nautical miles apart on a square grid is quite large. Thus a tighter grid spacing has been chosen. If the number of grid points is doubled on the square grid, each grid point represents about 6400 square nautical miles. This is near the limit of the useful precision from the presently available wind data. A similar, though larger, grid-point system is used in most numerical dynamical weather prediction work.

The time scale chosen must be compatible with the space scale such that the fastest wave components do not skip grid points during a particular time step. Because of the general smoothness of both the wind and wave fields for the longer faster components, the time constraint can be relaxed slightly without serious error. The time interval was therefore chosen as 2 hours to be an integral divisor of the normal 6-hourly meteorological charts.

Since the fields of wave spectra adequately represent an integration of all that has happened previously, the initial value problem given by a single initial wave field and the wind field for the next time interval are adequate to compute the change for the next time step. Thus, only the latest information is used for successive time steps.

Since wave energy propagates along great circle routes, it would be desirable for the lines connecting the grid points to be great circles. Then the energy could travel directly from one grid point to another grid point. It would also be desirable for the spacing of these points to be constant for the entire ocean and for as many directions as possible.

To have a conformal, equal-area, equidistant representation of the earth with great circles shown as simple curves (say, circular arcs or straight lines) we would be forced to work on a sphere. However, a usable network of nearly equally-spaced grid points which are defined by great circles cannot be developed

On plane projections, both geometrical and conventional (i.e., mathematical), the straight-line great circle property is the rarest one of the desired properties. It occurs only in the gnomonic projection, a geometric projection of no more than a hemisphere onto a tangent plane by means of diverging rays originating at the center of the sphere. On this projection, great circles are represented by straight lines. The primary disadvantage is that of great areal distortion. Many other projections exist which preserve either angles and small shapes, or areas, or distances; but on all of them the problems of accounting for the three properties not preserved is very complicated. On the other hand, the gnomonic projection possesses radial symmetry and the various distortions can be expressed as relatively simple functions of the distance between the point of tangency of the projection and the point in question. The equations, describing this projection and its superimposed grid, can be found in Baer and Adamo (25) or developed relatively easily if one is familiar with spherical geometry.



In order that each grid point will represent approximately the same area, a multiple-projection system was developed. This will keep the maximum distortion on each segment of the projection relatively small. But, the problem of bridging between the segments of the multiple-projection is thus introduced. The system, named an "icosahedral-gnomonic projection," is a separate gnomonic projection, "is a separate gnomonic projection on each face of an icosahedron circumscribed about the earth. A similar projection was described by Fisher (26) but emphasized land and was therefore not suitable for the present use.

Various properties of one of these triangular gnomonic projections are listed in Table 1. Figure 7 illustrates the areal distortion relative to 1.00 of the location of mean areal distortion for an equilateral triangular projection. Note that the distortion ranges only from -19% to +62% at the extremes and this positive extreme applies only to the apexes.

Table 1  
Properties of the Gnomonic Projection on a Face of an Icosahedron

|                                                            |                                  |
|------------------------------------------------------------|----------------------------------|
| Area, 1/20th of the earth's surface                        | $7.45 \times 10^6$ sq naut mi    |
| On the Earth:*                                             |                                  |
| Length of side                                             | $63^\circ 26.1'$ or 3806 naut mi |
| Length of altitude                                         | $58^\circ 16.9'$ or 3497 naut mi |
| Vertex angle (spherical)                                   | $72^\circ$                       |
| On the Plane:                                              |                                  |
| Length of side                                             | 4552 naut mi                     |
| Length of altitude                                         | 3942 naut mi                     |
| Vertex angle                                               | $60^\circ$                       |
| Distortion relative to 1.00 at the tangent point:          |                                  |
| Radial - maximum (at vertexes)                             | 1.58                             |
| - mean                                                     | 1.13                             |
| Transverse - maximum                                       | 1.26                             |
| - mean                                                     | 1.06                             |
| Areal - maximum                                            | 1.99                             |
| - mean                                                     | 1.20                             |
| Distortion relative to 1.00 at the midpoints of the sides: |                                  |
| Radial - maximum (at vertexes)                             | 1.37                             |
| - minimum (at tangent point)                               | 0.87                             |
| Transverse - maximum                                       | 1.18                             |
| - minimum                                                  | 0.93                             |
| Areal - maximum                                            | 1.62                             |
| - minimum                                                  | 0.81                             |

\*Assumes a spherical earth of radius 3440.19 nautical miles (from Ref. 25).

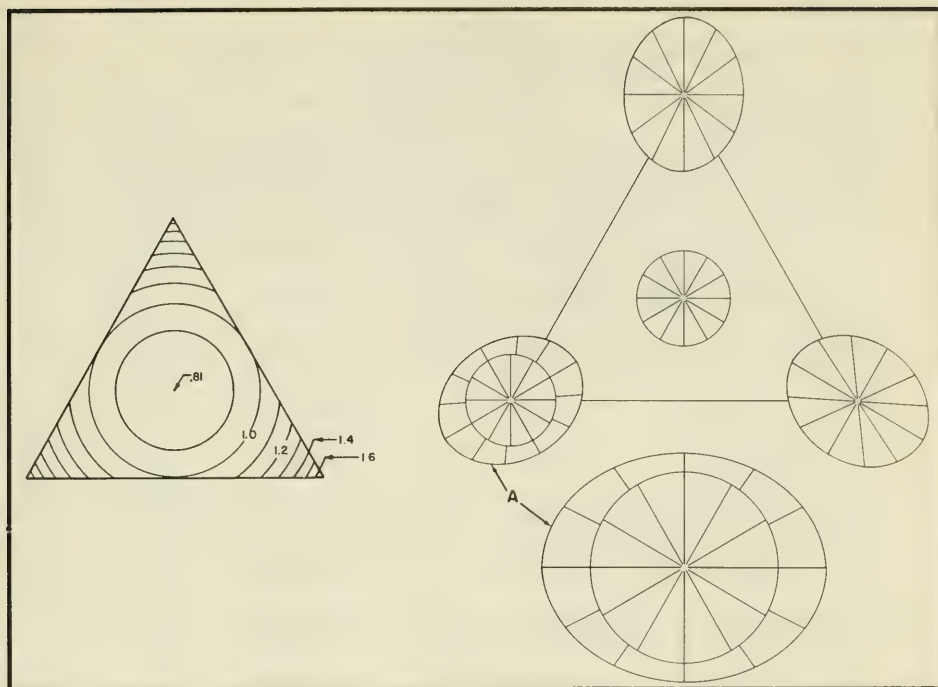


Fig. 7 - Areal and direction distortion for the icosahedral-gnomonic projection

The orientation of the icosahedron relative to the earth sphere was optimized for wave forecasting purposes by (a) having as many vertices as possible on land, (b) having as many edges as possible along shore lines, (c) using as few faces as possible for the ocean area, and (d) optimizing for the Northern oceans where a choice is necessary. To accomplish this, a clear plastic icosahedron to circumscribe a globe was prepared. For several interesting orientations thus approximated, detailed computations were made and a "best" one chosen. The final location of the 20 centers and orientations are listed in Table 2, and the triangles are shown on a Mercator projection in Fig. 8. It must be remembered that each triangular face represents the same area, since a presentation is subject to severe distortion. A further simplification is to put small extensions on the few triangles which have edges only slightly offshore. This eliminates bridging without significantly increasing distortion. Two possible arrangements of the polynomic projection on a plane are shown in Figs. 9a and 9b. An inset in Fig. 9a shows how the map folds into an icosahedron, and Fig. 9b is the working projection that shows the coordinates of all grid points.

A 60-degree coordinate system was selected for use on the icosahedral-gnomonic projection. In such a system the grid axes will coincide with two sides of the triangles and the third side will fall precisely on a line of grid



Table 2  
Location of the Tangent Points and Orientation  
of the Icosahedral Faces

| North<br>Latitude | East<br>Longitude | Rotation<br>Angle of the Y Axis<br>from North |
|-------------------|-------------------|-----------------------------------------------|
| 14.5075           | 139.3548          | -15.2623                                      |
| 20.5427           | 182.8676          | 31.0135                                       |
| 53.8644           | 216.0875          | -50.3112                                      |
| 27.7428           | 261.0318          | -21.3279                                      |
| -15.8715          | 203.7902          | -29.8942                                      |
| -11.6923          | 246.6958          | 40.8091                                       |
| 25.8444           | 308.1104          | 16.4492                                       |
| -40.1788          | 281.4638          | -3.1092                                       |
| 49.4441           | 353.0577          | 47.5038                                       |
| 81.6866           | 86.9816           | 16.6547                                       |
| -14.5075          | 319.3548          | -44.7377                                      |
| -20.5427          | 2.8675            | 28.9865                                       |
| -53.8644          | 36.0875           | -9.6889                                       |
| -27.7428          | 81.0318           | -38.6720                                      |
| 15.8715           | 23.7902           | -30.1058                                      |
| 11.6923           | 66.6958           | 19.1909                                       |
| -25.8444          | 128.1104          | 43.5507                                       |
| 40.1788           | 101.4638          | -56.8908                                      |
| -49.4441          | 173.0577          | 12.4962                                       |
| -81.6866          | 266.9816          | 43.3453                                       |

\*The Y axis is parallel to the line connecting the center and an apex.

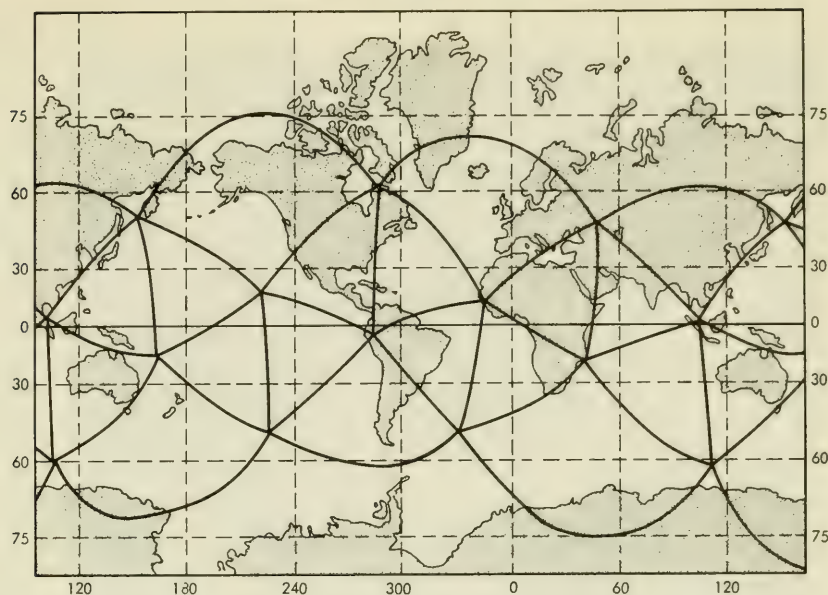


Fig. 8 - The locations on a Mercator projection of the triangles that constitute the icosahedral-gnomonic projection

points. While this is an important advantage of the system, it is not the only one. The area represented by each point is hexagonal, as shown in Fig. 10, which minimizes the maximum distance from the grid point to any point in the area. Hence, the value of some parameter such as wind speed at the grid point may be a better description of the entire area represented by the grid point than for, say, square unit areas. Also, each grid point in the hexagonal system has six natural directions associated with its neighbors. Twelve directions are exactly defined within 1.7 grid units. This compares with only eight exact directions in the square grid. And for the hexagonal system, the 12 can be increased to 24 if we accept a reasonable approximation. To achieve the desired representation for each grid point, each side of the triangle is divided into 48 intervals of 94.8 nautical miles by 49 grid points. Thus, an average grid point represents 6469 square nautical miles of the earth's surface. Extremes are therefore  $6469/0.83$  and  $6469/1.66$ .

Since the gnomonic projection is not conformal, the change in direction from true geographical direction and the change in scale in the various directions between "radial" and "transverse" must be considered. For a general triangular projection of this size Fig. 7 also illustrates the changes in scales and the changes in directions at the vertices of the triangle compared with an infinitesimal circle at true scale at the point of tangency. The circle is transformed to an ellipse at any point other than the tangent point with the major axis in a radial direction.

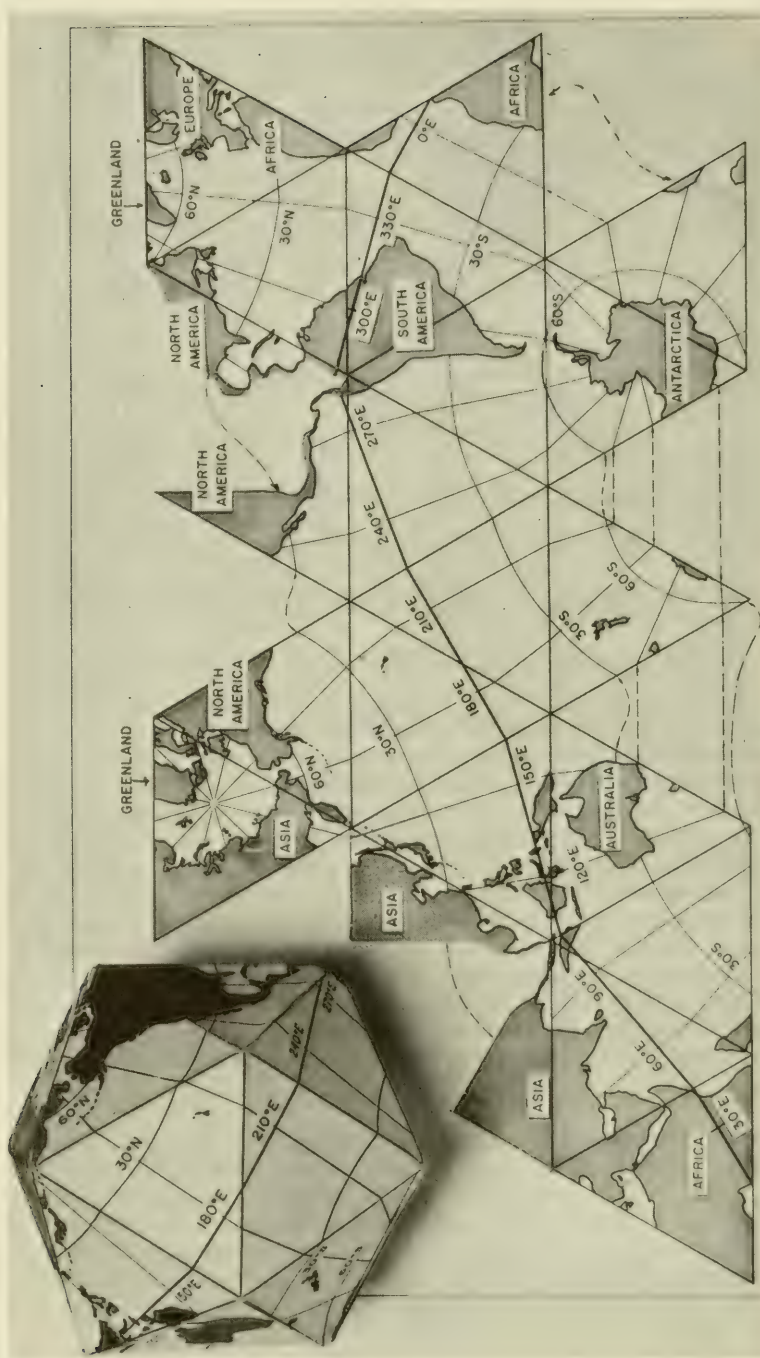


Fig. 9a - One form of the icosahedral-gnomonic projection and an inset showing how it can be folded to approximate a sphere

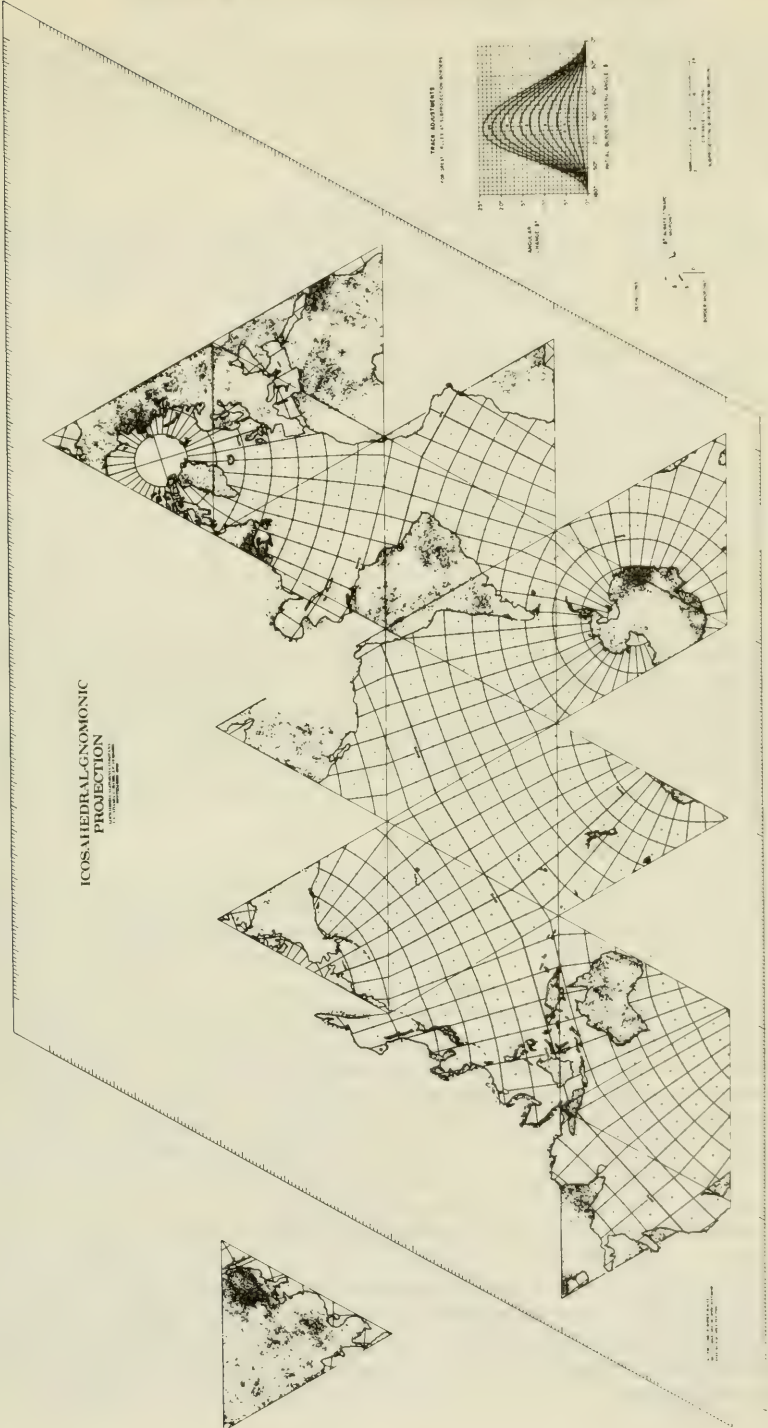


Fig. 9b - Map of the world ocean showing a nearly continuous Atlantic and a grid system used in numbering the points



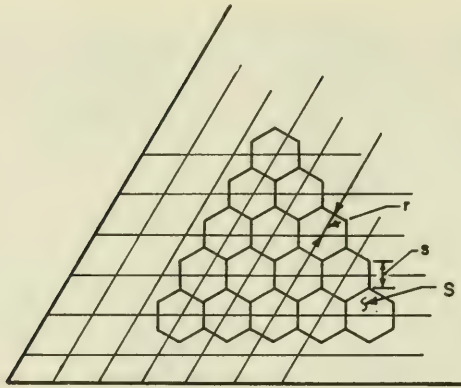


Fig. 10 - The hexagonal unit areas and the  $60^\circ$  grid system (from Ref. 25)

Each frequency-direction component propagates along a great circle tangent to its principal direction at a speed depending on its frequency, in accordance with linear wave theory. These tangent great circles connect nearby grid points in the chosen grid system. Further, this is relatively simple because great circles are straight lines within each triangular subprojection. There are, however, complications which occur because of scale and direction variations, bridging between subprojections, spatial discontinuities in the spectral components, and the discrete representation for the spectra.

Neither wave nor wave component fields representing the spatial distribution are continuous to first order. For example, regions of zero spectral variance are adjacent to regions where waves exist. Allowance for these possible discontinuities is a serious problem. As shown by Baer (2), propagation by multiplying the linear gradient as computed from grid points by the velocity to get the change at the grid point is inadequate for such discontinuities when repeated many times. Use of higher order gradients can cause other more serious errors. Allowance for discontinuities was accomplished in the previous North Atlantic work by assuming that all values were discontinuous. This was possible because in the region of the projection that was used the scale did not vary significantly throughout the field. It was thus possible to "jump" the values from all grid points in the field to the adjoining ones simultaneously after, of course, waiting the required time for the wave components to travel that distance. This caused some temporary irregularities in the component fields.

In the present case, using the icosahedral-gnomonic projection, variations in true distance between grid points are slightly over 50%, which is too much for such a simple system. A system has therefore been established which keeps track of the location of each discontinuity and uses the velocity gradient method within continuous regions.

The location of the discontinuity is remembered by keeping a field representing the fractional distance of any discontinuity to the next downstream grid point in a parallel storage position to that of the spectral variance itself. This simple system forces some assumptions that do not seem to have a significant effect.

Typically, the velocity-gradient propagation on grid points with a linear assumption results in errors near maxima and minima. This has been at least partially avoided in the present case at little expense by simply treating all maxima and minima as discontinuities.

It was pointed out previously that the  $60^\circ$  grid provides six primary directions, six secondary directions, and 12 tertiary directions. Propagation along the six primary directions is relatively simple in that the energy moves from grid point to adjacent grid point. However, in the other directions, the great circle along which the energy propagates skips either one or two rows of grid points. Thus, if the great circle path is taken too literally, winds a long way from the grid point of interest will have an effect rather than just winds at the adjoining grid points. Since both the wind and wave fields are relatively smooth, and since the accuracy of the method cannot exceed the grid point spacing, and to alleviate this proximity problem, we use a zigzag propagation scheme.

This zigzag is accomplished by choosing the location from which the energy travels by alternating between the grid points straddling the desired great circle path at successive time steps as illustrated in Fig. 11. For the secondary directions the two intermediate grid points are used alternately. For the tertiary, the grid point to one side is used thrice and the other once. This results in a slight but unimportant error because of scale changes.

In the above it is implicit that a system of grid directions are used in which rhumb lines and great circles are the same within a subprojection. This is similar to what was done for the North Atlantic computations. It thus becomes necessary to translate between grid coordinates and true earth coordinates at input and output and along the internal borders.

There are discontinuities in the particular projection chosen along the three internal triangle sides. As a great circle crosses these lines it changes direction discontinuously. These direction changes can most easily be seen by reference to the breaks in the latitude and longitude lines in Fig. 9. Near the center of the triangle sides there is little or no effect, but near an apex there is up to a  $30^\circ$  change depending on the direction of the great circle. This change in direction must therefore be taken into account.

At any grid point the spectral variance assigned to a particular frequency-direction component represents the integrated value from the complete directional spectrum over the region represented by the component. Since the change in grid direction which is required when a boundary is crossed is not usually an integral change of components, an approximation is required. Further, the angle representing a  $15^\circ$  grid increment on one projection will not always map a  $15^\circ$  increment on the adjoining projection.

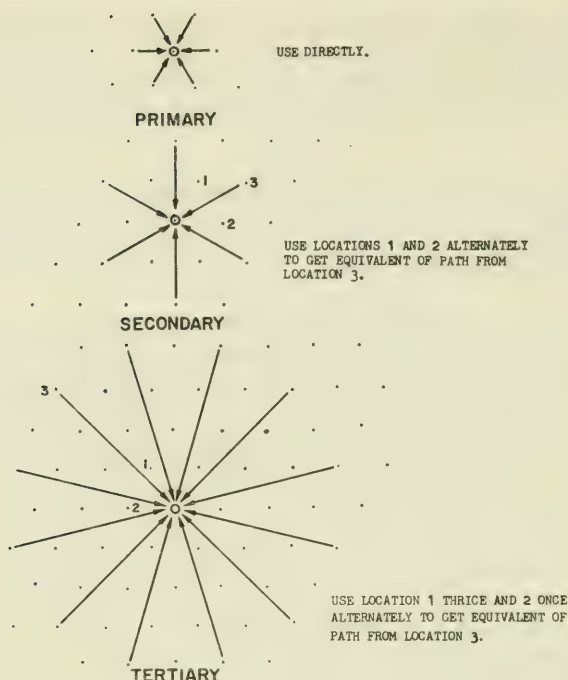


Fig. 11 - Propagation scheme for the 24 different directions

To facilitate solution of this problem a set of grid points has been chosen such that they fall exactly along the borders. The 24 direction increments are then split into two parts so that 12 are in the coordinates associated with each of the adjacent projections. Specifically, the 180 components representing outgoing directions are in the coordinates of the projection. This is simplified, because the two directions along the discontinuity are common to both projections. Before allowing energy to enter the region of the projection the incoming spectral components are curve-fitted (piecewise) and integrated over the appropriate directional limits found from the  $\pm 7.5^\circ$  limits within the projection. This introduces some smoothing and error which would be significant if repeated many times. However, for the entire North Pacific only a maximum of three such corrections are required.

For some situations with discontinuities, forward interpolation is required. Important here is that if a discontinuity is associated with an incoming direction at a border point and passes the next adjacent internal grid point in the time step, information must be provided in the adjoining projection to allow this forward interpolation. That is to say, instead of working with a triangle of only 49 grid points on a side there must be 51 grid points including a one-grid-point overlap of the adjoining projections. This overlap is integrated in much the same manner as the interpolation to allow for the changed location of the desired point.



The discrete representation of the spectra allows a possibility of false irregularities to develop in the variance fields, because truly continuous components are forced to propagate at the mean velocity for the frequency interval. Having a tight grid and many small components tends to minimize this effect. Also, the smoother and larger scale the wind field and the field for the spectral component is, the less the problem. But it must be allowed for in large area forecasts. The solution might be the use of a smoothing function while considering the location of the component discontinuities.

With experience it may also be found that the changing winds generate or dissipate a significant number of real discontinuities. The discontinuity location fields must therefore be updated. This is accomplished by testing the variance fields often enough that newly generated discontinuities will not be smoothed out.

### The Combined Steps

The computation procedure is to read in a wind field for a certain synoptic time, and carry out the steps of growth and dissipation at each grid point. Then the values of  $S(f_i, \theta_j)$  are propagated in their appropriate directions at their appropriate group velocities, and the new spectra are reassembled, completing a 2-hour time step. This is repeated twice more for the same wind field; then the next wind field is read in and the process is continued.

The usual starting point is a flat calm ocean,  $S(f_i, \theta_j) = 0$ , for all frequencies and directions, and it takes about one week for the North Atlantic to begin to behave realistically. The initial condition for each time step is the spectrum at each grid point, and the computations then proceed without interruption. Interesting effects have been observed in the results for the North Atlantic obtained to date. At times, several bands of swell are present plus a local sea. The swell off of North Africa is quite evident. Sometimes spectral values are present for all  $360^\circ$  of wave direction.

These procedures will be tested against data from the Weather Explorer, from Argus Island, and from FLIP (1) as kindly provided by Dr. Walter Munk. This will provide an independent check against the instrument used in measuring the waves.

### PROGRAMMING AND COMPUTATIONS

The flat earth approximation, the total number of data points (approximately 100,000), and the type of calculations being performed made the programming problem for the North Atlantic case a fairly simple one. Since the computations are of fairly low accuracy and the data are in a small range, it was possible to deal with the 100,000 data points within the structure of a 32,000 machine by packing four to a word. This packing and unpacking operation was efficient and its cost is small. The propagation portion involves simply a translation of the vector containing a frequency-direction field by some predetermined amount. The language it was written in was a rather high quality FORTRAN. However, the compiler which translated this does not produce a very efficient code. Even



so, the running time we have been able to achieve with a rather obsolete computer was 50 seconds for a 3-hour time step. It was estimated that the running time could be reduced to about 35 seconds if the program is recoded in machine language.

The North Pacific, on the other hand, does present some very difficult programming problems. The amount of active data is approximately one million numbers. Even with the latest computers having large core memories, large word sizes, and packing, it would not be possible to hold all the data in the main store. Use of external lower speed mass storage is required. In addition, for reasons indicated in previous sections, the program itself will be very large.

Since the basic nature of the operations required is that of having a matrix first in row order for growth and dissipation and then in column order for propagation, a careful layout of the data on this external storage medium is essential if the program is not to spend an inordinate amount of time exchanging the data between the main memory and the mass memory. One of the latest of the new generation of computers will be required for operational use.

Making the computations for a global grid is intrinsically no more difficult than that for the Pacific. The same techniques for moving data off and to the external memory will work for this larger grid. There will be considerably more programming to deal with the various marginal seas and with island arcs, and for special propagation procedures in the various parts of the world, but none of these present very great obstacles. The running time for the calculation at this stage becomes an important factor. Extrapolating from the past assures one that the needed computer speeds will be available by the time the program is ready.

A global forecast system using spacecraft measurements as input involves the use of a large data network and a great deal of message switching and display capability. The time constraints and the volume of computation to be handled will require a data system of a very sophisticated and highly automatic variety. These requirements, however, are not unique to the oceanographic forecasting. The space program, and even the planned management information systems, all require the same type of performance within the same type of constraints. One may also expect that the technology and experiences will be available at the time the system is ready for full-scale implementation.

The preparation of the wind fields present the usual problems of dealing with data from a remote source. The errors in the format have to be discovered and corrected, and data must be removed from the transmission medium into the computer with a minimum of human intervention. These, of course, are now standard problems, and most any organization can deal with them.

## DATA FROM A SPACECRAFT

It seems possible to measure a wave property which will be related to either the wind speed or to the wave height at a point on the ocean by means of a radar scatterometer on a polar orbiting spacecraft. Such a radar scatterometer

would measure radar scattering cross section,  $\sigma_0$  versus the angle  $\phi$  of the radar beam to the sea surface and present a graph of  $\sigma_0$  versus  $\phi$ . If  $\phi$  is  $90^\circ$  when the beam points straight down, then  $\sigma_0$  is high for low waves at  $\phi = 90^\circ$  and low at small  $\phi$ . Conversely for large waves  $\sigma_0$  is low for  $\phi = 90^\circ$  and high for small  $\phi$ . The variation of  $\sigma_0$  for  $\phi$  defined in this way is shown in Fig. 12. The difficulty is that present measurements are presented equally often in terms of either wind speed or the ill-defined term, sea state.

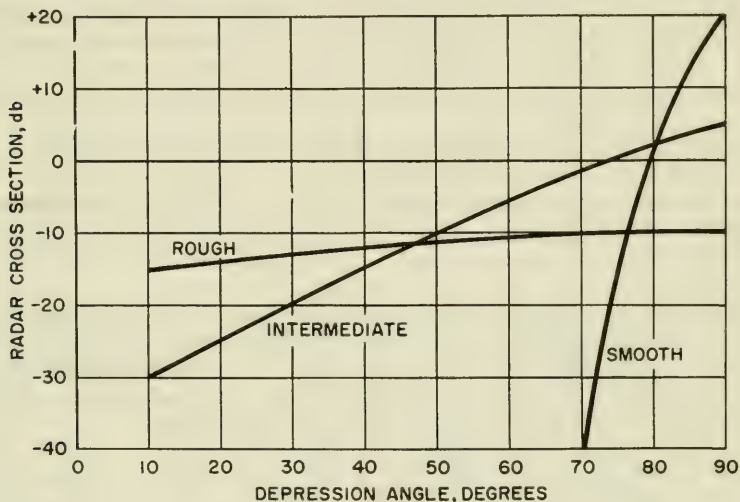


Fig. 12 - Radar scattering cross section versus the angle that the beam makes with a line parallel to the sea surface for various sea states according to Katz. (The convention here is opposite to the wording of the text.) (From Ref. 27.)

For a radar wavelength of 50 cm or so, it is not clear whether  $\sigma_0$  versus  $\theta$  is predominantly a function of the entire wave spectrum, in which case the scatterometer measures wave height, or of some part of the high wave number ( $k = \omega^2/g$ ) portion of the wave spectrum, in which case wind speed is measured, because the high-frequency part of the spectrum responds to the local wind very quickly.

These problems have been discussed in Ewing (27) and by Moore and Pier-son (28). A program of measurement from a NASA aircraft along with simultaneous wave measurements is underway at the Spacecraft Oceanography Project of the Naval Oceanographic Office. These problems should be resolved shortly.

The purpose of the preparation of the above global wave forecasting and hindcasting (or specification) computer-based procedure is to be ready to specify the wave and wind fields over the world ocean in a way that will permit the use

of satellite data to improve on the initial specification of the conditions over the ocean so as to obtain improved forecasts of the weather and wave conditions.

A spacecraft in a near polar orbit can complete an orbit in about 90 minutes. Each orbit is about  $22.5^{\circ}$  away from the preceding orbit, and due to "dwell" the higher latitudes are sampled more densely on an areal basis. Two complete surveys of the ocean would be made each day. The six, or seven, successive orbits that would provide the data for the North Pacific, say, would be obtained in 10 hours. With a wind speed, or a wave height, every 90 nautical miles along the subsatellite track, the result would be the equivalent of about 300 ships rather evenly spaced over the ocean and reporting on conditions. For all oceans, in one day the satellite would be equivalent to about 2800 ship reports spaced fairly uniformly over the oceans but concentrated at the higher latitudes.

For a particular pass, one can think of the data as a series of values along the line describing the subsatellite path. For the North Pacific again, as an example, one pass would be completed in 20 minutes and would be essentially synoptic. Were the satellite to pass over an extra tropical cyclone, the data would provide either a wave section or wind speed section through the cyclone.

The data can be compared with the winds and waves obtained from a conventional wind field analysis and the wave computed from the above procedures. A discrepancy would indicate a correction for the wind field to bring the results into agreement, and the final result would be an improved surface wind field that would yield improved meteorological and wave predictions.

After the development of the procedures described for spectral calculations and wind field calculations is completed, radar scatterometer data will be simulated and procedures for integrating it into improved wave forecasts and improved wind fields will be developed.

The final goal of this effort will be to use actual data from a spacecraft in real time to generate wave forecasts and wind field forecasts based on wind fields and initial wave conditions corrected by means of the spacecraft data.

## ACKNOWLEDGMENTS

This research is being supported by two separate Task Orders with the Naval Oceanographic Office under Contract N62306-1042. The purpose of one Task Order as funded by Code RAAD-22, Airframe Design Branch of the Air Systems Command, is to develop a wave hindcasting procedure for the North Pacific and produce a wave climatology for one year of weather data. The purpose of the second Task Order, monitored by the Spacecraft Oceanography Project of Navoceano, is to extend the results to the entire globe and develop ways to integrate data from a satellite into the problem of describing and predicting the surface winds and waves.

We have been assisted in this work by Miss Catherine Vail, Mr. Louis Adamo, Mr. Tokujiro Inoue, Miss Ming Shun Chang, Mr. Vincent Cardone, and Mr. Hong Chin. The personnel of the Naval Oceanographic Office have been



helpful in gathering data, carrying out supplemented investigations, and evaluating the results that have been obtained to date. We are indebted to Mr. Tim Barrett for discussions of his work in this area and to Mr. Lionel Moskowitz for his interest and support.

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## DISCUSSION

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I shall confine my comments to the discrepancies between the theoretical (15)\* and observationally inferred (17,18,24) values of the energy-transfer coefficient, say  $\beta$ , that governs the exponential growth of wind-generated waves. The most likely causes of these discrepancies appear to be (a) the neglect of wave-induced components of the Reynolds stresses in the turbulent shear flow and (b) the departure of the mean wind velocity from the logarithmic profile assumed in the original theoretical model. Both of these discrepancies are likely to be especially significant for those parts of the spectrum in which the critical layer, say  $z = z_c$ , at which wave speed equals wind speed, is at a relatively large elevation, say  $kz_c \approx 1$ ; this last condition appears to be typical in the neighborhood of the spectral peak for a fully developed sea. On the other hand, recent laboratory measurements, by Hsu and his colleagues at Stanford, appear to yield values of  $\beta$  that are consistent with the original theoretical calculations for  $kz_c \ll 1$ .

It seems quite unlikely that the profile discrepancy, (b) above, could account for an order-of-magnitude increase of 8 in  $\beta$ , relative to the value calculated on the hypothesis that the mean flow can be regarded as laminar. It is, however, entirely plausible that the wave-induced turbulent Reynolds stresses could account for this discrepancy, at least for  $kz_c \ll 1$ , although it seems unlikely that the energy-transfer processes associated with the quasi-laminar and turbulent mechanisms would be effective in the same proportion for all  $kz_c$ .

Further progress in our basic understanding of the physical processes involved in the generation of waves by wind almost certainly depends on controlled laboratory experiments. Further theoretical development also is possible, but our very limited understanding of turbulence suggests that this development is

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\*Numbers refer to references given by Pierson et al.

likely to rest on ad hoc hypotheses, the validity of which can be established only by a series of careful measurements. This will require time; meanwhile, practical necessity dictates the use of semiempirical models, in which the theoretical developments can be accommodated in the best tradition of engineering. Dr. Pierson and his colleagues have led the way in the development and testing of such models and are to be congratulated on their success.

\* \* \*

## DISCUSSION

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and

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I congratulate the authors on this paper, which I feel represents an appreciable advance in the art of wave prediction. It may be recalled that at the time of the last Ocean Wave Conference at Easton, in 1961, wave forecasting was in a very unsatisfactory state. It was one of the principal conclusions at that meeting that a reliable wave forecasting technique would need to be based on the numerical integration of the radiative transfer equation, and I feel that the authors have given very encouraging support to this view. It may be mentioned that similar forecasting techniques have been developed by Gelci and collaborators (the "DSA" method, cf. Gelci and Cazale, *J. d. Math. Phys. de l'Atmosph.* 2(No. 16): 15-41, 1962, Fons, *Cah. Ocean.* 18:16-33, 1966) and Barnett (Ph.D. thesis, Scripps Inst. of Oceanography, 1966). The most urgent need now appears to be for more detailed measurements of wave growth, as the source functions used by different authors in the transfer equation still differ appreciably. As regards the significance of the nonlinear energy transfer, it should be pointed out that the recent measurements of Snodgrass et al. (1) have clearly demonstrated that it is important and will need to be included in the source function.

\* \* \*

## REPLY TO DISCUSSION

Willard J. Pierson, Jr., Leo J. Tick, and Ledolph Baer

Since the preparation of our paper we have had the opportunity to study the two possible sources, suggested by Dr. Miles, of the discrepancies we found.



He has put his finger on the exact sources of the difficulties we encountered in applying the then available results of Dr. Phillips and him. We are most pleased to be able to report that by combining the original work of Miles (15) and Phillips (16) with the material in a recent text by Phillips (29)\* and with a variety of results on the wind profile and on the roughness length over a wind sea, Inoue (30) is now able to compute a realistic wave spectral growth in which the discrepancy between Miles' theory and the observations is reduced to a factor of less than 3 and in which the fit depends on the constant relating friction velocity to roughness length.

The new work of Phillips (29) provides realistic exponential growth rates for the low-frequency end of a wind sea spectrum by means of wind-induced components of the Reynolds stress. The original work of Miles (15) with a factor of less than 3 is then capable of accounting for the high-frequency end using an improved version based on the work of Barnett (24) of the linear growth term of Phillips. The earlier factor of 8 resulted from the need to get reasonable results at low frequencies and overdrove the high frequencies. An abstract of the results of Inoue (30) can be found in the Spring 1967 program of the Washington, D. C., meeting of the American Geophysical Union. The results should appear shortly in greater detail.

The departure of the wind profile from a simple logarithmic form was also of concern to us. Cardone (31) has, we believe, made progress on this part of the problem by introducing the form for the roughness length  $Z_0$  proposed by Kitaigorodskii and Volkov (32) and by including the effects of the Richardson number and the thermal wind. Variation in stability and in  $Z_0$  (as derived solely from the wave spectrum) makes it possible to obtain as selective subsets all of the various previously obtained curves on how the drag coefficient should vary with the wind speed. The natural variability of the underlying waves and the effect of atmospheric stability are quite capable of producing the wide scatter of data points usually associated with attempts to study  $C_{10}$ . The results are again given in abstract form in the program of the Washington AGU meeting and should appear shortly in published form.

There are a few more places in the structure of our numerical model where traces of semiempirical techniques can be found, but they are no longer as glaring as that factor of 8 that has now been considerably reduced.

In reply to Dr. Hasselmann's comments, it is instructive to point out how much of the wave forecasting theory available prior to the Easton conference seems to be borne out by our present results. As an example, the growth of significant wave height versus either duration or fetch turns out to be remarkably close to the curves given originally by Sverdrup and Munk. Also, once the difficulty of working with winds at a variety of elevations is clarified, the partially developed spectra derivable from the co-cumulative spectra in HO 603 turns out to be quite close to the more recent results. All frequencies start to grow at the same time, but the low frequencies take so much longer to reach the explosive exponential growth stage that the spectra appear to grow from high

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\*Additional references to those of the paper are given at the end of this reply.



frequencies toward low frequencies. It would be useful to discuss the ways these modern methods actually improve on earlier methods. They are substantial, but, nevertheless, the earlier methods had considerable validity.

With reference to nonlinear energy transfer, there are contradicting theories as to what actually happens to trains of long-crested intersecting waves. There is some evidence that a simple frequency adjustment is all that is needed to obtain realistic steady-state third-order results. The cited work of Snodgrass et al. (1), in our opinion, does not conclusively demonstrate the existence of this phenomenon on a geophysical scale. More definitive theories and observations are both needed.

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Monday, October 3, 1966

Afternoon Session

## OCEAN WAVES

Chairman: J. K. Lunde

Skipsmodelltanken  
Trondheim, Norway

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# INTERNAL WAVE INTERACTIONS

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## ABSTRACT

The second-order resonant interaction equations are derived for three internal wave modes in a fluid with constant Brunt-Väisälä frequency,  $N$ . These specify the energy interchanges among the modes, the redistribution of energy from one wavenumber to another. Integrals are shown to exist, representing the partition of energy among the modes and the conservation of energy among all three modes.

One particularly interesting type of interaction is that among two internal wave modes and a zero frequency horizontal drifting motion in the stratified fluid. Simple solutions are derived, and it is shown that the energy densities of the two wave modes are proportional to  $\cos^2(\sigma t)$  and  $\sin^2(\sigma t)$ , respectively, where  $\sigma$  is proportional to the vorticity in the steady horizontal motion. The energy exchange depends on, but does not involve, the horizontal motion; the latter acts as a catalyst for the interactions. From these results, it is inferred that an irregular vertical profile of horizontal velocity results in a horizontal channeling of low frequency internal wave motion, restricting their vertical spread even when the Brunt-Väisälä frequency is constant and much greater than the wave frequency.

## INTRODUCTION

The interactions among modes in wave motions have been studied considerably in recent years, and it is becoming evident that they are involved in a surprising number of physical phenomena, some of which have long been observed but not understood. Probably the most striking such example is the instability of the classical Stokes wave discovered by Benjamin and Feir (1) and described in the paper before the preceding one. The *effect* is well known; it is common knowledge among wave-tank people that if a plunger generates a smooth train of finite amplitude waves in a long tank, then as the waves propagate down the tank, they gradually degenerate into chaos, and that the steeper the initial wave, the sooner this happens. This has been regarded as a nuisance, probably "something to do with the plunger," and it was not until Benjamin and Feir's important work that this has been recognized as a genuine hydrodynamic instability of the wave train itself. Although their analysis is set up in terms of small perturbations to the Stokes wave train, with wavenumbers parallel and almost equal to that of the primary wave, it has since been shown (2) that the Benjamin-Feir



instability can be interpreted in terms of the theory of resonant interactions and that it is not restricted to purely two-dimensional motion.

Other manifestations of resonant interactions among surface waves have been studied theoretically by Benney, Hasselmann, Saffman and others; there have been unambiguous experimental verification of the existence of the interaction and the confirmation of its salient properties by both Longuet-Higgins and Smith (3) and by McGoldrick, Phillips, Huang, and Hodgson (4). The same type of interaction occurs in other dispersive wave motions and some have been considered in detail. In various contexts, they can be responsible for the generation of internal waves at considerable depth by the interaction between surface waves of nearly equal wavenumber (5,6), for the transfer of energy from capillary to longer but still short gravity waves (7), and for the transfer of energy to longer waves in a wind-generated sea (8).

In this paper, I propose to describe the resonant interactions among internal gravity waves in a fluid of constant stratification (i.e., constant Brunt-Väisälä frequency). One simple but important case will be considered in detail, namely, the case when one of the wave components reduces to a zero frequency wave, or a steady horizontal shearing motion with a periodic variation of velocity along the vertical. Motions of this kind appear to be widespread in stratified fluids, and it is found that their effect can be to prevent the vertical propagation of other internal wave modes — to restrict a disturbance to a zone surrounding the horizontal plane to the height of the source. This effect is clearly relevant to questions involving the vertical spread of internal wave energy in both oceans and atmosphere. It is described later in this paper, but the first step is to set up the general interaction equation from which these results are specialized.

## THE INTERACTION EQUATION

Suppose that, in the ambient state, the fluid is stratified with a uniform Brunt-Väisälä frequency

$$N = \left( - \frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \right)^{1/2}, \quad (1)$$

the  $z$  axis being vertically upward and  $\rho_0$  being the reference density. If the velocity field is represented as  $\mathbf{u} = (\mathbf{q}, w)$ , where  $\mathbf{q}$  represents the horizontal vectorial component of the velocity and  $w$  the vertical component, then the disturbances to the fluid are governed (9) by the equation

$$\frac{\partial^2}{\partial t^2} \nabla^2 w + N^2 \nabla_h^2 w = \frac{\partial^3}{\partial x_\alpha \partial z \partial t} \left( u_j \frac{\partial q_\alpha}{\partial x_j} \right) - \nabla_h^2 \left[ u_j \frac{\partial b}{\partial x_j} + \frac{\partial}{\partial t} \left( u_j \frac{\partial w}{\partial x_j} \right) \right], \quad (2)$$

where  $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ , and where the summation convention is used with  $j$  taking values 1, 2, 3 and  $\alpha$  taking values 1, 2. The coordinate specifications  $x, y, z$  and  $x_1, x_2, x_3$  are used interchangeably. The fluctuation  $b$  in buoyancy is related to that  $\rho'$  in density by

$$b = -g\rho'/\rho_0.$$

An elementary infinitesimal disturbance satisfying this equation with the non-linear terms neglected is of the form

$$\mathbf{u} = \mathbf{a} \cos X, \quad (3a)$$

$$b = N^2 \frac{\mathbf{a} \cdot \boldsymbol{\nu}}{n} \sin X, \quad (3b)$$

where  $X = \mathbf{k} \cdot \mathbf{x} - nt + \epsilon$  and  $\boldsymbol{\nu}$  is a unit vector vertically upward. The vector  $\mathbf{a}$  specifies the maximum particle velocity in the wave motion and, in virtue of the incompressibility condition  $\nabla \cdot \mathbf{u} = 0$ , is always normal to  $\mathbf{k}$ :

$$\mathbf{a} \cdot \mathbf{k} = 0.$$

Note that the particle trajectories are in the vertical plane, so that  $[\mathbf{a}, \mathbf{k}, \boldsymbol{\nu}] = 0$ . The frequency of the wave motion is given by

$$n = N \cos \theta, \quad (4)$$

where  $\theta$  is the angle between the wavenumber vector and the horizontal in the same vertical plane. Note also that

$$\mathbf{a} \cdot \boldsymbol{\nu} = a \cos \theta = |w|_{\max}. \quad (5)$$

These wave components are capable of resonant interaction; energy can be exchanged among sets of waves whose wavenumbers  $k_1, k_2, k_3$  and frequencies  $n_1, n_2, n_3$  satisfy the conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad (6a)$$

$$n_1 + n_2 = n_3. \quad (6b)$$

The last of these conditions is equivalent to

$$\cos \theta_1 + \cos \theta_2 = \cos \theta_3. \quad (7)$$

It has been shown (2) that provided the rates of shear in the internal wave modes are small compared with the Brunt-Väisälä frequency, the energy exchange is limited to this set of wavenumbers; the other components generated by the interaction (for example, the component with wavenumber  $\mathbf{k}_1 - \mathbf{k}_2$ ) are bounded in amplitude and remain small. They play no part in the resonant interaction and in the continuing energy exchange from one mode to another, at least to this order.

We consider, then, a set of three interacting internal wave components:

$$\mathbf{u} = \sum_{r=1}^3 \alpha_r \cos X_r + \hat{\mathbf{u}}, \quad (8a)$$

$$b = \sum_{r=1}^3 \frac{\alpha_r \cdot \nu}{n_r} \sin X_r + \hat{b}, \quad (8b)$$

where  $X_r = \mathbf{k}_r \cdot \mathbf{x} - n_r t + \epsilon_r$ , and  $\hat{u}$  and  $\hat{b}$  represent the small forced (nonresonant) products of the interaction. The "amplitudes"  $\alpha_r$  of the three primary waves will be supposed to be slowly varying functions of time, in general, as energy is exchanged in the interaction. The phase angle  $\epsilon_r$  may also vary slowly as a result of the interaction. Equations that express the variations of these quantities are obtained by the substitution of Eqs. (8) into Eq. (2). The linear terms on the left are proportional to  $\cos X_r$  and  $\sin X_r$ , and the quadratic terms on the right generate quantities proportional to  $\cos(X_r + X_s)$  and  $\cos(X_r - X_s)$ , some of which, in view of the resonance conditions (6), have the same spatial and temporal behavior as the primary waves themselves. Since

$$X_1 + X_2 = X_3 + \epsilon, \quad \epsilon = \epsilon_1 + \epsilon_2 - \epsilon_3,$$

then, in particular, the terms with wavenumber  $\mathbf{k}_1$  and frequency  $n_1$  must balance. This condition leads, after some algebra, to

$$\begin{aligned} & k_1^2 (-2n_1 \dot{\alpha}_1 \cdot \nu + \ddot{\epsilon}_1 \alpha_1 \cdot \nu + 2\dot{\epsilon}_1 \dot{\alpha}_1 \cdot \nu) \sin X_1 \\ & - k_1^2 (2\dot{\epsilon}_1 n_1 \alpha_1 \cdot \nu + \ddot{\alpha}_1 \cdot \nu - \dot{\epsilon}_1^2 \alpha_1 \cdot \nu) \cos X_1 \\ & = G_{23} \alpha_2 \alpha_3 \cos(X_2 - X_3) = G_{23} \alpha_2 \alpha_3 \cos(X_1 - \epsilon), \end{aligned} \quad (9)$$

where

$$\begin{aligned} G_{rs} &= G_{sr} = (n_r - n_s)(\mathbf{k}_r - \mathbf{k}_s) \cdot \nu k_r k_s \cos \theta_{rs} \cos \theta_{sr} \\ &+ \frac{1}{2} (n_r - n_s)(\mathbf{k}_r - \mathbf{k}_s)^2 (k_r \cos \theta_{rs} \cos \theta_r - k_s \cos \theta_{sr} \cos \theta_s) \\ &+ \frac{1}{2} N \left\{ (\mathbf{k}_r - \mathbf{k}_s)^2 - [(\mathbf{k}_r - \mathbf{k}_s) \cdot \nu]^2 \right\} (k_r \cos \theta_{rs} + k_s \cos \theta_{sr}) \end{aligned} \quad (10)$$

(no summation), with  $\theta_r, \theta_{rs}$  being defined by

$$\mathbf{k}_r \cdot \alpha_s = k_r \alpha_s \cos \theta_{rs}, \quad (11a)$$

$$\alpha_r \cdot \nu = \alpha_r \cos \theta_r. \quad (11b)$$

In general, in the weak interaction, the time scale of the variation of the wave amplitudes is long compared with the wave period:

$$|\dot{\alpha}_1 / n_1 \alpha_1| \ll 1,$$

so that the terms on the left side of Eq. (9) of the second order in the time derivative can be neglected. This then reduces to the pair of first-order equations

$$2n_1 k_1^2 \cos \theta_1 \dot{a}_1 = -G_{23} a_2 a_3 \sin \epsilon, \quad (12a)$$

$$2n_1 k_1^2 \cos \theta_1 a_1 \dot{\epsilon}_1 = -G_{23} a_2 a_3 \cos \epsilon. \quad (12b)$$

Similar pairs of equations specifying the rate of change of amplitude and phase of the other two components can be found in a similar way. They are

$$2n_2 k_2^2 \cos \theta_2 \dot{a}_2 = -G_{31} a_3 a_1 \sin \epsilon, \quad (13a)$$

$$2n_2 k_2^2 \cos \theta_2 a_2 \dot{\epsilon}_2 = -G_{31} a_3 a_1 \cos \epsilon \quad (13b)$$

and

$$2n_3 k_3^2 \cos \theta_3 \dot{a}_3 = H_{12} a_1 a_2 \sin \epsilon, \quad (14a)$$

$$2n_3 k_3^2 \cos \theta_3 a_3 \dot{\epsilon}_3 = -H_{12} a_1 a_2 \cos \epsilon, \quad (14b)$$

where

$$\begin{aligned} H_{rs} = & -(n_r + n_s)(\mathbf{k}_r + \mathbf{k}_s) \cdot \boldsymbol{\nu} k_r k_s \cos \theta_{rs} \cos \theta_{sr} \\ & + \frac{1}{2} (n_r + n_s)(\mathbf{k}_r + \mathbf{k}_s)^2 (k_r \cos \theta_{rs} \cos \theta_r + k_s \cos \theta_{sr} \cos \theta_s) \\ & + \frac{1}{2} N \left\{ (\mathbf{k}_r + \mathbf{k}_s)^2 - [(\mathbf{k}_r + \mathbf{k}_s) \cdot \boldsymbol{\nu}]^2 \right\} (k_r \cos \theta_{rs} + k_s \cos \theta_{sr}). \end{aligned} \quad (15)$$

The individual phase angles  $\epsilon_1, \epsilon_2, \epsilon_3$  can be eliminated from the second of each of these sets, giving

$$\begin{aligned} 2N a_1 a_2 a_3 \dot{\epsilon} &= 2N a_1 a_2 a_3 (\dot{\epsilon}_1 + \dot{\epsilon}_2 - \dot{\epsilon}_3) \\ &= - \left( G_{23} \frac{a_2^2 a_3^2}{k_1^2 \cos^2 \theta_1} + G_{31} \frac{a_3^2 a_1^2}{k_2^2 \cos^2 \theta_2} - H_{12} \frac{a_1^2 a_2^2}{k_3^2 \cos^2 \theta_3} \right), \end{aligned} \quad (16)$$

since  $n_r = N \cos \theta_r$ . This, together with

$$2N k_1^2 \cos^2 \theta_1 \dot{a}_1 = -G_{23} \sin \epsilon a_2 a_3, \quad (17a)$$

$$2N k_2^2 \cos^2 \theta_2 \dot{a}_2 = -G_{31} \sin \epsilon a_3 a_1, \quad (17b)$$

$$2N k_3^2 \cos^2 \theta_3 \dot{a}_3 = H_{12} \sin \epsilon a_1 a_2, \quad (17c)$$

provide four equations for the four unknowns,  $\epsilon$  the relative phase and the three amplitudes  $a_1, a_2, a_3$  of the wave components involved. They are characteristic



of those describing second-order resonant interactions, other examples having been given by McGoldrick (7) and Thorpe (6). From arbitrary initial conditions, the development in time can be traced and the full solution given in terms of elliptic functions (as in the manner described by Bretherton (10).

Some simple properties of the interaction can, however, be demonstrated without recourse to the general solution. It is evident that there are two stationary solutions for the phase angle, namely,  $\pm\pi/2$ , the value  $\epsilon = \pi/2$  being stable if the coefficient in the large parentheses of Eq. (16) is of the opposite sign to  $\alpha_1\alpha_2\alpha_3$ , and the value  $\epsilon = -\pi/2$  being stable if the coefficient is the same. When any one of  $\alpha_1$ ,  $\alpha_2$ , or  $\alpha_3$  passes through zero, the relative phase can change by  $\pi$ .

Two partition integrals follow immediately from Eqs. (17). From Eqs. (17a) and (17c),

$$(H_{12}k_1^2 \cos^2\theta_1)\alpha_1^2 + (G_{23}k_3^2 \cos^2\theta_3)\alpha_3^2 = \text{const.}, \quad (18)$$

and similarly, from Eqs. (17b) and (17c),

$$(H_{12}k_2^2 \cos^2\theta_2)\alpha_2^2 + (G_{31}k_3^2 \cos^2\theta_3)\alpha_3^2 = \text{const.} \quad (19)$$

Since  $\alpha_r^2$  represents the mean energy in the component with wavenumber  $k_r$ , these evidently describe the partition of energy among the three component modes. Not all of  $G_{23}$ ,  $G_{31}$ , and  $H_{12}$  can have the same sign. If, say,  $H_{12}$  and  $G_{23}$  do have the same sign, then as the energy of the  $k_1$  component increases as a result of the interaction, that of the  $k_2$  and  $k_3$  components decreases; this continues until (from Eqs. (17)) either  $\alpha_2$  or  $\alpha_3$  vanishes. At this point  $\dot{\alpha}_1 = 0$ ; the energy in the 1 component is a maximum and decreases at later times as energy is transferred back to the 2 and 3 components.

The conservation of total energy in the three wave components can be demonstrated, either from these partition integrals, or directly from Eqs. (17):

$$N \frac{d}{dt} (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = - \sin \epsilon \left( \frac{G_{23}}{k_1^2 \cos^2 \theta_1} + \frac{G_{31}}{k_2^2 \cos^2 \theta_2} - \frac{H_{12}}{k_3^2 \cos^2 \theta_3} \right) \alpha_1 \alpha_2 \alpha_3.$$

It is perhaps not obvious that the term containing the interaction coefficients on the right vanishes identically. The proof of this involves some elementary but tedious algebra and requires the use of the auxiliary conditions (6),  $\alpha \cdot \nu = 0$ ,  $[\alpha, k, \nu] = 0$ , and the set

$$k_1 \cos \theta_{12} = k_3 \cos \theta_{32}, \quad (20a)$$

$$k_2 \cos \theta_{23} = -k_1 \cos \theta_{13}, \quad (20b)$$

$$k_3 \cos \theta_{31} = k_2 \cos \theta_{21}. \quad (20c)$$

These last conditions are readily derived from

## Internal Wave Interactions

$$\mathbf{k}_1 \cdot \mathbf{a}_2 = (\mathbf{k}_3 - \mathbf{k}_2) \cdot \mathbf{a}_2 = \mathbf{k}_3 \cdot \mathbf{a}_2$$

(since  $\mathbf{k}_r \cdot \mathbf{a}_r = 0$ ), together with its permutations. The details will not be given here; suffice it to say that the right side vanishes identically and that

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \text{const.} \quad (21)$$

These interactions provide the means of energy exchange in a field of weak interacting gravity waves and play a role in determining the spectral properties of an ensemble of such waves. In this paper, however, we are concerned with a more restricted problem: the interaction between these waves and steady, horizontal shearing motions, with a periodic variation in the vertical.

### INTERACTIONS WITH STEADY, HORIZONTAL SHEAR

A very slowly varying, horizontal shearing motion can be considered as the limit of an internal wave whose frequency,  $n_1$ , say, approaches zero as the wavenumber approaches the vertical. The frequency condition for resonance then reduces to

$$n_2 = n_3, \quad (22a)$$

$$\cos \theta_2 = \cos \theta_3. \quad (22b)$$

Consequently, the internal waves capable of resonant interaction with the steady motion have wavenumbers that form the two equal sides of an isosceles triangle as indicated in the following figure. Clearly,

$$\theta_1 = \frac{1}{2} \pi,$$

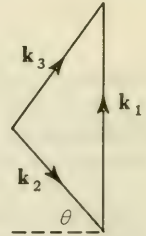
$$-\theta_2 = \theta_3 = \theta, \text{ say,}$$

$$\mathbf{k}_1 \cdot \boldsymbol{\nu} = k_1,$$

$$\mathbf{k}_2 \cdot \boldsymbol{\nu} = k_2 \sin \theta_2 = -k_2 \sin \theta,$$

$$\mathbf{k}_3 \cdot \boldsymbol{\nu} = k_3 \sin \theta_3 = k_3 \sin \theta,$$

$$k_2 = k_3 = \frac{k_1}{2 \sin \theta}.$$



The motion then represents the interaction of two internal waves of the same frequency but of opposite inclination to the vertical with a steady, horizontal motion whose variation in the vertical is specified by  $k_1$ . Under these conditions, the interaction coefficients simplify considerably:

$$G_{23} = 0, \quad (23a)$$

$$G_{31} = H_{12} = \frac{N k_1^3 \cos^3 \theta}{4 \sin \theta}. \quad (23b)$$

The first of the interaction equations, Eq. (17a), vanishes identically, since  $\cos \theta_1 = 0$  also, and the other two, Eqs. (17b) and (17c), become

$$2\dot{\alpha}_2 = -k_1 \cos \theta \sin \theta \alpha_3 \alpha_1, \quad (24a)$$

$$2\dot{\alpha}_3 = k_1 \cos \theta \sin \theta \alpha_1 \alpha_2. \quad (24b)$$

Clearly,

$$\alpha_2^2 + \alpha_3^2 = \text{const.}, \quad (25)$$

so that from Eq. (21)  $\alpha_1 = \text{const.}$  The energy density of the shearing motion is unaltered in time; though it plays a central role in the interaction, it does not partake of the energy exchange.

The complete solution to the interaction equations is very simple in this case.

$$\begin{aligned} \ddot{\alpha}_2 &= -\left(\frac{1}{2} k_1 \cos \theta \sin \theta \alpha_1\right) \dot{\alpha}_3 \\ &= -\left(\frac{1}{2} k_1 \cos \theta \sin \theta \alpha_1\right)^2 \alpha_2, \end{aligned}$$

so that if  $\alpha_2 = 0$  and  $\alpha_3 = A$  at  $t = 0$ ,

$$\alpha_2 = -A \sin \sigma t, \quad (26a)$$

$$\alpha_3 = A \cos \sigma t, \quad (26b)$$

where

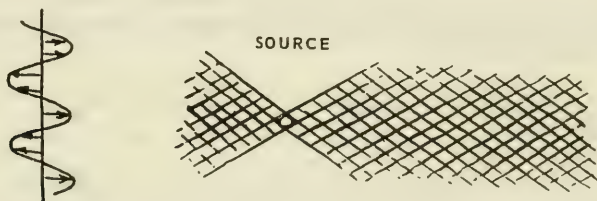
$$\sigma = \frac{1}{2} \cos \theta \sin \theta \alpha_1 k_1. \quad (27)$$

The energy density of the component with wavenumber  $k_2$  then varies as  $\sin^2 \sigma t$  and that of the  $k_3$  component as  $\cos^2 \sigma t$ ; the energy oscillates completely from one component to the other with a frequency that is proportional to the vorticity in the steady shearing motion.

This kind of behavior is remarkable in several respects. The steady shearing motion takes part in the interaction but does not share in the energy exchange; it acts as a kind of catalyst for the interaction. As far as I know, this is the first kind of interaction in which this effect has been demonstrated. It has some interesting physical consequences also. Suppose, in the ocean thermocline, there is some distribution of horizontal velocity with an internal wave with wavenumber  $k_2$  propagating upward through this region. This wave will select from the steady shearing motion the Fourier component with wavenumber  $k_1 = 2k_2 \sin \theta$  with which to interact (resonant interactions with other components being not possible). As a result of the interaction between this wave and the steady motion, energy begins to drain from the first wave component and generates another with the same frequency but moving in the downward direction



with equal inclination; this continues until all the energy is extracted from the first wave and has drained off to the second one. This, in turn, interacts with the shear, producing the first wave again traveling upward. As a result of this continual interchange between components whose propagation directions are inclined upward and downward, the internal waves are confined to a horizontal channel, in which the energy density in the upward and downward moving components are, on the average, equal. The kind of channeling is quite distinct from the well-known effect of the thermocline itself in restricting internal gravity waves to regions where  $n \leq N$ , and may have consequences that are equally important. If a local source generates internal gravity waves in the presence of steady motion, with an irregular velocity distribution, the resulting disturbance is limited to a range of depths surrounding the source, as shown in the following figure, even though



the stratification may be uniform far beyond. This range can be estimated simply from Eq. (27). The interaction time is  $T = 1/(1/2) \cos \theta \sin \theta \alpha_1 k_1$ , and the group velocity is  $c_g = (n/k_2) \sin \theta$  in the direction normal to  $k_2$ , or  $(n/k_2) \cos \theta \sin \theta$  in the vertical. The distance that the disturbance propagates in the interaction time is therefore

$$Z = c_g \cos \theta T$$

$$= \frac{2n_2}{\alpha_1 k_1 k_2} = \frac{4N \cos \theta \sin \theta}{\alpha_1 k_1^2} . \quad (28)$$

The maximum depth of the disturbance zone is found when  $\theta = 45^\circ$ , and the disturbance frequency equals  $N/\sqrt{2}$ ; in this case  $Z = 2N/\alpha_1 k_1^2$ . When  $n \rightarrow 0$ ,  $\theta \rightarrow \pi/2$ , and when  $n \rightarrow N$ ,  $\theta \rightarrow 0$ ; in either case  $Z \rightarrow 0$  and the disturbance is concentrated at the level of the source. The low-frequency limit is of particular interest in the oceanic and atmospheric case. It implies that low-frequency internal wave disturbances distribute themselves horizontally but very little vertically — a result that is consistent with the commonly observed zones of horizontal motions in the atmosphere, whose vertical extent is very limited.

Another interesting corollary of these results is that if  $\alpha_1 = 0$ , then  $\alpha_2 = \text{const.}$ ,  $\alpha_3 = \text{const.}$  Two internal waves of the same frequency pass through one another without interaction in a uniformly stratified ambient fluid at rest. The time seems to have come when controlled experiments on these effects would be of great value.



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## DISCUSSION

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(on leave from the University of Cambridge)

In his interesting paper Prof. Phillips has demonstrated analytically that internal waves can be channeled horizontally by interaction with a periodically layered distribution of horizontal currents. This result seems to have considerable potential value as regards the interpretation of internal-wave motions in the ocean and in the atmosphere. I wish now to summarize an alternative analysis of the mechanism of channeling, proceeding on lines quite different from

the application of general interaction theory made in the paper. Use is made of a method that has previously been applied to various other problems of wave propagation in periodically structured media, as discussed, for example, in Brillouin's well-known treatise.\* It will be shown that, for a given current distribution and given wave frequency, channeling of waves occurs not only at the particular wavenumber determined in the paper but over a set of wavenumber bands whose widths increase with the amplitude of the current.

## PERTURBATION EQUATIONS

The basic state is taken to be a stratified fluid with constant Väisälä frequency  $N$ , as defined by Eq. (1) in the paper, and with a steady velocity  $U(z)$  in the horizontal  $x$  direction. The wave motion is considered as a small perturbation from this basic state: thus the velocity vector is expressed in the form

$$\mathbf{u} = \mathbf{h}U(z) + \mathbf{u}'(x, y, z, t), \quad (\text{D1})$$

where  $\mathbf{h}$  is a unit vector in the  $x$  direction, and then the equations of motion are linearized in  $\mathbf{u}'$ .

Let us now assume  $\mathbf{u}'$  to be periodic in time and in the horizontal coordinates  $x$  and  $y$ , hence seek to determine its possible dependence on the vertical coordinate  $z$ . When  $\mathbf{u}'$  is found to be a periodic function of  $z$ , free propagation in a direction with a vertical component is in effect demonstrated. But the appearance of solutions attenuating with  $\pm z$  indicates that horizontal channeling of the wave motion must occur.

Accordingly we write

$$\mathbf{u}' = \hat{\mathbf{u}}'(z) e^{i(\alpha x + \beta y - nt)}, \quad (\text{D2})$$

and from the linearized equations of motion we may derive a set of ordinary differential equations for the components  $(\hat{u}, \hat{v}, \hat{w})$  of  $\hat{\mathbf{u}}'(z)$ . Some simplification of the resulting equations is justified on the basis of the assumption (made implicitly in the paper) that  $-\rho_0^{-1} d\rho/dz = N^2 g \ll k$ , where  $k$  is a typical wavenumber. (Alternatively, we may consider the linearized form of Eq. (1) in the paper, also using the incompressibility condition  $\nabla \cdot \mathbf{u}' = 0$  and the equation

$$\frac{\partial b}{\partial t} + U \frac{\partial b}{\partial x} = -N^2 w$$

corresponding to Eq. (3b).) Then it is found that the vertical component  $\hat{w}(z)$  satisfies

$$\frac{d^2 \hat{w}}{dz^2} + \left[ \gamma^2 + \frac{(2f - f^2) k^2}{(1 - f)^2} + \frac{1}{1 - f} \frac{d^2 f}{dz^2} \right] \hat{w} = 0, \quad (\text{D3})$$

\*L. Brillouin, "Wave Propagation in Periodic Structures," Dover, 1953.

where

$$\gamma^2 = k^2 \sin^2 \theta = \left( \frac{N^2}{n^2} - 1 \right) (\alpha^2 + \beta^2) \quad (\text{D4})$$

and

$$f = \frac{\alpha}{n} U(z). \quad (\text{D5})$$

Note that in the absence of the steady current ( $f = 0$ ) Eq. (D3) shows  $\gamma$  to be simply the vertical wavenumber, and Eq. (D4) gives the dispersion relation

$$n = N \left( \frac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2 + \gamma^2} \right)^{1/2} = N \cos \theta, \quad (\text{D6})$$

which was presented as Eq. (4) in the paper.

## ANALYSIS

If  $f(z)$  is any periodic function, then Eq. (D3) becomes a form of Hill's equation and resort may be made to the extensive theory available for equations of this general type. However, let us keep to the simple example treated in the paper. Thus we take

$$U = \hat{U} \cos k_1 z, \quad (\text{D7})$$

and further assume that  $\hat{U}$  is small in comparison with the phase velocity  $n/\alpha$  in the  $x$  direction, so that  $\hat{f} = \alpha \hat{U}/n \ll 1$ . Hence Eq. (D3) becomes approximately

$$\frac{d^2 \hat{w}}{dz^2} + (\gamma^2 + k_1^2 \hat{a} \cos k_1 z) \hat{w} = 0, \quad (\text{D8})$$

with

$$\hat{a} = \left( \frac{2}{\sin^2 \theta} \frac{\gamma^2}{k_1^2} - 1 \right) \hat{f}. \quad (\text{D9})$$

This is Mathieu's equation, and so the problem is now cast in a very familiar form.

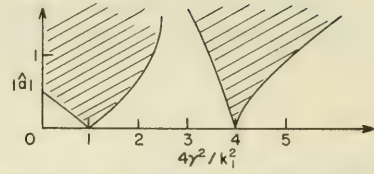
According to the well-known theory of Mathieu's equation,\* the solutions of Eq. (D8) are *periodic* for most values of  $\gamma^2 > 0$  when  $\hat{a}$  is small. In this case, free transmission upward or downward is possible. The exceptional case occurs when the parameters  $4\gamma^2/k_1^2$  and  $\hat{a}$  lie in certain ranges indicated by the shaded areas in the following diagram:

\*See, for example, Chapter IV of "Theory and Application of Mathieu Functions," by N. W. McLachlan (Oxford University Press, 1947).

The solutions then take the form

$$\hat{w} = e^{\pm \kappa z} \times (\text{periodic function}), \quad (\text{D10})$$

where  $\kappa$  is a real constant. This case represents the horizontal channeling of waves: the negative exponential in (10) describes the attenuation of wave amplitude in the upward direction ( $z$  increasing), and the positive exponential describes the downward attenuation. (Note that this is not, of course, a dissipative attenuation. It corresponds, rather, to the attenuation along a lossless transmission line excited at a frequency above the cutoff value.)



The first range in which vertical transmission is suppressed occurs when  $\gamma$  is near  $k_1/2$ . For small  $\hat{a}$ , the limits of this range are approximately

$$\gamma = \frac{1}{2} k_1 (1 \pm \hat{a}), \quad (\text{D11})$$

and  $\kappa$  is a maximum at approximately the center of it, i.e., at  $\gamma = k_1/2$ . The latter case is precisely the one considered in the paper. The maximum value of  $\kappa$  is

$$\begin{aligned} \kappa_m &= \frac{1}{2} k_1 |\hat{a}| = \frac{1}{2} \left| \frac{1}{2 \sin^2 \theta} - 1 \right| k_1 \hat{f} \\ &= \frac{k_1 \alpha \hat{U}}{4n} \left| \frac{\cos 2\theta}{\sin^2 \theta} \right|; \end{aligned} \quad (\text{D12})$$

hence, in view of Eq. (D6) and since

$$\alpha = \gamma \cot \theta \cos \phi = \frac{1}{2} k_1 \cot \theta \cos \phi, \quad (\text{D13})$$

where  $\phi$  is the angle between the  $x$  direction and the horizontal component of the wavenumber vector  $\mathbf{k} = (\alpha, \beta, \gamma)$ , we have

$$Z_m = \frac{1}{\kappa_m} = \frac{8N}{k_1^2 (\hat{U} \cos \phi)} \left| \frac{\sin^3 \theta}{\cos 2\theta} \right|. \quad (\text{D14})$$

This expression for the minimum value of the "penetration depth"  $Z$  may be compared with Eq. (7) in the paper (here  $\hat{U} \cos \phi = \alpha_1$  in the paper), but unfortunately the two results are not in agreement as regards the factors depending on  $\theta$ . The reason for the disagreement has not yet been fully uncovered. Note that, as would be expected for obvious reasons, the penetration depth depends on the mean-velocity component  $\hat{U} \cos \phi$  in the horizontal direction of wave propagation.



Using Eq. (D6) again, we can rewrite Eq. (D14) in the form

$$Z_m = \frac{8 (N^2 - n^2)^{3/2}}{k_1^2 (U \cos \phi) |2n^2 - N^2|}, \quad (D15)$$

which shows clearly that the penetration depth for  $\gamma = k_1/2$  is least when the wave frequency approaches the Vâisälâ frequency. Note also that  $Z_m \rightarrow \infty$  (i.e., there is a window for transmission upward or downward) when  $n \rightarrow N/\sqrt{2}$  and so  $\theta \rightarrow \pi/4$ . At a given frequency the horizontal wavenumbers  $\alpha$  and  $\beta$  corresponding to  $\gamma = k_1/2$  are given by

$$(\alpha, \beta) = \frac{k_1 n}{2 (N^2 - n^2)^{1/2}} (\cos \phi, \sin \phi), \quad (D16)$$

and the limits of the channeled range of wavenumbers are given approximately by this expression times  $1 \pm \hat{a}$ .

Channeling also occurs for small  $\hat{f}$  when  $\gamma$  is near the values

$$\gamma = \frac{s}{2} k_1, \quad s = 2, 3, 4, \dots \quad (D17)$$

But the widths of these higher ranges become progressively narrower with increasing  $s$ . Also the respective minimum penetration depths corresponding to Eq. (D14) are proportional to  $\hat{f}^{-s}$  approximately for small  $\hat{f}$ , and so they greatly exceed  $Z_m$  in the first range. The higher wavenumber ranges in which channeling occurs appear to be much less significant, therefore, than the first range discussed in the preceding paragraphs.

\* \* \*

## REPLY TO DISCUSSION

O. M. Phillips

In these comments, Dr. Benjamin makes a most valuable contribution. The problem is viewed here from the point of view of single scattering theory, whereas the interaction problem corresponds more to multiple scattering. Nevertheless, one might have expected the results to have been in qualitative agreement, and the appearance of the window in Dr. Benjamin's analysis but not in mine is a difference that must be resolved.

Further work (which will be described elsewhere in due course) has indicated that the discrepancy has arisen as a result of different specifications of the current structure, which in both cases is periodic in the vertical direction.

In the paper, I considered the steady motion as the limit of a zero frequency internal wave. In an internal gravity wave in a uniformly stratified fluid, the magnitude of the variations in density  $\delta\rho$  and velocity  $\delta\mathbf{u}$  are intimately related:

$$|\delta\rho| = \frac{|\delta\mathbf{u}|}{N} \frac{\partial\bar{\rho}}{\partial z},$$

and this relation is preserved as the frequency approaches zero. Consequently, in the analysis that I presented, there are coupled vertical variations in both the density gradient and the horizontal velocity field. In Dr. Benjamin's analysis, only velocity variations are considered. In actual fact, either or both can occur independently; it will be of interest to see how the two effects separately influence the nature of the trapping. Preliminary results indicate that Dr. Benjamin's result can be recovered when the density gradient is uniform and that the results presented in the paper appear when the variations in density and velocity are related according to the equation above. These further results will be presented in the literature in due course.

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# RADIATION AND DISPERSION OF INTERNAL WAVES

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## ABSTRACT

For small perturbations of a basic flow having a prescribed stratification of entropy and a vertical shear of horizontal free stream in a gravity field, a linearized equation for the vertical component of velocity is derived. The dispersion function is examined in detail for the special case of incompressible stratified flows with a uniform specific-density gradient and without shear. This dispersion relationship is applied to consider the radiation of internal gravity waves due to (a) an oscillating dipole and (b) a steadily moving singularity. Analyses of these two problems are also carried out in order to present a self-contained discussion of the results.

## INTRODUCTION

Consideration of the propagation of gravity and infrasonic waves in the atmosphere or ocean generally requires full-wave methods. Previous treatments of this class of flow problems have shown that the integral representations of the solutions of steady and quasi-steady flows involve, in general, some branch points on the path of integration. The indeterminacy of the branch cuts had been a subject of discussion for some time (see, for example, a review by Yih (1), p. 65 ff.), until a clarification by the approach of an initial value problem was given by Crapper (2). It is of interest to examine if the solution can also be made determinate by the consideration of the propagation of wave energy alone. One of the purposes of the present investigation is to discuss and compare these two approaches.

For the general purpose, a linearized equation is derived for the vertical velocity component of a perturbed flow while the basic flow is entropy-stratified and has a vertical shear. The gravity-acoustic mode and the effect of shear are not considered further here; discussions of some phases of these problems can be found in the recent literature, e.g., see Bjerknes (3), Hines (4) on isothermal atmosphere without winds, Weston (5), Pierce (6) on isothermal atmosphere with constant winds. The dispersion relation is investigated in detail for an incompressible flow with a constant specific-density gradient in the absence of shear.



It is found that the group velocity and phase velocity of these internal gravity waves can differ drastically from each other in both magnitude and direction. Based on this dispersion relation, the radiation of internal waves is traced for two specific problems: (a) an oscillating dipole and (b) the three-dimensional fundamental solution of a steady flow. From the comparison with the corresponding analytical solutions, it is found that the usual dynamic concept of the group velocity associated with the propagation of wave energy remains valid here.

## BASIC EQUATIONS AND BOUNDARY CONDITIONS

When the effects of viscosity, heat conduction, and mass diffusion can be neglected, the basic equations of motion for a compressible fluid are:

$$\text{continuity} \quad \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{q} = \rho Q(\mathbf{x}, t), \quad (1)$$

$$\text{momentum} \quad \frac{d\mathbf{q}}{dt} = -\frac{1}{\rho} \operatorname{grad} p + \mathbf{X}(\mathbf{x}, t), \quad (2)$$

$$\text{energy} \quad \frac{dS}{dt} = 0, \quad (3)$$

$$\text{state} \quad f(p, \rho, T) = 0, \quad (4)$$

in which  $t$  denotes the time,  $\mathbf{x} = (x, y, z)$  the Cartesian coordinates fixed in an inertial frame,  $\mathbf{q}$  the flow velocity,  $\rho$  the density,  $p$  the pressure,  $T$  the temperature,  $S$  the specific entropy of the fluid,  $Q$  the fluid source strength,  $\mathbf{X}$  the external force acting on a unit mass of the fluid, and  $d/dt = \partial/\partial t + \mathbf{q} \cdot \nabla$ . Equation (3) states that the process is isentropic along a material line, which is a valid assumption for not too rapid change of state in the absence of heat conduction.

The class of flows of our present interest is characterized by a uniform gravity field acting in the direction of  $z$  decreasing, or  $\mathbf{X} = (0, 0, -g)$ , on a basic reference flow which has a specified vertical shear and entropy stratification

$$\mathbf{q} = (U(z), V(z), 0), \quad S = S_0(z). \quad (5)$$

It is assumed that  $S'_0(z) \equiv dS_0/dz > 0$  to ensure thermodynamic stability, e.g., see Landau and Lifshitz (7) (but we shall not go here into the condition relating  $U'(z)$ ,  $V'(z)$ , and  $S'_0(z)$  for dynamic stability).

At the surface of a solid boundary the condition, as usual, is that the velocity component normal to the surface vanishes,

$$\mathbf{q} \cdot \mathbf{n} = 0 \quad (6)$$

on solid surface. At a free surface, say described by  $F(\mathbf{x}, t) = 0$ , across which the tangential velocity, density, temperature, and entropy may have a jump, there are two conditions:

$$dF/dt = 0 \text{ on } F = 0, \quad (7)$$

$$p \text{ is continuous across } F = 0. \quad (8)$$

In (8) the surface tension effect, if any, is not included.

The following special cases are of particular interest.

### Incompressible Stratified Flows

When the effect of compressibility of the fluid is negligible in flows of concern,  $S$  is not considered, and Eq. (3) is replaced by the equation of incompressibility, which is, in general,

$$d\rho/dt = 0, \quad (9)$$

stating that  $\rho$  remains constant along a material line. Then Eq. (1) reduces to

$$\operatorname{div} \mathbf{q} = Q(\mathbf{x}, t). \quad (10)$$

The momentum Eq. (2) remains the same in form, with  $\mathbf{X} = -g\mathbf{e}_z$ . The dual role of both hydrodynamic and thermodynamic nature played by  $p$  in the general case now reduces to one of only the hydrodynamic origin in incompressible flows. The basic flow is now characterized by a shear and a density stratification

$$\mathbf{q} = (U(z), V(z), 0), \quad \rho = \rho_0(z), \quad (11)$$

in which  $\rho'_0(z)$  is assumed to be  $\leq 0$  for static stability. A sufficient condition for stability is that  $-4g\rho'_0/\rho_0 > (U')^2 + (V')^2$  (see Howard (8)).

### Steady Stratified Flows

When the flow is steady the time derivatives of all flow quantities drop out from the above set of equations. In this case there exists a useful transformation, due to Yih (1,9,10), by which flows of nonhomogeneous fluids can be reduced to flows of homogeneous ones (called the related flows) under certain conditions. However, the related flow involves, in general, a nonuniform force field and an additional shear in the free stream.

For steady, two-dimensional incompressible flows, with  $\mathbf{q} = (u, 0, w)$ , a stream function  $\psi(x, z)$  exists such that

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x. \quad (12)$$

Then density  $\rho$  and the total head  $H$  remain constant along each streamline,

$$\rho = \rho(\psi), \quad H(\psi) = p + \frac{1}{2}\rho(u^2 + w^2) + g\rho z. \quad (13)$$

The differential equation for  $\psi$  (in a source free region) can be written

$$\rho \nabla^2 \psi + \rho'(\psi) \left[ \frac{1}{2} (\nabla \psi)^2 + gz \right] - H'(\psi) = 0, \quad (14)$$

which is due to Dubreil-Jacotin (11) and Long (12). In terms of the pseudo-stream-function  $\psi(x, y)$  of Yih (10), such that

$$\rho^{1/2} u = \tilde{\psi}_z, \quad \rho^{1/2} w = -\tilde{\psi}_x, \quad (15)$$

Eq. (14) becomes, somewhat simpler in form,

$$\nabla^2 \tilde{\psi} + gz \rho'(\tilde{\psi}) - H'(\tilde{\psi}) = 0. \quad (16)$$

These equations have been often used as a starting point for studying steady flows of finite amplitude.

#### INFINITESIMAL DISTURBANCES; LINEARIZED FORMULATION

If the disturbed flow differs only slightly from the basic flow, we may introduce the perturbation quantities; for the general case of compressible flows of an ideal gas, we write

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1, \quad \rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad S = S_0 + S_1, \quad (17)$$

in which  $\mathbf{q}_0 = (U, V, 0)$ , the perturbation velocity  $\mathbf{q}_1$  has the components  $(u_1, v_1, w_1)$ ,  $\rho_0$ ,  $p_0$ ,  $S_0$  are related to each other by

$$p'_0(z) = -g\rho_0 \quad (18)$$

and

$$\rho_0 p_0^{-1/\gamma} e^{S_0/c_p} = \rho p^{-1/\gamma} e^{S/c_p} = \text{const}, \quad (19)$$

where  $\gamma = c_p/c_v$  is the ratio of the specific heats. The local speed of sound relative to the medium of the basic flow is  $c_0$ , where

$$c_0^2 = (dp_0/d\rho_0)_{S_0} = \gamma p_0/\rho_0. \quad (20)$$

Substituting Eq. (17) into Eqs. (1-3), assuming  $\mathbf{q}_1$ ,  $\rho_1$ ,  $p_1$ ,  $S_1$  all small compared with their appropriate counterparts of the basic flow, we obtain the linearized equations as:

$$\rho_0 \operatorname{div} \mathbf{q}_1 + (D\rho_1 + w_1 \rho'_0) = \rho_0 Q. \quad (21)$$

$$\rho_0 \left[ D\mathbf{q}_1 + w_1 (U' \mathbf{e}_x + V' \mathbf{e}_y) \right] = -\nabla p_1 - g\rho_1 \mathbf{e}_z, \quad (22)$$

$$D\rho_1 + w_1 \rho'_0 = (Dp_1 - g\rho_0 w_1)/c_0^2, \quad (23)$$

where

$$D = \frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} + V(z) \frac{\partial}{\partial y} . \quad (24)$$

Now  $\rho_1$ ,  $u_1$ ,  $v_1$  can be readily eliminated from Eqs. (21), (22), and (23) (e.g., first eliminate  $\rho_1$  in (21) and (22) by substitution of (23), then eliminate  $(\partial u_1/\partial x + \partial v_1/\partial y)$  calculated from (22) by using (21)), giving

$$\left[ D \left( \frac{\partial}{\partial z} - \frac{g}{c_0^2} \right) - D' \right] w_1 = \frac{1}{\rho_0} \left[ \nabla_2^2 - \frac{1}{c_0^2} D^2 \right] p_1 + DQ \quad (25)$$

$$(D^2 + N^2) w_1 = - \frac{1}{\rho_0} D \left( \frac{\partial}{\partial z} + \frac{g}{c_0^2} \right) p_1 \quad (26)$$

where

$$\nabla_2^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \quad D' = U' \frac{\partial}{\partial x} + V' \frac{\partial}{\partial y} , \quad N^2 = - \frac{g\rho_0'}{\rho_0} - \left( \frac{g}{c_0} \right)^2 , \quad (27)$$

$N(z)$  being known as Väisälä frequency. Further elimination of  $p_1$  or  $v_1$  can be effected by cross differentiation and by using again (25) or (26), giving, after some manipulation,

$$(D^2 + N^2) \left( \nabla_2^2 - \frac{D^2}{c_0^2} \right)^2 w_1 + D \left[ \left( \nabla_2^2 - \frac{D^2}{c_0^2} \right) \left( \frac{\partial}{\partial z} - \frac{N^2}{g} \right) + 2D \frac{c_0 D' - c_0' D}{c_0^3} \right] (Lw_1 - DQ) = 0 , \quad (28)$$

where

$$L = D \left( \frac{\partial}{\partial z} - \frac{g}{c_0^2} \right) - D' .$$

If a horizontal free surface is present, represented by  $z = \zeta(x, y, t)$ , then

$$w_1(x, y, 0, t) = D\zeta . \quad (29)$$

The dynamic condition  $dp/dt = 0$  on  $z = \zeta$  is linearized to give

$$Dp_1 - g\rho_0 w_1 = 0 \quad \text{on} \quad z = 0 , \quad (30)$$

which becomes, upon using (25) or (26),

$$\left[ D^2 \frac{\partial}{\partial z} - g \nabla_2^2 - DD' \right] w_1 = 0 \quad \text{on} \quad z = 0 \quad (31)$$

or

$$\left[ D^2 + g \frac{\partial}{\partial z} - \frac{g}{\rho_0} \rho_0' \right] p_1 = 0 \quad \text{on} \quad z = 0 . \quad (32)$$



It is noteworthy that the compressibility effect does not appear explicitly in the above boundary conditions on free surface.

## Incompressible Flows of Small Amplitude

The linearized differential equations for incompressible flows with density stratification and shear can be deduced from the above case by passing to the limit  $c_0^2 \rightarrow \infty$  and  $dc_0^2/dz \rightarrow 0$ . In this limit, the order of Eq. (28) reduces by two and (28) becomes

$$\left\{ (D^2 + N^2) \nabla_2^2 + D \left( \frac{\partial}{\partial z} + \frac{\rho'_0}{\rho_0} \right) D \left[ \frac{\partial}{\partial z} - D' \right] \right\} w_1 = \frac{D}{\rho_0} \frac{\partial}{\partial z} (\rho_0 D Q), \quad (33)$$

or, after some rearrangement,

$$\left\{ D^2 \nabla^2 + \left[ \frac{\rho'_0}{\rho_0} \left( D^2 \cdot \frac{\partial}{\partial z} - g \nabla^2 \right) - DD'' - \frac{\rho'_0}{\rho_0} DD' \right] \right\} w_1 = \frac{D}{\rho_0} \frac{\partial}{\partial z} (\rho_0 DQ), \quad (34)$$

effect of:

inertia   gravity

density gradient

Shear

Shear-stratification  
interaction

in which

$$D' = U' \frac{\partial}{\partial x} + V' \frac{\partial}{\partial y}, \quad D'' = U'' \frac{\partial}{\partial x} + V'' \frac{\partial}{\partial y},$$

and  $\nabla^2$  is the Laplacian in  $x, y, z$ . The vertical displacement  $\zeta(x, y, z, t)$ , such that  $w_1 = D\zeta$ , satisfies the equation

$$\left\{ D^2 \nabla^2 + \left[ \frac{\rho'_0}{\rho_0} \left( D^2 \frac{\partial}{\partial z} - g \nabla_2^2 \right) - DD'' \right] \right\} \zeta = \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0 DQ). \quad (35)$$

This equation may be called the Love Equation (Lamb (13)), although Love (1891) first derived it for  $U' = V' = 0$ . For two-dimensional flows (in  $x$ - $z$  plane), all the  $y$ -derivatives in (33) and (34) of course drop out.

The different physical nature represented by different terms in (34) is indicated below (34). The effect of density gradient appears in two terms, one due to inertial and the other due to gravitational origin. The quantity  $D^2 \partial w_1 / \partial z$  gives a measure of the acceleration of rotation of a fluid bulk, and  $(-\rho_0 / \rho'_0)$  is related to the radius of gyration of the heterogeneous bulk. The gravity term represents the restoring force. The relative magnitude of these two terms gives rise to a Froude number,

$$O\left(D^2 \frac{\partial w_1}{\partial z} g \nabla_2^2 w_1\right) \sim U_0^2 / g \ell = Fr,$$

where  $U_0$  is a characteristic velocity and  $l$  a characteristic length, both appropriate to the particular problem in question. For small values of this  $Fr$ ,  $g \nabla^2 w$

is more important. Strictly speaking, this is true only in a region beyond a certain distance from the disturbance, for the inertial effect cannot be negligible at small distances. For large values of  $Fr$ ,  $D^2 \partial w_1 / \partial z$  is dominant near the disturbance. This relative importance of the inertial effect for large  $Fr$ , however, is not uniformly valid at infinity. The third term in the square bracket of (34) represents the effect of shear, which depends linearly on the shear gradient  $U''$ ,  $V''$  and is independent of the shear scale  $U'$ ,  $V'$ . The latter, however, may play a role in the interaction with the effect of density stratification.

## DISPERSION OF INTERNAL GRAVITY WAVES

The simplest case of density stratification is the uniform density gradient such that for a positive constant  $\beta$ ,

$$\rho_0 = \text{const } e^{-2\beta z} \quad \text{or} \quad -\rho'_0/\rho_0 = (\beta \geq 0). \quad (36)$$

For periodic motions in this medium (which is otherwise at rest),

$$U = V = 0, \quad w_1 = w(\mathbf{x}) e^{-i\omega t}, \quad Q = Q_0(\mathbf{x}) e^{-i\omega t}, \quad (37)$$

Eq. (34) further reduces to

$$\left[ \left( 1 - \frac{N^2}{\omega^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{\partial^2}{\partial z^2} - 2\beta \frac{\partial}{\partial z} \right] w = \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0 Q_0), \quad (38a)$$

where

$$N^2 = -\frac{g}{\rho_0} \rho'_0 = 2\beta g. \quad (38b)$$

Here the Väisälä frequency  $N$  is constant, corresponding to a period  $\tau = 2\pi/N = 2\pi \sqrt{g/H}$  ( $H = 1/2\beta$ ), the same as that of a pendulum of length  $H$ . The partial differential Eq. (37) of  $w(x, y, z)$  has constant coefficients; its type is

$$\begin{array}{ll} \text{elliptic} & \\ \text{parabolic} & \text{according as } \omega^2 \begin{array}{l} > \\ < \end{array} N^2. \\ \text{hyperbolic} & \end{array} \quad (39)$$

For  $\omega^2 < N^2$ , Eq. (37) has two real mathematical characteristics (straight lines):

for plane flows

$$x \pm \left( \frac{N^2}{\omega^2} - 1 \right)^{1/2} z = \text{const}; \quad (40a)$$

for axisymmetric flows

$$(x^2 + y^2)^{1/2} \pm \left( \frac{N^2}{\omega^2} - 1 \right)^{1/2} z = \text{const}. \quad (40b)$$

When  $\omega^2 \sim N^2$ , it is expected that the nonlinear effects will become important.

Although this is a highly idealized special case, its mathematical simplicity facilitates a thorough investigation of the detailed flow features, and these features may cast light on more complicated general problems. Some theoretical and experimental study of this type of flow has been done by Görtler (14).

### Dispersion Relation

We propose to investigate once more the elementary wave

$$\zeta = f(z) \exp[i(k_x x + k_y y - \omega t)] \quad (41)$$

propagating in an unbounded medium free of wave source. The solution of  $f(z)$  is

$$f = e^{\beta z} \left( A e^{i k_z z} + B e^{-i k_z z} \right), \quad (42)$$

where

$$k_z = \left[ \left( \frac{N^2}{\omega^2} - 1 \right) k_r^2 - \beta^2 \right]^{1/2}, \quad k_r^2 = k_x^2 + k_y^2. \quad (43)$$

Hence

$$\begin{aligned} k_z^2 &> 0 \\ \text{according as } \omega &\begin{cases} < \\ > \end{cases} \omega_c = \frac{N}{\sqrt{1 + \left( \frac{\beta}{k_r} \right)^2}}. \end{aligned} \quad (44)$$

$$k_z^2 \equiv -\sigma^2 < 0$$

Thus  $\omega = \omega_c$  separates the motion oscillatory in  $z$  from that aperiodic in  $z$ .

Rewriting (43) and (44), we obtain the basic dispersion relation as

$$\frac{\omega}{N} = \frac{k_r}{(k_r^2 + k_z^2 + \beta^2)^{1/2}} \quad (k_z^2 > 0) \quad (45a)$$

$$= \frac{k_r}{(k_r^2 + \beta^2 - \sigma^2)^{1/2}} \quad (0 < \sigma^2 < k_r^2 + \beta^2). \quad (45b)$$

For plane waves in  $x$ - $z$  plane,  $k_y = 0$ , one simply replaces  $k_r$  by  $k_x$  in these formulas. This dispersion relation (45) is plotted in Fig. 1, with level lines and paths of steepest ascent on the  $\omega$ -surface shown. The projections of these lines on the  $k_r$ - $k_z$  plane are shown in Fig. 2.

For a wave oscillatory in  $z$ , of given wave number vector  $\mathbf{k} = (k_x, k_y, k_z)$ , the phase velocity  $\mathbf{c}$  is

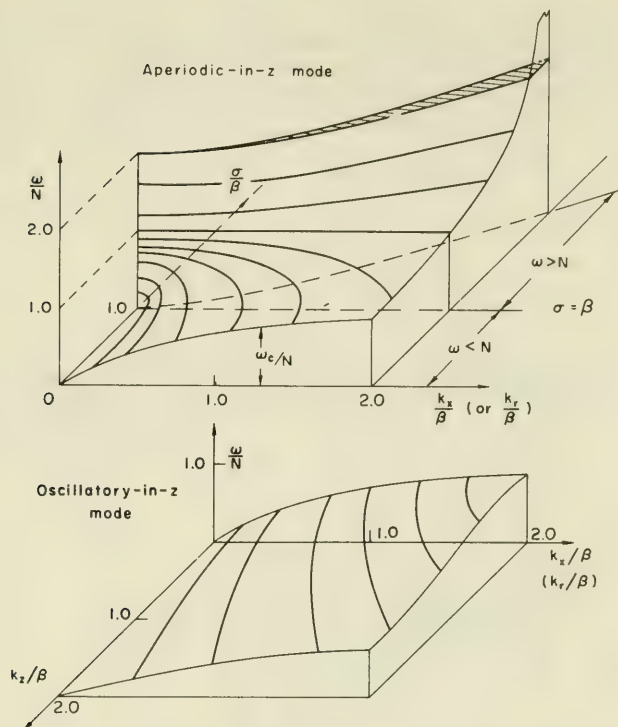


Fig. 1 - Plot of dispersion relation (Eq. (45)) showing level lines and paths of steepest ascent on the  $\omega$ -surface

$$c = \frac{\omega}{k} \mathbf{n} \quad (\mathbf{n} = \mathbf{k}/k), \quad (46a)$$

$$c = |\mathbf{c}| = \frac{\omega}{k} = \frac{N k_r}{k(k^2 + \beta^2)^{1/2}}, \quad (46b)$$

or

$$c = A \sin \theta, \quad A = N \sqrt{\epsilon/\beta}, \quad \epsilon = (1 + k^2/\beta^2)^{-1}, \quad (46c)$$

in terms of the polar coordinates

$$k_x = k \sin \theta \cos \varphi, \quad k_y = k \sin \theta \sin \varphi, \quad k_z = k \cos \theta. \quad (47)$$

Thus  $c$  increases with increasing  $\theta$  and decreasing  $k$ ; i.e., less inclined (to horizontal) longer waves travel faster. That  $c$  depends on  $k$  indicates that these internal gravity waves are dispersive.



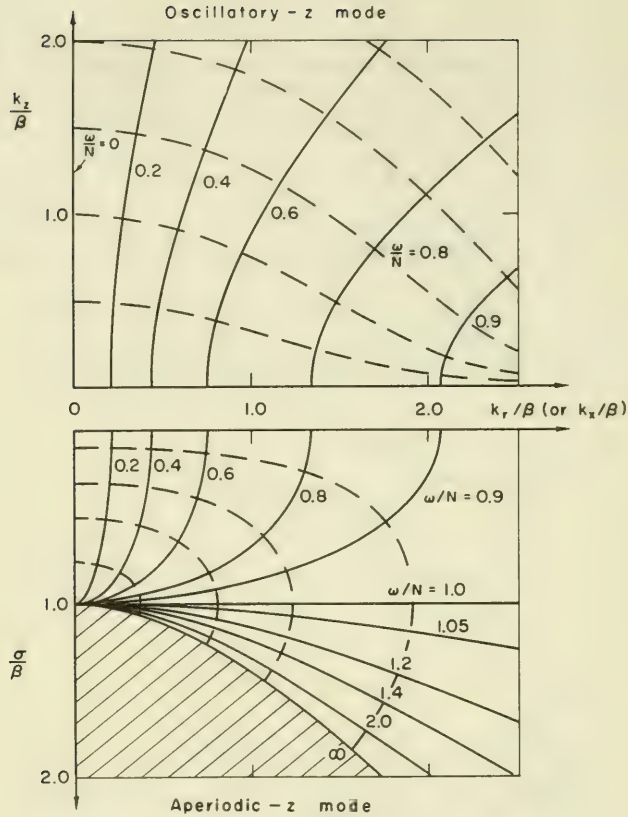


Fig. 2 - Projections on the  $k_r$ - $k_z$  plane of lines in Fig. 1

The group velocity  $c_g$  of these waves is

$$c_g = \frac{\partial \omega}{\partial \mathbf{k}} = \left( \frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z} \right), \quad (48)$$

which is the gradient of  $\omega$  in the  $\mathbf{k}$  space; hence  $c_g$  is everywhere orthogonal to  $\omega = \text{const.}$  surfaces, which are hyperbolic surfaces of revolution (about  $k_z$ -axis), in the direction of  $\omega$  increasing. From (45a) and (48),

$$\begin{aligned} c_{g_x} &= A \cos \varphi (k^2 \cos^2 \theta + \beta^2) / (k^2 + \beta^2), \\ c_{g_y} &= A \sin \varphi (k^2 \cos^2 \theta + \beta^2) / (k^2 + \beta^2), \\ c_{g_z} &= -A k^2 \sin \theta \cos \theta / (k^2 + \beta^2). \end{aligned} \quad (49)$$

It is significant to note that  $c_{gz}$  is always opposite in sign to  $c_z$ , and  $c_{gy}/c_{gx} = c_y/c_x$ . Since  $\mathbf{c}_g$  gives the direction and speed of propagation of wave energy, this means that  $\mathbf{c}$ ,  $\mathbf{c}_g$  and  $\mathbf{e}_z$  are always coplanar, and a wave group propagates with its vertical component in the direction opposite to the phase velocity of individual wavelets. Furthermore,  $\mathbf{c}_g$  has the magnitude

$$c_g = |\mathbf{c}_g| = A (\cos^2 \theta + \epsilon^2 \sin^2 \theta)^{1/2}. \quad (50a)$$

Hence for fixed  $k$ ,  $c_g > c$  for  $0 < \theta < \theta_e = \cot^{-1} \sqrt{1 - \epsilon^2}$ , and  $c_g < c$  for  $\theta_e < \theta < \pi/2$ . And

$$\mathbf{c}_g \cdot \mathbf{n} = \epsilon c, \quad \mathbf{c}_g \cdot \mathbf{e}_\theta = A \cos \theta. \quad (50b)$$

These basic features are illustrated in Fig. 3. These waves may be superimposed to give standing cellular waves, such as can be observed in a heated room. The amplitude of these waves increases upward, proportional to  $e^{\beta z}$ , so that the energy in each cell is the same.

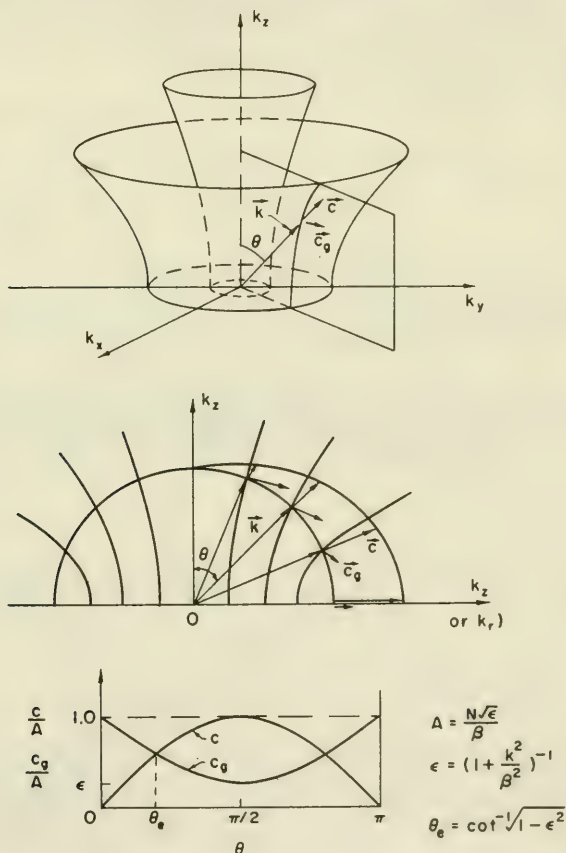


Fig. 3 - Dispersion of gravity waves

For the mode aperiodic in  $z$ , given by Eq. (45b), the phase velocity  $c$  is horizontal, and its value can be obtained much the same as before. However, calculation of  $c_g$  should depend on the physical circumstance of the problem. For instance, boundary conditions and nature of disturbances give rise to a relationship (often an eigenvalue problem) between  $\omega$ ,  $k_r$ , and  $\sigma$  (other than that in Eq. (45b)); consequently  $\sigma$  is no longer arbitrary. Even new modes may arise, such as the irrotational wave modes in a stratified deep ocean. Some classical examples can be found in Lamb (13) and Yih (1); another will be discussed later.

The above results of elementary waves can be used to construct qualitative solutions of more complicated form. The following are two typical examples.

### RADIATION OF INTERNAL WAVES DUE TO AN OSCILLATING SINGULARITY

Consider the flow due to a point singularity oscillating up and down with a fixed (circular) frequency  $\omega$  in a uniformly stratified fluid ( $\beta = \text{const.}$ ) which is unbounded and otherwise at rest. First we consider the case  $\omega < N$ . Since  $\omega$  is fixed, all the steady-state waves that are oscillating in  $z$  must have their wave vector  $k$  ending at the hyperbola (see Eq. (45a))

$$k_z^2 + \beta^2 = M^2 k_r^2 \quad \left\{ M = \left[ \left( \frac{N}{\omega} \right)^2 - 1 \right]^{1/2} \right\} \quad (51a)$$

or

$$\left( \frac{\beta}{k} \right)^2 = M^2 \sin^2 \theta - \cos^2 \theta. \quad (51b)$$

Evidently,  $\theta$  is bounded in the range

$$\theta_M \leq \theta \leq \pi - \theta_M \quad (\theta_M = \cot^{-1} M), \quad (52)$$

and (51b) gives  $k$  for each  $\theta$  lying in this range, as illustrated in Fig. 4. The group velocity of these waves therefore has the components (by substituting (51) in (49))

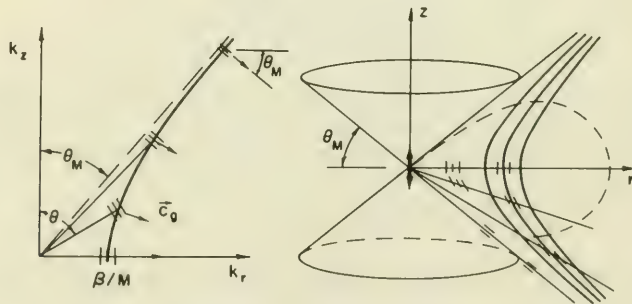


Fig. 4 - Radiation of internal waves due to an oscillating singularity

$$c_{gr} = \frac{N}{k(\theta)} \frac{M^2 \csc \theta}{(1+M^2)^{3/2}}, \quad c_{gz} = \frac{-N \cos \theta \csc^2 \theta}{(1+M^2)^{3/2}}. \quad (53)$$

The wavelet for each specific  $\theta$  is propagated along the straight line

$$\frac{z}{r} = \frac{c_{gz}}{c_{gr}} = -\frac{1}{M^2} \cot \theta, \quad (54)$$

and all these waves therefore exist only in the sector outside the vertical cone bounded by  $r = \pm Mz$ . Since the waves are all outgoing, with wave front normal to  $k$ , the slope of the constant phase surfaces (crests) is

$$\frac{dz}{dr} = -\tan \theta. \quad (55)$$

Upon elimination of  $\theta$  from (54) and (55), and integrating the resulting equation, we obtain the constant-phase surfaces as

$$M^2 z^2 = r^2 - r_0^2, \quad (56)$$

which are hyperbolic surfaces of revolution. Furthermore,  $c_g$  has the magnitude

$$c_g = \frac{\omega^3}{\beta N^2} (M^2 \sin^2 \theta - \cos^2 \theta)^{1/2} (M^4 \sin^2 \theta + \cos^2 \theta)^{1/2} \csc^2 \theta. \quad (57)$$

For large  $t$  ( $\gg N^{-1}$ ), the extent penetrated by these waves will be  $(c_g t)$ , which depends on  $\theta$ , as illustrated in Fig. 4. The solution is singular as  $\theta \rightarrow \theta_M$  (or  $r = \pm Mz$ ), which are the mathematical characteristics.

Furthermore, there are also waves with  $\omega < N$  that propagate in horizontal directions, with their amplitudes varying exponentially in  $z$ . The wave number  $k_r$  and exponential factor  $\sigma$  satisfy the relation (see (45b))

$$M^2 k_r^2 + \sigma^2 = \beta^2 \quad (0 < k_r < \beta/M, \quad 0 < |\sigma| < \beta). \quad (58)$$

Hence all real values of  $k_r$  lying in  $(0, \beta/M)$  are admissible to this class of waves, and the solution will be of the form

$$\zeta = e^{\beta z - i\omega t} \int_0^{\beta/M} e^{-\sigma |z| + i k_r r} f(k_r) dk_r, \quad (59)$$

in which  $\sigma = (\beta^2 - M^2 k_r^2)^{1/2}$  assumes the positive branch, and  $f(k_r)$  depends on the description of the singularity. For large values of  $r$  and  $|z|$ , the above integral may be evaluated by usual asymptotic methods, such as the method of steepest descent, by which it can be shown that



$$\zeta \sim e^{\beta z - i\omega t} \operatorname{fn}\left(\frac{\beta}{M} \sqrt{M^2 z^2 - r^2}\right) \quad (r < M|z|), \quad (60)$$

provided  $f(k_r)$  satisfies certain conditions of regularity.

For  $\omega > N$ , all admissible waves behave exponentially in  $z$  since

$$\sigma^2 = B^2 k_r^2 + \beta^2 \left[ B^2 = 1 - \left(\frac{N}{\omega}\right)^2 > 0 \right]. \quad (61)$$

Hence all positive real values of  $k_r$  may be used to construct the solution as formally expressed in (59), except the upper limit of integration is  $\infty$ . From this it follows that, as can be verified by using the method of steepest descent,

$$\zeta \sim e^{\beta z - i\omega t} \operatorname{fn}\left(\frac{\beta}{B} \sqrt{r^2 + B^2 z^2}\right) \quad \left(\beta \sqrt{r^2 + z^2} \gg 1\right). \quad (62)$$

The dependence of  $\zeta$  on  $r, z$  as given by (60) and (62) will be evaluated again analytically later.

## INTERNAL WAVES IN STEADY FLOWS

Suppose a localized disturbance, such as one produced by an obstacle of fixed shape, moves horizontally with a constant velocity  $U$  in the negative  $x$ -direction, while the fluid (with  $\beta = \text{const.}$ ) was originally at rest; and suppose the point disturbance reaches the origin at time  $t = 0$ . At an instant  $t$  earlier, the moving disturbance passed by the point  $Q$ , located at  $x = Ut$ , and emitted at that instant all components of internal gravity waves. These waves have been propagating away from  $Q$  in all directions after the disturbance left  $Q$ . For each specific wave number  $k$ , the waves with this  $k$ , if they had been carried along with phase velocity  $c$  (see (46c)), would be situated at  $t = 0$  on a surface of revolution which is obtained by revolving the circle having a horizontal diameter  $PQ$  about the vertical axis through  $Q$ , where  $PQ = c_k t = Nt/(k^2 + \beta^2)^{1/2}$ . The waves with different  $k$  will be located at the present instant on a different such surface of revolution. The farthestmost penetration of these waves is reached by the limiting component with  $k = 0$  (infinite wavelength), corresponding to  $P_0Q = Nt/\beta$ . Of all the waves generated at  $Q$ , those that may remain stationary relative to the moving disturbance must have their wave number vector  $\mathbf{k}$  satisfy the relation

$$\mathbf{k} \cdot \mathbf{U} = \omega = ck, \quad (63)$$

where  $\mathbf{U}$  is the velocity vector of the moving disturbance, so that the frequency of these stationary waves will appear to be zero with respect to the disturbance. But  $\mathbf{U} \cdot \mathbf{k} = -Uk \sin \theta \cos \varphi$  (see Fig. 5), hence from (63)

$$U \cos \varphi = -N/(k^2 + \beta^2)^{1/2}. \quad (64)$$

The solution  $k$  of (64) is real if  $U |\cos \varphi| < N/\beta$  and  $\pi/2 < \varphi < 3\pi/2$ ; then

# Radiation and Dispersion of Internal Waves

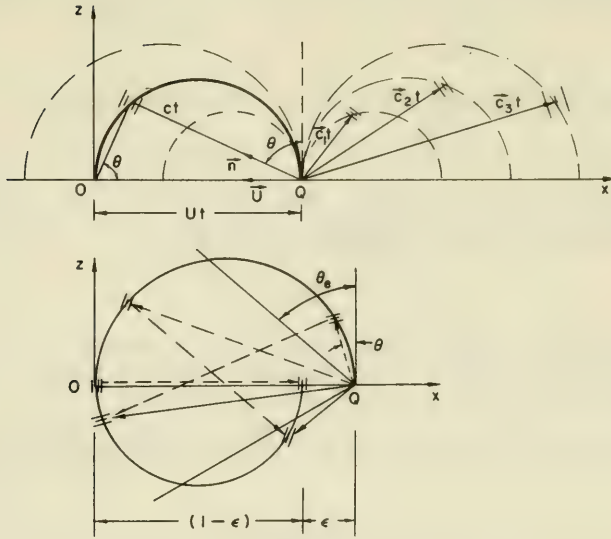


Fig. 5 - Internal waves in steady flows

$$k = \left[ \left( \frac{N}{U \cos \varphi} \right)^2 - \beta^2 \right]^{1/2} \quad (U |\cos \varphi| < N/\beta = [-2g\rho_0/\rho'_0]^{1/2}), \quad (65a)$$

for  $\varphi = 0$ , and

$$\lambda = \frac{2\pi}{k} = 2\pi U \left( \frac{H}{g} \right)^{1/2} \left( 1 - \frac{U^2}{4gH} \right)^{1/2} \quad (H = -\rho_0/\rho'_0). \quad (65b)$$

When the condition  $U < N/\beta$  is met, (63) indicates that the stationary waves would be located on the entire spherical surface with diameter OQ if these waves had been propagated with phase velocity  $c$ . It is of interest to note that the wavelength  $\lambda$  of the stationary internal waves depends only on the property of medium and free stream velocity, and, in contrast with ship waves in a homogeneous water, is independent of the angle of wave propagation. Generally  $\beta$  is very small ( $H = 1/2\beta \sim 8000$  meters for atmosphere), hence for sufficiently small  $U$  the last factor of (65b) in the square bracket is nearly unity, then  $\lambda$  is proportional to  $U$ . This feature is also in contrast with two-dimensional irrotational small waves in deep water, which have wavelength  $\lambda = 2\pi U^2/g$  in steady motion.

Since the wave energy is propagated with velocity equal to  $c_g$ , the final configuration of the stationary waves must be determined on this basis. Consider first the two-dimensional motions in the  $x$ - $z$  plane. The final location of a stationary wave packet emitted at Q in the direction making an angle  $\theta$  with the vertical evidently has the coordinates

Wu

$$\begin{aligned}x &= Ut - |c_{gx}|t = Ut(1 - \epsilon) \sin^2 \theta, \\z &= c_{gz}t = -Ut(1 - \epsilon) \sin \theta \cos \theta,\end{aligned}\quad (66)$$

in which use has been made of (49) (with  $\varphi = \pi$ ) and (64), and

$$\epsilon = \beta^2/(k^2 + \beta^2) = (U\beta/N)^2, \quad (67)$$

which is generally small. Eliminating  $\theta$  in (66), we obtain the locus of the stationary waves produced by the disturbance at Q as

$$x^2 + z^2 = Ut(1 - \epsilon)x, \quad (68)$$

which is a circle with diameter joining the points  $x = 0$  and  $x = Ut(1 - \epsilon)$ . This result is shown in Fig. 5 (with  $Ut$  normalized to 1); also shown in Fig. 5 is the locus of constant phase lines for several wave elements.

Next we consider the three dimensional case. The equations analogous to (66) are:

$$\begin{aligned}x &= Ut - |c_{gx}|t = Ut \left\{ 1 - \cos^2 \varphi [(1 - \epsilon) \cos^2 \theta + \epsilon] \right\}, \\y &= c_{gy}t = Ut \sin \varphi |\cos \varphi| [(1 - \epsilon) \cos^2 \theta + \epsilon], \\z &= c_{gz}t = -Ut |\cos \varphi| (1 - \epsilon) \sin \theta \cos \theta,\end{aligned}\quad (69)$$

in which  $\epsilon$  is given by (67). Upon elimination of  $\theta$  and  $\varphi$  from the above, we obtain

$$z^2 = \left[ 1 - \frac{Ut(Ut - x)}{(Ut - x)^2 + y^2} \right] [x(Ut - x) - y^2], \quad (70)$$

which is the equation of a closed surface bounded in the sphere  $y^2 + z^2 = x(Ut - x)$ . It is seen that the waves are nearly horizontal directly above and below the origin, and vertical along the inner rim of the surface near  $x = Ut$ .

We proceed to treat a few typical problems analytically in order to achieve an overall understanding of the analytical features of the problem and the physical arguments presented above.

## DIPOLE RADIATION

In order to avoid the indeterminacy of the quasi-steady solution, we consider a corresponding initial value problem of a plane flow so that the vertical velocity component  $w$  satisfies (see Eq. (34))

$$\left[ D^2 \nabla^2 - N^2 \left( \frac{D^2}{g} \frac{\partial}{\partial z} - \frac{\partial^2}{\partial x^2} \right) \right] w = D^2 \frac{1}{\rho_0} \frac{\partial}{\partial z} [\rho_0 \delta(x) \delta(z)] H(t) e^{-i\omega t}. \quad (71)$$

where  $D = \partial/\partial t$ ,  $\nabla^2$  is the Laplacian in  $(x, z)$ ,  $N^2$  again given by (38b), and  $H(t)$

is the Heaviside step function. By applying the Fourier transform (denoted with a tilde over  $w$ ) with respect to  $x$ , and the Laplace transform (denoted with a bar with respect to  $t$ , as defined by

$$\tilde{w}(k, z, s) = \int_0^\infty e^{-st} dt \int_{-\infty}^\infty e^{-ikx} w(x, z, t) dx, \quad (72)$$

Eq. (71) becomes

$$w(x, z, t) = \left( \frac{\partial}{\partial z} - 2\beta \right) G(x, z, t), \quad (73)$$

$$\left[ \frac{d^2}{dz^2} - 2\beta \frac{d}{dz} - k^2 \left( 1 + \frac{N^2}{s^2} \right) \right] \tilde{G} = \frac{1}{s + i\omega} \delta(z). \quad (74)$$

The solution of  $\tilde{G}$  of (74) which is continuous at  $z = 0$  and satisfies the required jump in  $d\tilde{G}/dz$  across  $z = 0$  is

$$\tilde{G} = -\frac{1}{2} \frac{1}{(s + i\omega)m} e^{\beta z - m|z|}, \quad (75a)$$

and

$$m(k, s) = \left[ k^2 \left( 1 + \frac{N^2}{s^2} \right) + \beta^2 \right]^{1/2}. \quad (75b)$$

The function  $m(k, s)$  has two branch points in the complex  $k$ -plane. An appropriate branch cut is to be introduced such that  $m \rightarrow |k|$  as  $k \rightarrow \pm\infty$  along the real  $k$ -axis in order to ensure the convergence of the Fourier inverse integral. By using the translation  $s_1 = s + i\omega$  in the Laplace inversion, we have

$$\tilde{G}(k, z, t) = \frac{1}{2\pi i} e^{\beta z - i\omega t} \int_{\Gamma} e^{s_1 t} F(k, z, s_1) \frac{ds_1}{s_1}, \quad (76)$$

where

$$F(k, z, s_1) = \exp[-|z|m(k, s_1 - i\omega)] / [-2m(k, s_1 - i\omega)]$$

and  $\Gamma$  is an integration path parallel to the imaginary  $s_1$ -axis in the right half plane. The large time solution of  $\tilde{G}(k, z, t)$  can be evaluated by applying the Tauberian theorem which states that

$$\lim_{t \rightarrow \infty} \tilde{G}(k, z, t) = e^{\beta z - i\omega t} \lim_{s_1 \rightarrow +0} F(k, z, s_1), \quad (77)$$

provided that a certain necessary and sufficient condition is satisfied (which can be verified separately). Now for small positive  $s_1$  the branch points of  $m$  are



$$k = \pm \frac{\beta\omega}{(N^2 - \omega^2)^{1/2}} \left[ 1 + \frac{iN^2 s_1}{\omega(N^2 - \omega^2)} + O(s_1^2) \right]$$

Hence, (a) for  $\omega < N$ , the two branch points are in the first and third quadrant of the  $k$ -plane; (b) for  $\omega > N$ , they lie very close to the imaginary  $k$ -axis; and they are symmetric with respect to  $k = 0$  in both cases. The appropriate branch cut for these two cases may be introduced as shown in Fig. 6. Therefore in the limit as  $t \rightarrow +\infty$ , we obtain the following solutions after applying (77) and Fourier inversion.

(I)  $\omega < N$ :

$$\begin{aligned} G(x, z, t) &= \frac{1}{2\pi} e^{\beta z - i\omega t} \int_{-\infty}^{\infty} e^{ikx} F(k, z, 0+) dk \\ &= -\frac{1}{4\pi} e^{\beta z - i\omega t} \int_{-\infty}^{\infty} e^{ikx - |z|\sqrt{\beta^2 - M^2 k^2}} \frac{dk}{\sqrt{\beta^2 - M^2 k^2}} \end{aligned}$$

where  $M^2$  is defined in (51a). With the branch cut so chosen and the integral path indented as shown in Fig. 6, it follows that

$$G(x, z, t) = -\frac{1}{4M} e^{\beta z - i\omega t} H_0^{(1)} \left( \frac{\beta}{M} \sqrt{x^2 - M^2 z^2} \right) \quad (x^2 > M^2 z^2) \quad (78)$$

$$= -\frac{1}{2\pi i M} e^{\beta z - i\omega t} K_0 \left( \frac{\beta}{M} \sqrt{M^2 z^2 - x^2} \right) \quad (x^2 < M^2 z^2), \quad (79)$$

where  $H_0^{(1)}$  denotes the Hankel function, and  $K_0$  the modified Bessel function of the second kind. Equation (78) shows that there is an outgoing wave in the region  $x^2 > M^2 z^2$ , its phase being constant along  $x^2 - M^2 z^2 = \text{const.}$  at any instant. These salient features have already been predicted by our previous geometric construction of the waves. In the region  $x^2 < M^2 z^2$ , however, our previous approach was only strong enough to predict the argument of the functional dependence.

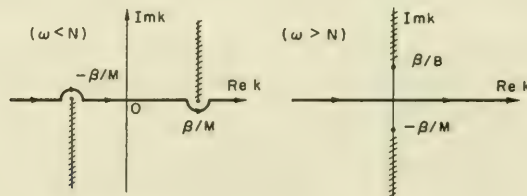


Fig. 6 - Branch cuts of  $m$  for  $\omega < N$  and  $\omega > N$

(II)  $\omega > N$ :

$$\begin{aligned}
 G(x, z, t) &= -\frac{1}{4\pi} e^{\beta z - i\omega t} \int_{-\infty}^{\infty} e^{ikx - |z| \sqrt{\beta^2 + B^2 k^2}} \frac{dk}{\sqrt{\beta^2 + B^2 k^2}} \\
 &= -\frac{1}{2\pi B} e^{\beta z - i\omega t} K_0 \left( \frac{\beta}{B} \sqrt{x^2 + B^2 z^2} \right), \quad (80)
 \end{aligned}$$

where  $B^2 = 1 - (N/\omega)^2$ . The argument of the functional dependence of  $G$  in this case also has been correctly predicted earlier (see (62)).

This problem has also been investigated by Görtler (14) and Wong (15).

## FUNDAMENTAL SOLUTION OF STEADY INTERNAL WAVES

The fundamental solution of steady two-dimensional internal waves due to a point source moving underneath a free surface was discussed earlier by Mei and Wu (16) for the cases (a)  $\beta = \text{const.}$ , (b)  $\beta$  arbitrary but  $g/(\beta U^2)$  large. An error contained in the solution of one case due to the indeterminacy of making the necessary branch cuts has been corrected subsequently (Wu and Mei (17)). The three-dimensional case has been discussed by Wu (18) for  $\beta = \text{const.}$  We propose to review here briefly the solution of three-dimensional internal waves in order to make a comparison with the approach based on the propagation of wave groups. The two-dimensional solution then appears to be deducible by a descent in dimension.

For the steady three-dimensional flow of a stratified fluid (with  $\beta = \text{const.}$  and the fluid unbounded in extent) having a uniform velocity  $U$  past a point source  $Q = \delta(x) \delta(y) \delta(z)$ , the vertical velocity component  $w$  satisfies the equation (see (34))

$$\left[ \frac{\partial^2}{\partial x^2} \nabla^2 - 2\beta \frac{\partial^3}{\partial x^2 \partial z} + \alpha^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] G = \frac{\partial^2}{\partial x^2} \delta(x) \delta(y) \delta(z), \quad (81a)$$

$$w(x, y, z) = \left( \frac{\partial}{\partial z} - 2\beta \right) G(x, y, z), \quad (81b)$$

where

$$\alpha^2 = N^2/U^2 = (g/U^2)(-\rho'_0/\rho_0), \quad \beta = -\frac{1}{2} \rho'_0/\rho_0. \quad (82)$$

By applying to (81a) the double Fourier transform

$$\tilde{G}(k_x, k_y, z) = \iint_{-\infty}^{\infty} e^{-ik_x x - ik_y y} G(x, y, z) dx dy, \quad (83)$$

under the assumption that  $G$  vanishes sufficiently rapidly at upstream infinity, (81a) is transformed to

$$\left[ \frac{d^2}{dz^2} - 2\beta \frac{d}{dz} - (k^2 - \alpha^2 \csc^2 \varphi) \right] \tilde{G} = \delta(z), \quad (84)$$

where

$$k_x = k \sin \varphi, \quad k_y = k \cos \varphi. \quad (85)$$

Hence

$$\tilde{G} = -\frac{1}{2m} e^{\beta z - m|z|} \quad (86a)$$

with

$$m(k, \varphi) = (k^2 - \alpha^2 \csc^2 \varphi + \beta^2)^{1/2}, \quad (86b)$$

the positive branch of  $m$  (so that  $m \rightarrow k$  as  $k \rightarrow +\infty$ ) being understood. Regarding  $\varphi$  to be real and  $k$  complex,  $m$  has two branch points at

$$k = \pm b, \quad b = (\alpha^2 \csc^2 \varphi - \beta^2)^{1/2}. \quad (87)$$

Hence  $b$  is real for  $\alpha > \beta$ . For  $\alpha < \beta$ ,  $b$  is real or imaginary according as  $\sin^2 \varphi < \text{or} > \alpha^2/\beta^2$ . Ordinary,  $\alpha \gg \beta$  in most of the natural circumstances. We shall only consider the case  $\alpha > \beta$ . A branch cut may be introduced between  $k = -b$  and  $+b$  along the real  $k$ -axis. By inversion, we have

$$G = -\frac{1}{4\pi^2} e^{\beta z} \operatorname{Re} \int_0^\pi d\varphi \int_{0\Gamma}^\infty e^{ikr \sin(\varphi - \theta) - m|z|} \frac{k dk}{m}, \quad (88)$$

where

$$x = r \cos \theta, \quad y = r \sin \theta, \quad (89)$$

and the path  $\Gamma$  is taken along the positive real  $k$ -axis except that it lies below the branch cut from  $k = 0$  to  $k = b$ ; this choice of  $\Gamma$  can be justified by the large time limit of a corresponding initial value problem (see Wu (18)). The path of integration  $\Gamma$  may be further deformed for  $-(\pi - \theta) < \varphi < \theta$  to proceed along the entire negative imaginary  $k$ -axis from  $k = -i0$  to  $-i\infty$ . For  $\theta < \varphi < (\pi + \theta)$ ,  $\Gamma$  may be deformed to circumvent the branch cut from  $k = i0$  around the point  $k = b$  to  $k = +i0$ , and then along the positive imaginary axis to  $k = +i\infty$  (see Fig. 7).

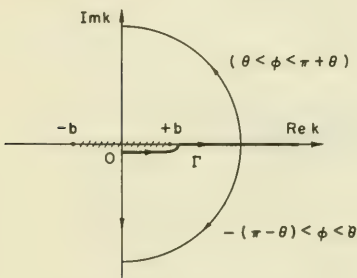


Fig. 7 - Path of integration  $\Gamma$

At large distances from the singularity ( $\alpha r \gg 1$ ), the asymptotic values of these resulting integrals can be evaluated by the method of stationary phase or by that of

steepest descent. The final result for  $\alpha \gg \beta$  may be cited here as

$$G \sim \frac{1}{2\pi} e^{\beta z} H(x) A(x, y, z) \sin(\alpha \Psi) \left[ 1 + O\left(\frac{1}{\alpha r}\right) \right]^{1/2}, \quad (90)$$

where  $H(x)$  is the step function,

$$\Psi(x, y, z) = |z| \left( \frac{x^2 + y^2 + z^2}{y^2 + z^2} \right)^{1/2}, \quad (91)$$

$$A(x, y, z) = \frac{x|z|}{(x^2 + y^2 + z^2)^{1/2} [(y^2 + z^2)^2 + x^2 y^2]^{1/2}}, \quad (92)$$

and there are no internal waves on the upstream side ( $x < 0$ ). A characteristic wavelength of  $G$  is

$$\lambda = \frac{2\pi}{\alpha} = 2\pi \frac{U}{N} = 2\pi U \left( -\frac{\rho_0}{g\rho_0'} \right)^{1/2}, \quad (93)$$

which agrees with (65b). The equiphase surfaces  $\Psi_n = \text{const.}$  are given by

$$\Psi_n = \frac{\lambda}{2} \left( n + \frac{1}{2} \right) \quad (n = 0, 1, 2, \dots), \quad (94)$$

they being the wave crests. The surface  $\Psi_1 = \text{const.}$  is plotted in Fig. 8, which is typical of all  $\Psi_n = \text{const.}$  They all descend downward in the streamwise direction to form a long-trenched valley, starting from  $x_n = (2n + 1)\lambda/4$  to  $\infty$ . The steepest path of descent is the circular arc in the symmetry plane  $y = 0$ . The level lines on  $\Psi_n = \text{const.}$  are hyperbolas, along each of which the amplitude  $A$  decreases monotonically with increasing  $x$ . These salient features are in qualitative agreement with the previous discussion of the wave field based on the propagation of wave energy, and it should be noted that some of the fine structure

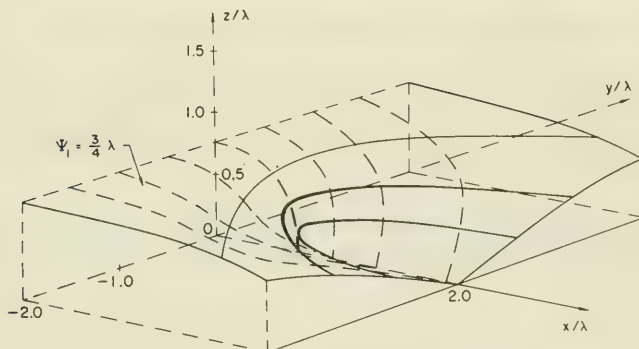


Fig. 8 - The surface of  $\Psi = \text{constant}$



of the flow field is lost in the above analytical result under the assumption of  $\alpha \gg \beta$ .

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# DAMPING OF GRAVITY WAVES BY SURFACE FILMS

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## INTRODUCTION

We consider the damping of gravity waves in the presence of surface contamination. Our primary goal is an estimate of the effect of such contamination on the generation of waves by wind on an unbounded surface; however, in order to facilitate comparison with laboratory experiments, we consider waves in a cylindrical basin, with an unbounded surface as a limiting case. We give rather more detailed results for the surface-film damping coefficient than is warranted by our present knowledge of the physical parameters, and in the final analysis we consider only the limiting cases of an uncontaminated surface and an inextensible film. We find that the minimum wind speed required to initiate waves of significant magnitude on a contaminated surface may be roughly an order of magnitude larger than for a clean surface. This conclusion is consistent with the available observations.

Let  $k$  and  $\sigma$  denote the wave number and angular frequency of an oscillation determined on the hypothesis of an inviscid liquid, and let  $\nu$  be the kinematic viscosity of the actual liquid. Then  $1/k$  is an appropriate length scale for the irrotational component of the velocity field, and

$$\delta = (2\nu/\sigma)^{1/2}, \quad (1)$$

the thickness of the boundary layer on an oscillating flat plate, is an appropriate length scale for the viscous component. We base our discussion on the hypothesis that

$$\epsilon = k\delta \ll 1, \quad kh, \quad (2)$$

where  $h$  is the mean depth of the liquid. The dispersion relation between  $\sigma$  and  $k$  then is

$$\sigma^2 = gk \tanh kh [1 + O(\epsilon)], \quad (3)$$

where  $g$  is the acceleration due to gravity.

The contribution of viscous dissipation in the boundary layers on the walls of a cylindrical basin to the logarithmic decrement has been calculated by Ursell (1) (see Benjamin and Ursell (2) for the corrected result) and has the form

$$\alpha_w = A\epsilon + O(\epsilon^2), \quad (4)$$

where  $A$  depends only on the geometrical shape of the container and the particular mode; see Eq. (32) and Eqs. (33).

The logarithmic decrement for a *clean, unbounded* surface is given by (Stokes (3) and Lamb (4a))

$$\alpha_S^{(0)} = 2\pi\epsilon^2. \quad (5)$$

It can be shown (cf. Ursell (1) and Case and Parkinson (5)) that this result also is valid, within a factor of  $1 + O(\epsilon)$ , for a bounded surface, but it then is negligible compared with  $\alpha_w$ . (Case and Parkinson and others have retained  $\alpha_S^{(0)}$  in the calculation of  $\alpha$  for closed basins, but this is clearly inconsistent unless terms of  $O(\epsilon^2)$  and  $O(\epsilon)$  are included in the dissipation in the boundary layer on  $w$  and in the mean energy of the oscillation, respectively.

Available measurements (Benjamin and Ursell (2), Case and Parkinson (5), Keulegan (6), Van Dorn (7)) reveal that (the theoretical value of)  $\alpha_w$  does not provide an adequate approximation to  $\alpha$  for gravity waves in laboratory-size basins except under special and carefully controlled conditions and that the observed values of  $\alpha/\alpha_w$  may be as large as three.

Van Dorn (7) attributed the observed discrepancies to a surface film produced by spontaneous contamination. He found that "while the observed attenuation agreed with that computed for the solid boundaries [ $\alpha_w$ ] when the water was fresh, the former tended to increase with time to some higher limiting value, usually within an hour." He also reported that "the fully contaminated surfaces exhibited no obvious visual manifestation" and that "the water always appear[ed] to be in every respect as clear and fresh as when uncontaminated." He obtained an adequate approximation to the limiting value of  $\alpha$  simply by adding to  $\alpha_w$  the value of  $\alpha$  calculated on the hypothesis of an inextensible film, namely (Lamb (4b))

$$\alpha_S^{(1)} = \frac{1}{2}\pi\epsilon. \quad (6)$$

This agreement notwithstanding, we suggest that capillary hysteresis could have been significant in Van Dorn's experiments, especially since some of his measurements were made in lucite tanks (see following description of Keulegan's experiments (6)).

The observation that surfactants, such as oil on water, can lead to appreciable damping of surface waves goes back to antiquity (Pliny the Elder, in the first century A.D., is cited by Levich (8) and by Davies and Vose (9), and Benjamin Franklin (10) attempted a phenomenological explanation in 1774). The



first satisfactory explanations appear to have been given by Reynolds (11) and Aitken (12), each of whom invoked the variations of surface tension with wave motion; Aitken also carried out laboratory experiments. Lamb provided the necessary mathematical analysis in support of the Reynolds-Aitken model in the second (1895) edition of his "hydrodynamics," but abbreviated the analysis to that for an inextensible film in the sixth edition (4b). Subsequent analyses, dealing with the physicochemical as well as the hydrodynamical problem, have been given by Levich (8,13), Dorrestein (14), Goodrich (15), and Tempel and Riet (16).

Levich considered both insoluble and soluble surfactants, but appears to have assumed (as, implicitly, did Lamb) that damping would increase monotonically with concentration to the limiting value  $\alpha_S^{(1)}$ . This tacit assumption led him to overlook the fact that his general formulation for an insoluble film yields the maximum value  $\alpha_S = 2\alpha_S^{(1)}$  at an intermediate concentration, a result established independently by Dorrestein. Levich considered various limiting cases of his general formulation, but his approximations do not appear to be entirely systematic and, in any event, do not give a clear picture of the interrelations among such parameters as frequency, surface modulus of compression, and solubility. Dorrestein considered only insoluble surfactants, but included surface viscosity in his analysis (Levich included surface diffusion of the surfactant, which is essentially similar to surface viscosity in its effect on damping). Tempel and Riet attributed differences among the results of Levich, Dorrestein, and Goodrich to "the use of an incorrect boundary condition for the tangential stress." In fact, Tempel and Riet's formulation appears to agree with those of Levich and Dorrestein when appropriate comparisons are made; detailed comparisons of their analytical results are not possible, as Tempel and Riet used an electronic computer to solve the algebraic equations implied by their boundary conditions.

None of the formulations of Levich, Dorrestein, Goodrich, and Tempel and Riet is directly applicable to gravity waves in closed basins. In the following three sections we present a derivation of  $\alpha_S$  for either soluble or insoluble films that, by virtue of its direct appeal to boundary-layer approximations, is not only simpler than its antecedents, but also is applicable to closed basins.

Davies and Vose (9) have made careful measurements of the damping of 150-cps capillary waves by various types of surface films. Their experiments were carried out in a "draught-free darkroom"; their apparatus was "cleaned with hot chromic acid followed by washings with syrupy phosphoric acid, hot tap water and distilled water" before each run. In their experiments on clean water, "the surface was cleaned prior to the observations by spreading ignited talc on the surface and then sucking off the talc and any contamination through a fine capillary," a procedure that was repeated between observations. They monitored surface tension continuously. They found that  $\alpha$  did approximate  $\alpha_S^{(0)}$  for clean water and also confirmed Dorrestein's prediction that  $\alpha$  increases from  $\alpha_S^{(0)}$  to approximately  $2\alpha_S^{(1)}$  and then decreases to the limiting value  $\alpha_S^{(1)}$  with increasing concentration of an insoluble surfactant. They also obtained qualitative confirmation of Dorrestein's calculations for the effect of surface viscosity. They found that  $\alpha_S$  was typically smaller for soluble surfactants provided that the relaxation frequency for solution was above that of the capillary waves; however, their results were not in complete agreement with the



predictions of Levich (we recall that Levich did not establish, at least in any systematic way, regimes of validity for his various approximations).

Benjamin and Ursell (2), Case and Parkinson (5), and Keulegan (6) all attributed the observed discrepancies between  $\alpha$  and  $\alpha_w$  to capillary hysteresis at the contact line of the meniscus, but none of them attempted a direct calculation of this effect. Although Benjamin and Ursell appear to have been the first to recognize the significance of capillary hysteresis for surface-wave damping, the basic phenomenon is well known in the literature of surface chemistry (see, e.g., Schwartz et al. (17)), and Adam (18) refers to A. A. Milne (19) for an everyday observation.

Keulegan (6) established that damping in a hydrophobic basin may be substantially larger than in a geometrically similar, hydrophilic basin. Subsequently, Keulegan and Brockman (20) carried out a series of measurements aimed at isolating the effect and concluded, at least tentatively, that both the advance and the recession of a meniscus are opposed by constant forces that depend only on the material properties of the three-phase interface. This conclusion agrees with that generally accepted in the literature of surface chemistry.

We have reexamined the results of Case and Parkinson (5) and Keulegan (6) and regard it as probable that both surface contamination and capillary hysteresis were significant in their experiments. We emphasize, however, that these two effects are not likely to be entirely independent, for both the modulus of compression for the surface film and capillary hysteresis at the meniscus depend on the contamination of the free surface through mechanisms that are far from being fully understood.

## FLUID MOTION

We consider a cylindrical container filled to a mean depth  $h$  with a slightly viscous liquid of density  $\rho$  and designate the upper surface of the liquid by  $S$  and the wetted boundary of the container by  $W$ . The velocity field associated with a wave motion on  $S$  will be essentially irrotational outside of viscous boundary layers at  $S$  and  $W$  and can be derived from a velocity potential, say  $\phi = \nabla\phi$ . Choosing cylindrical coordinates, with  $z$  measured vertically downward from the mean position of  $S$ , we pose this potential in the form (Lamb (4c))

$$\phi(x, y, z, t) = \psi(x, y) \operatorname{sech} kh \cosh [k(z - h)] e^{i\sigma t}, \quad (7)$$

where  $\psi$  is real and satisfies the two-dimensional Helmholtz equation

$$\nabla^2 \psi + k^2 \psi = 0. \quad (8)$$

The usual conventions hold in respect to complex, time-dependent variables, such as  $\phi$ .

The wave number  $k$  is determined (as a member of an infinite, discrete set of eigenvalues) by the requirement that the normal derivative of  $\psi$  vanish on  $W$ . The angular frequency,  $\sigma$ , is given by Eq. (3). We regard  $\sigma$  as real, with the

implication that  $\psi$  is a slowly varying function of time for free oscillations. We then account for the time dependence of  $\psi$  only indirectly, by introducing a dissipation function (see the section on Dissipation). The consequent errors in the kinematical description are  $O(\epsilon)$ .

The tangential component of the velocity in the boundary layers at S and W can be calculated by analogy with the flow over a flat plate that moves with the tangential velocity

$$\mathbf{v} = \nabla\phi_0 - \mathbf{u}, \quad (9)$$

where  $\nabla\phi_0$  denotes the tangential component of  $\nabla\phi$  at the boundary, and  $\mathbf{u}$  denotes the actual tangential velocity at the boundary ( $\mathbf{u} = 0$  on W). Evaluating  $\nabla\phi_0$  at  $z = 0$  from Eq. (7) and remarking that the velocity in the surface film must be parallel to  $\nabla\phi_0$ , say  $\mathbf{u} = (1 - C)\nabla\phi_0$ , we obtain

$$\mathbf{u} = (1 - C)\nabla\psi e^{i\sigma t}, \quad (10a)$$

$$\mathbf{v} = C\nabla\psi e^{i\sigma t} \quad (z = 0), \quad (10b)$$

where  $C$  is a constant that is determined (in next section) by the equality between the viscous shearing stress in the liquid and the tangential stress in the surface film. Following Stokes (3), who remarked that "the effect of [viscosity] may be calculated with a very close degree of approximation by regarding each element of the [boundary] as an element of an infinite plane oscillating with the same linear velocity," we obtain (cf. Landau and Lifshitz (21a))

$$\tau = -\rho(i\nu\sigma)^{1/2} \mathbf{v} \quad (11a)$$

$$= -C\rho(i\nu\sigma)^{1/2} \nabla\psi e^{i\sigma t} \quad (11b)$$

as the boundary-layer approximation to the viscous shearing stress

$$\left[ i^{1/2} \equiv 2^{-1/2}(1 + i) \right].$$

## SURFACE FILM

We neglect surface viscosity (a more detailed investigation reveals that surface viscosity is likely to be significant only for very short capillary waves) and assume that the surface tension is uniquely specified by the superficial concentration of the surface film, say  $T = T(\Gamma)$ . The shearing stress associated with small variations of  $\Gamma$  about the equilibrium concentration, say  $\Gamma_0$ , then is given by

$$\tau = \nabla T \approx (dT/d\Gamma)_0 \nabla(\Gamma - \Gamma_0). \quad (12)$$

The calculation of  $\Gamma - \Gamma_0$  for an insoluble film follows directly from the continuity equation

$$\partial\Gamma/\partial t + \Gamma\nabla \cdot \mathbf{u} = 0. \quad (13)$$

Linearizing Eq. (13) with respect to perturbations about  $\Gamma = \Gamma_0$  and  $\mathbf{u} = 0$ , we obtain

$$\Gamma - \Gamma_0 = -(i\sigma)^{-1} \Gamma_0 \nabla \cdot \mathbf{u} . \quad (14)$$

Substituting Eq. (14) into Eq. (12) and invoking Eq. (10a) and Eq. (8), we obtain

$$\boldsymbol{\tau} = -(i\sigma)^{-1} \Gamma_0 (dT/d\Gamma)_0 \nabla \nabla \cdot \mathbf{u} \quad (15a)$$

$$= (1 - C)(i\sigma)^{-1} \Gamma_0 (dT/d\Gamma)_0 k^2 \nabla \psi e^{i\sigma t} . \quad (15b)$$

Equating Eq. (15b) and Eq. (11b) and solving for  $C$ , we place the result in the form

$$C = \xi / (\xi - 1 + i) , \quad (16)$$

where

$$\xi = -\rho^{-1} \Gamma_0 (dT/d\Gamma)_0 (2/\nu)^{1/2} k^2 \sigma^{-3/2} . \quad (17)$$

We extend this last result to soluble surfactants on the hypothesis that the relaxation time for equilibrium between the material in the film and the dissolved material in the underlying liquid is negligible compared with  $2\pi/\sigma$  (there can be little departure from this assumption for gravity waves; cf. the relaxation times given by Davies and Vose (9)). We then can relate the changes in surface concentration to the corresponding changes in bulk concentration, say  $\gamma - \gamma_0$ , by the linearized equilibrium equation

$$\Gamma - \Gamma_0 \approx (d\Gamma/d\gamma)_0 (\gamma - \gamma_0) , \quad (18)$$

where  $\gamma$  satisfies the diffusion equation\*

$$\partial\gamma/\partial t = D\partial^2\gamma/\partial z^2 , \quad (19)$$

and the right-hand side of Eq. (13) is replaced by the transfer rate  $D(\partial\gamma/\partial z)_0$ ;  $D$  is the bulk diffusion coefficient. Solving this set of equations for  $\gamma - \gamma_0$  and  $\Gamma - \Gamma_0$ , we obtain results that differ from Eq. (14) and Eq. (15) only by an additional factor of  $[1 + 1/2(1 - i)\eta]^{-1}$  on the right-hand side of each, where

$$\eta = (2D/\sigma)^{1/2} (d\Gamma/d\gamma)_0^{-1} . \quad (20)$$

The corresponding generalization of Eq. (16) is

$$C = \xi / [\xi - 1 + i(1 + \eta)] . \quad (21)$$

The quantities  $\Gamma_0 (dT/d\Gamma)_0$  and  $(d\Gamma/d\gamma)_0$  must be determined either experimentally or from surface-chemistry considerations. Invoking the semiempirical equations of Szyszkowski and Langmuir for soluble surfactants (18a), we find that

\*This is a boundary-layer approximation ( $\nabla^2\gamma \approx \partial^2\gamma/\partial z^2$ ). Typically,  $D \ll \nu$ , so that the diffusion boundary layer lies inside of the viscous boundary layer.

## Damping of Gravity Waves by Surface Films

$$\Gamma_0(dT/d\Gamma)_0 = -AT(0) (\gamma_0/\gamma_1) \quad (21a)$$

$$\approx T(\Gamma_0) - T(0) \quad (\gamma_0 \ll \gamma_1) \quad (21b)$$

$$(d\Gamma/d\gamma)_0 = AT(0) (R\theta)^{-1} \gamma_1(\gamma_0 + \gamma_1)^{-2} \quad (22a)$$

$$\approx (A/R\theta\gamma_1) T(0) \quad (\gamma_0 \ll \gamma_1), \quad (22b)$$

where  $A$  is a dimensionless constant ( $A = 0.18$  for fatty acids),  $\gamma_1$  is that value of  $\gamma_0$  at which  $\Gamma_0$  attains half its saturation ( $\gamma_0 \rightarrow \infty$ ) value,  $R$  is the gas constant, and  $\theta$  is the absolute temperature. The results (Eqs. (21a,b)) are not, of course, applicable to insoluble surfactants.

## DISSIPATION

Let  $E_0$  be the total energy of the oscillation on the hypothesis  $\epsilon = 0$ , and let  $\alpha$  be the logarithmic decrement of the damped oscillation as  $\epsilon \rightarrow 0$ . Invoking the hypothesis of small oscillations, which implies that the energy must be quadratic in the amplitude, we infer that the mean energy must decay like

$$\langle E \rangle = E_0 \exp(-\alpha \sigma t / \pi), \quad (23)$$

where  $\langle \rangle$  implies an average over one cycle of the oscillation, say  $t = (0, 2\pi/\sigma)$ . Introducing the Rayleigh dissipation function

$$D = -\frac{1}{2} dE/dt, \quad (24)$$

equating its mean value to the corresponding mean value implied by Eq. (23), and anticipating that  $\alpha = O(\epsilon)$ , we obtain

$$\alpha = (2\pi/\sigma E_0) \langle D \rangle [1 + O(\epsilon)]. \quad (25)$$

The total energy of the undamped oscillation is equal to twice the mean kinetic energy and is given by

$$E_0 = \rho \iiint \langle \nabla \phi \rangle^2 dV \quad (26a)$$

$$= \frac{1}{2} \rho I k^{-1} \tanh kh, \quad (26b)$$

where

$$I = \iint (\nabla \psi)^2 dS = k^2 \iint \psi^2 dS. \quad (27)$$

The two forms of  $I$  are equivalent by Green's theorem.

The viscous dissipation in the boundary layer on  $S$ , say  $\dot{E}_1$ , is equal to that for a flat plate oscillating with the velocity  $\mathbf{v}$  and is given by (21b)



$$\dot{E}_1 = -\rho \frac{1}{2} \sigma \nu^{1/2} \iint_S \mathbf{v}^2 dS. \quad (28)$$

To this we must add the dissipation associated with the motion in the film, namely,

$$\dot{E}_2 = \iint_S \boldsymbol{\tau} \cdot \mathbf{u} dS. \quad (29)$$

Substituting  $\boldsymbol{\tau}$  from Eq. (11a) into Eq. (29), adding the result to Eq. (28), and averaging with respect to  $t$ , we obtain

$$\langle D_S \rangle = -\frac{1}{2} \langle \dot{E}_1 + \dot{E}_2 \rangle \quad (30a)$$

$$= \frac{1}{4} \rho \left( \frac{1}{2} \sigma \nu \right)^{1/2} \operatorname{Re} \iint_S \left[ |\mathbf{v}|^2 + (1+i) \mathbf{v} \cdot \mathbf{u}^* \right] dS \quad (30b)$$

$$= \frac{1}{4} \rho \left( \frac{1}{2} \sigma \nu \right)^{1/2} (C_r - C_i) I, \quad (30c)$$

where Eq. (30c) follows from Eq. (30b) by virtue of Eqs. (10a,b) and Eq. (27). Substituting Eq. (26b) and Eq. (30c) into Eq. (25) and invoking Eq. (1) and Eq. (2), we obtain

$$\alpha_S = \frac{1}{2} \pi \epsilon (C_r - C_i) \coth kh. \quad (31)$$

The contribution of the boundary layer on  $W$  to  $\alpha$ , namely  $\alpha_W$  can be derived by replacing  $S$  by  $W$  in Eq. (28), substituting  $\mathbf{v} = \nabla \phi_0$  from Eq. (7), and again invoking Eq. (25); the result is

$$\alpha_W = \frac{1}{2} \pi \epsilon \left\{ \frac{1}{2} (J + K) + [2 - kh (J - K)] \operatorname{csch} 2kh \right\}, \quad (32)$$

where

$$J = \frac{k}{I} \int_L \psi^2 dL, \quad (33a)$$

$$K = \frac{1}{kI} \int_L \left( \frac{\partial \psi}{\partial L} \right)^2 dL \quad (33b)$$

are essentially form factors.

The parameter  $C_r - C_i$  is a measure of the damping that owes its existence to the presence of the film; more precisely, it is the ratio of this damping to the damping that would be produced by an inextensible film ( $\xi = \infty$ , in which limit

$C_r - C_i \rightarrow 1$ ). Referring to Eq. (21), we obtain

$$C_r - C_i = \xi(\xi + \eta) / [(\xi - 1)^2 + (1 + \eta)^2], \quad (34)$$

which is plotted vs  $\xi$  in Fig. 1 for  $\eta = 0, 1, 2$ .

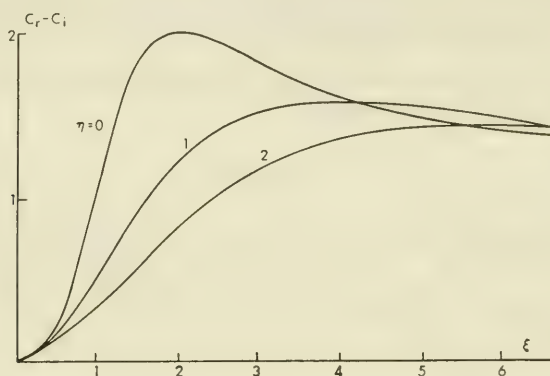


Fig. 1 - The surface-damping parameter  $C_r - C_i$  as given by Eq. (34)

The parameter  $\xi$  vanishes like  $\sigma^{5/2}$  as  $\sigma \rightarrow 0$  (i.e.,  $\xi \propto \sigma^{5/2}$  for long gravity waves)  $\sigma^{-1/6}$  as  $\sigma \rightarrow \infty$  (very short capillary waves) and is typically smaller for gravity waves than for capillary waves. The parameter  $\eta$  varies monotonically like  $\sigma^{-1/2}$  and therefore is more important for gravity waves. This is consistent both with the observations of Davies and Vose (9), who found that bulk solubility of the film material has only a slight effect on the damping of capillary waves in the absence of relaxation effects (i.e., in the frequency range for which Eq. (18) is valid), and with the empirical (and ancient) observation that soluble oils are less effective in the damping of gravity waves than are insoluble oils. We remark that  $\langle \dot{E}_2 \rangle$  is proportional to  $\eta$  and therefore vanishes for an insoluble film; more generally,

$$\langle \dot{E}_2 \rangle / \langle \dot{E}_1 \rangle = (C_r - C_i - |C|^2) / |C|^2 = \eta / \xi. \quad (35)$$

The maximum value of  $C_r - C_i$  is 2 and is attained for  $\xi = 2$  and  $\eta = 0$ . Dr. T. B. Benjamin (private communication) has pointed out that the film velocity  $\mathbf{u}$  is in quadrature with the irrotational velocity  $\nabla \phi_0$  at this point, in consequence of which  $|\mathbf{v}|^2 = 2 |\nabla \psi|^2$ , thereby doubling the dissipation rate for an inextensible film, for which  $\mathbf{u} = 0$  and  $|\mathbf{v}|^2 = |\nabla \psi|^2$ .

## WIND-GENERATED WAVES

We now consider the effect of an inextensible surface film on the generation of deep-water gravity waves by wind. That this effect is substantial is a matter

of common knowledge (see, e.g., Franklin (10)), and both Keulegan (22) and Van Dorn (23) found that the addition of detergent prevented the formation of sensible waves by winds of at least 12 m/sec. There appears to be little doubt that the primary effect of a surface film is to increase the dissipation, and hence the energy that must be supplied by the wind, although associated changes in the velocity profile of the wind also could be significant. (Dorrestein (14) suggested that a surface film inhibits the transfer of energy from wind to waves through tangential stresses, but it is now generally accepted that this transfer takes place primarily through normal pressures.)

We first assume that the aerodynamic energy transfer can be described by the (equivalent) logarithmic decrement (24)

$$-\alpha_a = \pi s (U_1/c)^2 \beta, \quad (36)$$

where  $s = \rho_a/\rho_w$  is the air/water density ratio,  $c = \sigma/k$  is the wave speed,  $U_1$  is a reference wind speed, and  $\beta$  is a dimensionless energy-transfer coefficient that depends on  $c/U_1$ . Equating  $-\alpha_a$  to  $\alpha_S^{(1)}$ , as given by Eq. (6), and solving for  $U_1$  as a function of  $c/U_1$ , we obtain

$$U_1 = (g\nu/2s^2)^{1/3} (c/U_1)^{1/3} \beta^{-2/3} \quad (37a)$$

$$= 150 (c/U_1)^{1/3} \beta^{-2/3} \text{ cm/sec}, \quad (37b)$$

where Eq. (37b) follows from Eq. (37a) after setting  $g = 980$ ,  $\nu = 10^{-2}$  and  $s = 1.2 \times 10^{-3}$  in cgs units. Invoking the theoretical model (24,25) for an equivalent laminar profile of the form  $U = U_1 \log y + \text{const.}$ , we find that the minimum value of  $U_1$  predicted by Eq. (37b) is about 1 m/sec at  $c/U_1 \approx 3$ , corresponding to a nominal wind speed of about 12 m/sec and a wavelength of about 6 m. The predicted critical wind speed of 12 m/sec is at least consistent with the observations of Keulegan (22) and Van Dorn (23) and is roughly an order of magnitude larger than that predicted by a similar calculation based on  $\alpha_S^{(0)}$  — namely,  $U_1 = 14\text{--}15$  cm/sec for gravity waves of 20–30 cm (25).

A rather different energy-transfer process, which depends on the existence of a viscous sublayer for the wind structure very close to the water, may be effective for  $c/U_1 < 3$  (26). Hidy (27) has obtained qualitative confirmation of the theoretical model, but found it necessary to postulate increased dissipation, relative to that based on  $\alpha_S^{(0)}$ , in order to obtain approximate, quantitative agreement. He commented that, "Although care was taken to minimize the contamination of the water surface . . . , small amounts of oil may have been unavoidably present [and] could easily account for the systematic deviation between [the theoretical and experimental results] shown in the data." The original theoretical predictions (26) were carried out for a wind profile that is linear in the viscous sublayer and logarithmic above this layer with a free-surface damping given by  $\alpha_S^{(0)}$ . Replacing  $\alpha_S^{(0)}$  by  $\alpha_S^{(1)}$  in these calculations, we find that waves of roughly 4–10 cm in length should be generated by a wind with a friction velocity  $U_*$  of 15 cm/sec, corresponding to a nominal wind speed of roughly 4–5 m/sec. This prediction is consistent with the observations of Hidy, but not with those of Keulegan (22).

## CONCLUSIONS

We conclude that, in the absence of very special precautions, surface contamination may contribute significantly to the damping of surface waves. The preceding analysis provides a rational estimate of the corresponding logarithmic decrement provided that the required parameters, especially the surface compressional modulus, are known. Further observations, including controlled experiments guided by the theoretical predictions, are necessary before we can say whether a reliable understanding and ability to predict have been achieved.

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# GENERATION OF WAVES BY TURBULENT WIND

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## ABSTRACT

Interactions between random gravity waves and the turbulent atmosphere boundary layer can be treated by an extension of wave-wave interaction theory. The energy transfer resulting from various interaction combinations can be characterized by transfer diagrams corresponding to the Feynman diagrams of wave-wave interactions. The complete set of lowest order transfer diagrams is shown to include Miles' and Phillips' theories of wave generation and a further set of wave-turbulence interactions which have not been considered previously. The predicted wave growth for the various interactions is compared with existing measurements. But a conclusive answer to the question of wave generation must await more detailed experiments correlating wave measurements with turbulence measurements in the atmosphere boundary layer.

## INTRODUCTION

The fundamental papers of Miles (1) and Phillips (2) marked an important advance in our understanding of the basic processes by which waves can be generated by wind. It has been widely hoped that the growth of waves could be largely accounted for by a superposition of these two processes. However, it is known that although the theories describe independent aspects of the wave-atmosphere interaction, they are not entirely complementary; there remain interactions which are included in neither theory. Miles considers the coupling of the wave field with the mean atmospheric boundary-layer flow, but ignores the wave-turbulence interactions. These are similarly neglected in Phillips' treatment, which is concerned only with the external excitation of the waves by the undisturbed turbulent field.

Apart from the general theoretical interest, a complete theory of wave-atmosphere interactions has now become particularly desirable through the field

study of Snyder and Cox (3), which indicates that the combined Miles-Phillips mechanisms fail to account for the observed wave growth by almost an order of magnitude.

We shall attempt here to outline a complete theory based on a systematic expansion of the coupled fields. Besides the Miles and Phillips processes, the theory yields three further processes at lowest order: a nonlinear interaction with the mean boundary-layer flow and two forms of wave-turbulence interaction.

The problem may be divided into two parts: the analysis of the coupling between the wave field and the turbulent boundary-layer flow, and the determination of the energy transfer due to the coupling. The first part concerns the details of the interaction expansion. The second part may be regarded as a particular application of a general transfer theory for random wave fields in weakly coupled systems.

## WAVE-ATMOSPHERE INTERACTIONS

We present here only the general structure of the interaction analysis; a detailed derivation is given in Ref. 4. Let  $\zeta$  be the surface displacement,  $\mathbf{u} = \mathbf{U} + \mathbf{u}'$  be the turbulent velocity field in the atmosphere, consisting of a mean flow  $\mathbf{U}$  and a superimposed fluctuating component  $\mathbf{u}'$  of zero mean, and  $\delta\mathbf{u}$  be the wave-induced perturbation of the turbulent velocity field. We assume that all fields are statistically homogeneous, so that they may be represented as a superposition of mutually statistically orthogonal Fourier components of amplitude  $\zeta_{\mathbf{k}}$ ,  $\mathbf{u}'_{\mathbf{k}}$ ,  $\delta\mathbf{u}_{\mathbf{k}}$ , where  $\mathbf{k}$  is the two-dimensional, horizontal wavenumber vector.

The equations of motion of the coupled wave-atmosphere system may then be expressed in the form

$$L(\delta\mathbf{u}_{\mathbf{k}}) = Q(\mathbf{u}'_{\mathbf{k}}, \delta\mathbf{u}_{\mathbf{k}}) \quad (z > 0), \quad (1)$$

$$\delta\mathbf{u}_{\mathbf{k}} = R[\mathbf{u}'_{\mathbf{k}}, \zeta_{\mathbf{k}}] \quad (z = 0), \quad (2)$$

$$\ddot{\zeta}_{\mathbf{k}} + \sigma^2 \zeta_{\mathbf{k}} = S[\zeta_{\mathbf{k}}, \mathbf{u}'_{\mathbf{k}}, \delta\mathbf{u}_{\mathbf{k}}] \quad (z = 0), \quad (3)$$

where  $\sigma = (gk \tanh kH)^{1/2}$ ,  $g$  is the gravitational acceleration,  $H$  is the water depth,  $z$  is the vertical coordinate, measured positive upward,  $L$  represents a linear (essentially the Orr-Sommerfeld) operator, and  $Q$ ,  $R$ , and  $S$  are nonlinear functionals of the coupled fields.

The wave-atmosphere interactions are proportional to the air-to-water density ratio and are therefore weak. This suggests a solution by iteration. To first order, the forcing function  $S$  in the harmonic-oscillator equation (Eq. (3)) can be neglected, yielding a stationary wave field of free, sinusoidal waves. The free-wave field can then be substituted in the boundary condition (Eq. (2)), which together with Eq. (1) determines the wave-induced velocity field  $\delta\mathbf{u}$ . Substitution of the solution  $\delta\mathbf{u}$  in the forcing function  $S$  then determines a second-order solution for the wave field, and so forth.



Miles and Phillips introduce additional simplifications. Phillips ignores the dependence of  $s$  on the fields  $\zeta_k$  and  $\delta u_k$ , so that Eq. (3) reduces to a purely external excitation by the turbulence field  $u'$  (physically,  $s$  corresponds in this case to the unmodified turbulent surface pressure). Miles ignores the nonlinear term  $Q$  in Eq. (1) (i.e., the wave-induced perturbation of the turbulent Reynolds stress) and the  $u'$ -dependence of  $R$  in Eq. (2). This reduces Eqs. (1) and (2) to linear, constant-coefficient equations, and the boundary-value problem of determining  $\delta u$  becomes tractable.

If the Miles approximation is regarded as an acceptable first-order solution, the general case can be approached by introducing a second iteration loop in which  $Q$  and the  $u'$  dependence of  $R$  are treated as further perturbations. The  $n$ th iteration is obtained by solving Eqs. (1) and (2) with the  $(n-1)$ th iteration substituted in the right-hand sides. In this manner, the wave-induced velocity field is obtained as a power series in the components  $\zeta_k$  and  $u'_k$ , and the forced-harmonic-oscillator equation (Eq. (3)) takes the form

$$\ddot{\zeta}_k + \sigma^2 \zeta_k = P_k + B_k \zeta_k + \sum_{k_1 + k_2 = k} C_{k_1 k_2} \zeta_{k_1} u'_{k_2} + \dots \quad (4)$$

The first two terms on the right correspond to the Phillips and Miles approximations, respectively;  $P_k$  denotes the external forcing term due to random turbulent pressure fluctuations; and  $B_k$ ,  $C_{k_1 k_2}$ ,  $\dots$  are coupling coefficients, which are determined by solving the Orr-Sommerfeld equation, Eq. (1), under boundary condition (2). In general, this is possible only by numerical methods or by restriction to simple boundary-layer models. (It is known that a simple constant-velocity or constant-slope profile is inadequate in the Miles approximation, in which the energy transfer is determined by the local profile curvature at the critical layer. However, the detailed properties of the velocity profile are probably less important for the higher order processes.)

Correlation measurements of wave height and surface pressure by Longuet-Higgins et al. (5) indicate that the Miles approximation does indeed yield a reasonable first-order description of the wave-induced fluctuations in the atmosphere. It should be noted, however, that this does not necessarily apply to the Miles transfer expression. In Miles' approximation, the wave-induced pressure fluctuations are almost 90 degrees out of phase with the wave height over most of the wave spectrum, so that only a small fraction of the pressure field is effective in generating waves. It is therefore conceivable that the higher order pressure fluctuations, although smaller in absolute magnitude, lead to a larger energy transfer.

## THE ENERGY TRANSFER

After determining the coefficients of the interaction expansion, the problem remains of evaluating the energy transfer resulting from the coupled equations (Eqs. (4)). The analysis is basically straightforward but involved algebraically. We summarize here only the results, referring to Refs. 4 and 6 for details. The problem may be regarded as a generalization of the theory of wave-wave interactions, first considered by Peierls (7) in his classic study on the heat conduction



in solids and now developed to a standard scattering formalism in various fields of physics. The theory has recently also found a number of geophysical applications (8).

The energy transfer arises from interactions between combinations of Fourier components whose wave numbers  $k_1, \dots, k_n$  and frequencies  $\omega_1, \dots, \omega_n$  satisfy the transfer conditions

$$\sum_{j=1}^{n-1} s_j k_j = k_n \quad (5)$$

and

$$\sum_{j=1}^{n-1} s_j \omega_j = \omega_n, \quad (6)$$

where  $s_j = \pm 1$ . Equation (5) follows from the homogeneity of the physical system and applies to all interactions. An energy transfer between the interacting components occurs only if the additional *resonance* condition of Eq. (6) is also satisfied.

The net energy transfer is found by summing the contributions from all combinations of resonant interactions. The final expression consists of a number of integrals containing various spectral products, which can conveniently be divided into two classes. Integrals in which the transfer conditions of Eqs. (5) and (6) occur as  $\delta$  factors are associated with *scattering* processes, the remaining integrals are associated with *parametric* processes.

To distinguish between the various transfer terms, it is further convenient to introduce a notation based on *transfer* diagrams. The transfer diagram for a scattering process consists of a number of wavenumber vectors  $k_1, \dots, k_{n-1}$  entering a vertex and a single wave component  $k_n$  leaving the vertex. The components satisfy the transfer conditions of Eqs. (5) and (6). Components associated with a negative sign  $s_j = -1$  are indicated by a cross stroke.

The transfer diagram for a parametric process consists only of ingoing components. There are no side conditions on the wavenumbers or frequencies. Parametric processes occur only in interacting systems in which the total energy and momentum of the wave fields are not conserved. They have no counterpart in the theory of wave-wave interactions, but there is a close analogy with the interactions occurring in nonlinear parametric amplifiers.

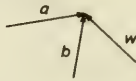
The structure of the various transfer expressions can be deduced from the transfer diagrams with the aid of a single transfer rule: the rate of change of the energy spectrum of any wave component in a transfer diagram is proportional to the product of the spectral densities of the ingoing components. Thus for any interacting system, the set of all transfer expressions for a particular wave field  $w$  is obtained by applying the transfer rule to all wave components  $w$  in all possible transfer diagrams (Fig. 1).

## Generation of Waves by Turbulent Wind



$$\frac{\partial F_n}{\partial t} = \int T_n F_a F_b F_1 \delta(k_a - k_b + k_1 - k_2) \delta(w_a - w_b + w_1 - w_2) dk_a dk_b dk_{\bar{n}} \quad (n=1,2; \bar{n}=2,1)$$

(i)



$$\frac{\partial F_w}{\partial t} = \int T_w F_a F_b F_w dk_a dk_b$$

(ii)

Fig. 1 - Examples of transfer diagrams and transfer expression for (i) a third-order scattering process and (ii) a third-order parametric process.  $a$  and  $b$  represent arbitrary field components;  $w_1$ ,  $w_2$ , and  $w$  represent wave components;  $T_n$ ,  $T_w$  are transfer functions; and the indices  $n$ ,  $\bar{n}$  refer to the wave components  $w_1$ ,  $w_2$ .

It is important to note that the transfer diagrams reflect only the structure of the transfer expressions. They are normally not directly related to the basic component-interactions responsible for the energy transfer. Thus although all transfer expressions are due entirely to resonant interactions, the resonant interaction conditions of Eqs. (5) and (6) occur only in the scattering, not the parametric transfer diagrams. The structure of the interaction analysis can be summarized independently in terms of *interaction* diagrams (6). (However, for conservative wave-wave interactions, the interaction and transfer diagrams are very simply interrelated (8).)

## APPLICATION TO WAVE-ATMOSPHERE INTERACTIONS

The complete set of lowest-order transfer diagrams in the case of wave-atmosphere interactions are shown in Fig. 2. The linear interaction with the mean boundary-layer flow according to Miles appears as the degenerate parametric diagram (i). Phillips' external excitation by the atmospheric turbulence field is represented by the diagrams (iii). (If the external field is expressed in terms of the turbulent pressure  $p^t$  instead of the turbulent velocity components  $u$ , these reduce to a single linear diagram.) The remaining processes represent a nonlinear interaction with the mean boundary-layer flow,

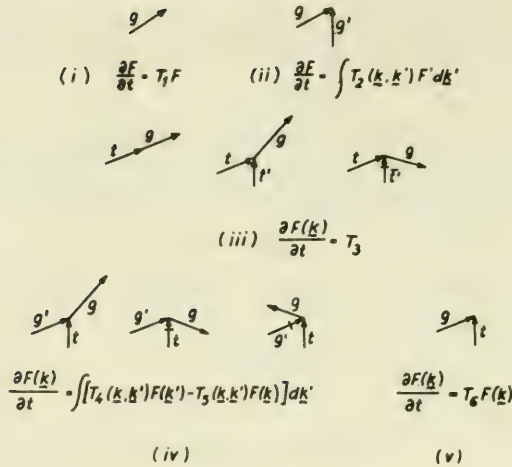


Fig. 2 - Lowest order transfer diagrams and transfer expressions for wave-atmosphere interactions: (i) Miles, (ii) nonlinear interaction with mean wind, (iii) Phillips, (iv) wave-turbulence scattering processes, and (v) wave-turbulence parametric process. The components  $g$ ,  $t$ , and  $p$  represent gravity-wave, turbulent-velocity and turbulent-pressure components, respectively.

diagram (ii), and wave-turbulence interactions, diagrams (iv) and (v). The transfer expressions derived from the transfer rule are also shown. Only the dependence on the wave spectrum is given explicitly. The transfer functions  $T_1, \dots, T_6$  depend on the coupling coefficients, and in the case of diagrams (iii), (iv), and (v), on the atmospheric turbulence spectra. The expressions are given in full in Ref. 4.

## THE PRESSURE SPECTRA

Figure 3 shows schematically a two-dimensional  $k_1 - \omega$  section of the three-dimensional surface pressure spectra  $F_p(k, \omega)$  associated with the various transfer processes. The scattering processes (iii) and (iv) of Fig. 2 correspond to three-dimensional pressure distributions, whereas the parametric processes (i), (ii), and (v) yield two-dimensional distributions concentrated on the gravity-wave dispersion surfaces  $\omega = \pm \sigma(k) = \pm (gk \tanh kh)^{1/2}$ . Only the pressure fluctuations in resonance with free gravity waves, i.e., on the dispersion surface, transfer energy to the wave field.

The three-dimensional turbulent pressure distribution is concentrated about the "convection surface"  $\omega + k_1 U_m = 0$ , where  $U_m$  is the mean



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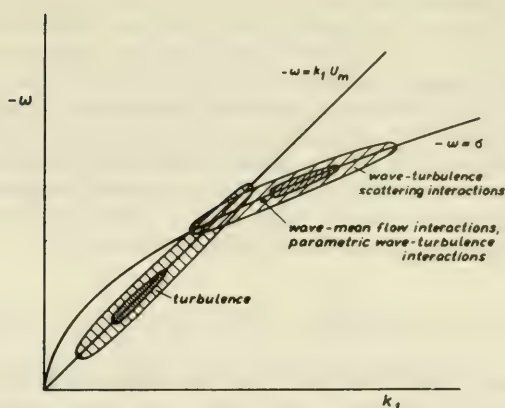


Fig. 3 - Two-dimensional  $k_1$ - $\omega$  section of the three-dimensional surface pressure spectra  $F_p(\mathbf{k}, \omega)$ . (The mean wind  $U_m$  is parallel to the  $x_1$  axis.)

("anemometer") wind speed and  $k_1$  is the wavenumber component parallel to the wind. The distribution follows from Taylor's hypothesis, according to which the wavenumber and frequency spectra of a turbulence field are approximately related as though the turbulence were a frozen special pattern convected bodily downstream with the mean flow velocity. It follows that the energy transfer due to Phillips' mechanism is appreciable only where the convection surface and dispersion surface intersect, i.e., for gravity waves whose phase velocities in wind direction are approximately equal to the wind speed (2) (see also Ref. 9 for the present interpretation of Phillips' result).

Linear wave interactions with the mean wind lead to pressure fluctuations of the same wavenumber and frequency as the wave components. Miles' pressure spectrum is therefore represented by a two-dimensional distribution on the dispersion surface.

The parametric processes (ii) and (v) also corresponds to two-dimensional pressure distributions. The process (ii) is due to a cubic wave-wind interaction involving a wave component  $(\mathbf{k}, \sigma)$  and a complex conjugate pair of wave components  $(\mathbf{k}', \sigma')$ ,  $(-\mathbf{k}', -\sigma')$ . This leads to a pressure fluctuation with the wavenumber and frequency of the first wave component. Process (iv) is due to a similar cubic interaction between a wave component  $(\mathbf{k}, \sigma)$  and a complex conjugate pair of turbulence components  $(\mathbf{k}', \omega')$ ,  $(-\mathbf{k}', -\omega')$ , again producing a pressure fluctuation of wavenumber  $\mathbf{k}$  and frequency  $\sigma$ .

The scattering processes (iv) are associated with quadratic interactions between wave components  $(\mathbf{k}, \sigma)$  and turbulence components  $(\mathbf{k}', \omega')$ . In this case the induced pressure fluctuations can have arbitrary wavenumbers  $\mathbf{k} + \mathbf{k}'$  and frequencies  $\sigma + \omega'$ , and the spectrum is a three-dimensional continuum.



Since atmospheric turbulence spectra are normally peaked at much lower wave-numbers and frequencies than wave spectra, it may be expected that for most interactions  $k' \ll k$ ,  $\omega' \ll \sigma$ , so that the resultant pressure distribution lies rather close to the dispersion curve. Furthermore, the maximum of the distribution will lie close to the wave spectral maximum. The strongest wave generation may therefore be expected for frequencies close to the wave spectral peak, in accordance with the observed sequential development of the wave spectrum from high to low frequencies, the waves growing only in a narrow frequency band about the momentary wave peak. (However, other explanations of the sequential wave growth have also been suggested.)

## CONCLUSIONS

The recent field study of Snyder and Cox (3) indicate that both Miles' and Phillips' theories are incapable of explaining the wave growth observed in the ocean, strongly suggesting that one or more of the remaining lowest-order processes, in particular the wave-turbulence interactions, are the principal source of wave energy. However, the question of wave generation must be regarded as open until further measurements and transfer computations have been made. Although a complete theory of expansible interactions has been developed, the expansions are valid only for weak spacially uniform interactions. Strong, local effects, such as flow separation at the wave crests, are therefore not included in the theory.

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Tuesday, October 4, 1966

Morning Session

## **SHIP-GENERATED WAVES AND SHIP WAVE RESISTANCE**

Chairman: G. P. Weinblum

Institut für Schiffbau der Universität Hamburg  
Hamburg, Germany

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# SECOND-ORDER WAVE THEORY FOR SUBMERGED TWO-DIMENSIONAL BODIES

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## ABSTRACT

The main object of this work has been to investigate the importance of nonlinear free-surface effects on flow past submerged two-dimensional bodies. For a foil-shaped body the surface waves and the wave resistance have been obtained both experimentally and analytically, including second-order effects.

Second-order wave theory has previously only been applied to submerged circular cylinders and only the wave resistance and the lift have been computed correct to the second order. This work therefore presents the first computations of wave profiles correct to the second order in wave amplitude for free-surface flow past a body. It is also the first time an experimental verification of the second-order wave theory has been carried out.

The wave profiles have been measured by a capacitance wire, and the wave resistance data have been obtained by three techniques: (a) drag measurements, (b) wave survey, accounting for second-order effects, and (c) second-order wave theory.

The work clearly proves the importance of the effect of nonlinearity at the free surface. At low speeds the second-order theory predicts wave heights several times the size given by the linear theory.

The results show, for low and moderate speeds, amazingly good agreement between second-order theory and experiments both for the wave resistance and the wave elevations. Especially encouraging is the excellent agreement at the very low speed, where the difference between linear and second-order theory is large. The theory gives less satisfactory agreement with experiments at high speeds, however.

Useful information and data are included on wave breaking for small submergences.



## INTRODUCTION

The determination of the wave resistance of ship hulls is one of the most important and challenging problems the naval architect has to contend with. In the last century, many outstanding scholars in the field have been engaged in this subject; moreover, the last several years have brought a renewed interest in the theory of ship waves. Due to the complexity of the problem, however, we are by no means able to predict theoretically the wave resistance of a ship with sufficient accuracy.

The three methods in use for obtaining the wave resistance are (a) model testing, subtracting the estimated viscous drag from the measured total drag force, (b) wave survey behind the model, assuming small waves and using linear wave theory, and (c) theoretical analysis, using linear theory and representing the hull by some singularity distribution. The correlations between these three techniques have been far from satisfactory. In many cases, the differences are as large as 40 to 50 percent (1). Fairly good agreements have been achieved only by introducing artificial correction factors; however, very little is known in general about how these factors are related to physical parameters.

The author believes that a main part of the discrepancies between analytical and experimental results could be due to the neglect of nonlinear effects at the free surface and that the viscous effect is probably not as important as often stated. This nonlinearity is investigated here, and in particular its effect on the wave resistance.

The first nonlinear wave theory for irrotational flow of an ideal fluid was derived by G. G. Stokes (2). Applying this theory to the problem of flow past submerged two-dimensional bodies, one can easily relate the wave resistance of the body to the far downstream wave height. On the other hand, no relationship between the shape of the body and the wave elevation or the wave resistance can be obtained from the Stokes wave. Appendix A gives a detailed discussion of "Stokes Waves and their Application to Wave Resistance Problems."

Only two investigators have applied a consistent second-order wave theory to the problem of free-surface effects on the flow past bodies: M. Bessho (3) and E. O. Tuck (4). Both restricted themselves to the simplified two-dimensional case of a submerged circular cylinder. Bessho in his fine work derived correctly the complex potential but unfortunately obtained the forces incorrectly such that the most important higher-order term was not included. His final result and many of his conclusions are therefore incorrect. Tuck, on the other hand, correctly obtained the wave resistance and the lift for the circular cylinder, and he also correctly stated the very opposite conclusion of Bessho, namely, that for a circular cylinder "it is more important to correct for nonlinearity at the free surface than for the fact that the boundary condition is not satisfied exactly by the first approximation on the body surface." This excellent paper by Tuck appears to be the only work to date in which the effect of nonlinearity at the free surface has been treated correctly to the second order.

Certain features of Tuck's results, however, indicate that the validity of this second-order theory may be questionable. The three following points will be mentioned here:

1. At a submergence equal to twice the diameter and at Froude number  $U/\sqrt{gb} = 0.55$ , the second-order theory gives twice the wave resistance obtained by linear theory, and at an even smaller Froude number,  $U/\sqrt{gb} = 0.47$ , his work predicts the resistance three times the linear theory. Results of this nature are not in agreement with the assumed converging perturbation series

$$w = \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \dots$$

and in this speed range at least, it is therefore doubtful if his theory is applicable.

2. The fact that the difference between the linear and second-order theory increases as the Froude number decreases may indicate that the second-order wave resistance approaches infinity as the Froude number approaches zero. Unfortunately, no data are shown by Tuck for Froude numbers smaller than 0.47.

3. The largest submergence investigated by Tuck was twice the diameter of the cylinder. When the body is as close as that to the free surface, the waves created by the cylinder will break, resulting in a highly nonlinear phenomenon which cannot be treated by second-order perturbation theory.

In addition to these points, it should be mentioned that the circular cylinder lends itself nicely to mathematical treatment and is therefore very useful for a preliminary mathematical investigation of nonlinear free-surface effects. But we must keep in mind that when assuming inviscid fluid flow past a *circular* cylinder we should not expect the mathematical solution to be "a good approximation to the true wave resistance" (as hoped by Tuck).

The object of this work has been to investigate the accuracy of second-order wave theory by a comparison of analytical and experimental results, and also to clear up some of the uncertainties in previous work. To perform such a comparison the author has applied second-order wave theory to a streamlined two-dimensional body and conducted experiments on an 11-foot-long strut of 13-inch chord length. Wave resistance data have been obtained by three techniques: (a) drag measurements, (b) wave survey, accounting for second-order effects, and (c) second-order theory. In addition to these data, the wave profiles have also been measured and computed for the selected body shape. This is the first time wave profiles correct to the second order in wave amplitude have been computed for flow past a body.

## MATHEMATICAL FORMULATION

An infinitely long cylinder is supposed to move with a constant velocity  $U$  in a direction perpendicular to its axis and at a fixed distance below the free surface. The problem is to determine the surface waves and the wave resistance.

The flow will be treated as steady in a coordinate system moving with the cylinder. A two-dimensional coordinate system will be used, with the  $y$  axis vertical, and the  $x$  axis a distance  $b$  below the undisturbed free surface. The direction of decreasing  $x$  coincides with the direction of motion of the cylinder. The fluid will be assumed to be inviscid, incompressible, and without surface tension. It will also be assumed that the flow is irrotational and that the disturbances at the free surface are all small.

The assumption of irrotationality ensures the existence of a velocity potential  $\Phi(x, y)$  from which the velocity field can be derived, i.e.

$$\mathbf{v} = \text{grad } \Phi = \Phi_x \mathbf{i} + \Phi_y \mathbf{j}. \quad (1)$$

In view of the equation of continuity, the velocity potential  $\Phi$  is a solution of the Laplace equation

$$\nabla^2 \Phi = \Phi_{xx} + \Phi_{yy} = 0. \quad (2)$$

Another significant consequence of the irrotational character is obtained by integrating the equation of motion, which leads to the Bernoulli equation:

$$\frac{1}{2} |\text{grad } \Phi|^2 + \frac{p}{\rho} + gy = C_1, \quad (3)$$

where  $p$  is the pressure,  $g$  is the gravitational acceleration, and  $\rho$  is the density.

There are two boundary conditions to be satisfied on the free surface. The kinematic condition states that any particle which is on the surface remains there. Thus if  $y = b + \eta(x)$  is the equation of the free surface and  $\Psi(x, y)$  the stream function, then

$$\Psi[x, b + \eta(x)] = \text{constant} = bU \quad (4)$$

on the free surface. The dynamical condition requires that the pressure above the free surface is constant, here set equal to zero. By Eq. (3), the Bernoulli equation, it therefore follows that

$$\frac{1}{2} |\text{grad } \Phi|^2 + gy = \text{constant on } y = b + \eta(x). \quad (5)$$

The nonlinear conditions above are often referred to as the "exact"\* free-surface conditions.

The boundary condition to be satisfied at the wall of the cylinder is clearly

$$\frac{\partial \Phi}{\partial n} = 0, \quad (6)$$

---

\*In this work, "exact" in quotation marks refers to exact within the potential flow theory.



in which  $\partial/\partial n$  represents differentiation along the normal to the surface wall. For two-dimensional bodies with a sharp trailing edge the Kutta-Jukowski condition specifying that the trailing edge is the stagnation point should also be satisfied.

The depth of the water is assumed to be infinite, which results in the condition

$$\lim_{y \rightarrow -\infty} (\text{grad } \Phi) = U i. \quad (7)$$

One additional condition is necessary to ensure uniqueness, namely the absence of waves far upstream. To make the list of boundary conditions complete, we should also mention the absence of wave reflection far downstream.

Introducing the complex variable  $z = x + iy$  and the complex velocity potential  $W(z) = \Phi(x, y) + i\Psi(x, y)$  and assuming that a perturbation about the flow past the body in an infinite fluid can be carried out, we write the expansion

$$W(z) = \left( Uz + w_{B_0} \right) + \epsilon \left( w_{F_1} + w_{B_1} \right) + \epsilon^2 \left( w_{F_2} + w_{B_2} \right) + \dots, \quad (8)$$

where  $\epsilon$  is some physical parameter which vanishes with the disturbance at the free surface. The potentials  $w_{B_0}$ ,  $w_{B_1}$ , etc., are analytic everywhere outside the body and are chosen so that the terms in parentheses in Eq. (8) satisfy the cylinder-wall condition exactly. On the other hand,  $w_{F_1}$  and  $w_{F_2}$  are analytic for all  $\text{Im } z < b$  and are the first-order and second-order free-surface contributions.

Assuming that the free-surface disturbances are small, of order  $\epsilon$ , we have the following expansion for the surface elevation:

$$\eta(x) = \epsilon \eta^{(1)}(x) + \epsilon^2 \eta^{(2)}(x) + \dots. \quad (9)$$

It also follows from the assumption of small disturbances at the free surface that the body singularities  $w_{B_0}$ ,  $w_{B_1}$ , etc., are of one order higher in  $\epsilon$  near the free surface than near the body. The expansion near the free surface, therefore, is of the form

$$W(z) = Uz + \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \dots, \quad (10)$$

with

$$\begin{aligned} \epsilon w^{(1)} &= w_{B_0} + \epsilon w_{F_1}, \\ \epsilon^2 w^{(2)} &= \epsilon w_{B_1} + \epsilon^2 w_{F_2}. \end{aligned}$$

Substitution of the expansion given by Eq. (10) in the "exact" free-surface conditions of Eqs. (4) and (5) and including only first-order terms gives the first-order free-surface condition



$$\operatorname{Re} \left[ w_z^{(1)} + i\nu w^{(1)} \right] = 0 \quad \text{at} \quad z = x + ib, \quad (11)$$

where  $\nu = g/U^2$  is the wavenumber. Including terms of second-order, one obtains after some manipulation that the second-order free-surface condition is

$$\operatorname{Re} \left( w_z^{(2)} + i\nu w^{(2)} \right) = \frac{1}{\rho U} f(x) \quad \text{at} \quad z = x + ib, \quad (12)$$

where

$$\frac{1}{\rho} f(x) = \frac{1}{2} \left| w_z^{(1)} \right|^2 - \operatorname{Im} w^{(1)} \operatorname{Re} \left[ i w_{zz}^{(1)} - \nu w_z^{(1)} \right].$$

It also follows from these substitutions that the corresponding approximations to the wave height are

$$\eta^{(1)}(x) = \frac{1}{U} \psi^{(1)}(x, b), \quad (13a)$$

$$\eta^{(2)}(x) = \frac{1}{U} \psi^{(2)}(x, b) + \frac{1}{U^2} \psi^{(1)}(x, b) \psi_y^{(1)}(x, b). \quad (13b)$$

In this work the complex potential will be used only to determine the wave elevation and to obtain the wave resistance from momentum considerations in the far field. We will not need, therefore, the general expansion given by Eq. (8) but only the expansion given by Eq. (10), which is valid near the free surface and in the far field. The expansions and boundary conditions to be used for the more general problem are discussed by Giesing (5). It is seen from Eq. (10) that we have two second-order terms,  $\epsilon_{wB_1}$  and  $\epsilon_{wF_2}^2$ . Tuck (4) has shown that in obtaining the wave resistance for a circular cylinder it is more important to include the second-order free-surface term  $\epsilon_{wF_2}^2$ , than the body correction in term  $\epsilon_{wB_1}$ . Assuming that the same applies to the problem treated here, we will disregard the term  $\epsilon_{wB_1}$  and simply represent the body by its singularity distribution in an infinite fluid. This implies that the cylinder-wall condition is satisfied only to the first order of approximation.

It can certainly be argued that the second-order theory presented here is not consistent; however, it is believed that the method used will give a solution far more realistic than the first-order theory and therefore that it is an improvement over the linear theory.

## MATHEMATICAL SOLUTION

### Complex Potential and Wave Elevation

The two-dimensional submerged body will be represented mathematically by a singularity distribution of the following strength and location: eleven sources equally spaced along the  $x$  axis between  $x = 0$  and  $x = 1.0$ , and with strengths  $m_j$  given by

# Submerged Two-Dimensional Bodies

$$\frac{m_j}{2\pi U} = +0.04 \quad \text{when } 0 \leq j \leq 2, \quad (14a)$$

$$= -\frac{0.04}{168} (65 - 3j) \quad \text{when } 3 \leq j \leq 10, \quad (14b)$$

where  $j = 0, 1, 2$ , etc., refer to the sources at  $x = 0, 0.1, 0.2$ , etc.

The stream function for this singularity distribution in a uniform flow (velocity =  $U$ ) with no free surface is

$$\Psi(x, y) = Uy + \sum_{j=0}^{10} \frac{m_j}{2\pi} \tan^{-1} \frac{y}{x - (j/10)}. \quad (15)$$

The body is given by the streamline  $\Psi = 0$ . Hence the equation of the body is

$$y = \pm \sum_{j=0}^{10} \left( \frac{m_j}{2\pi U} \right) \tan^{-1} \frac{y}{x - (j/10)}. \quad (16)$$

The cross section of the body is seen in Fig. 1. In terms of the selected unit length, the chord length is 1.09 and the thickness is 0.374. The unit length used here is, for numerical examples as well as for the experimental work, set equal to 1 foot.

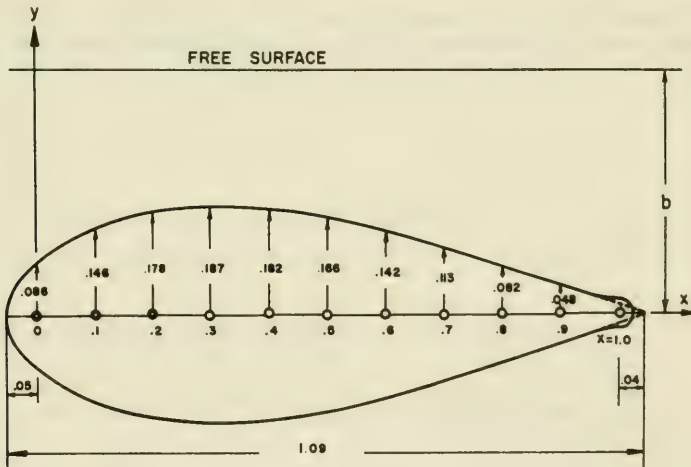


Fig. 1 - Singularity representation and the cross section of the body

The first-order complex potential for the given singularity distribution (Eqs. (14)) and satisfying the condition given by Eq. (11) can be obtained from Wehausen and Laitone (6a):

$$w^{(1)}(z) = \sum_{j=0}^{10} \frac{m_j}{2\pi} \ell_n \left( z - \frac{j}{10} \right) + \sum_{j=0}^{10} \frac{m_j}{2\pi} \ell_n \left( z - \frac{j}{10} - 2ib \right) + \sum_{j=0}^{10} \frac{m_j}{\pi} I \left[ i\nu \left( z - \frac{j}{10} - 2ib \right) \right], \quad (17)$$

where the exponential integral  $I(\zeta)$  is defined as

$$I(\zeta) \equiv -\text{P.V.} \int_0^{\infty} \frac{e^{-u\zeta}}{1-u} du - i\pi e^{-\zeta}, \quad (18a)$$

and has the expansion

$$I(\zeta) = -e^{-\zeta} \left( \gamma + \ell_n \zeta + i\pi + \sum_{n=1}^{\infty} \frac{\zeta^n}{nn'} \right), \quad (18b)$$

where Euler's constant is  $\gamma = 0.5772 \dots$ .

In accordance with the assumption that the cylinder-wall condition will be satisfied only to the first-order approximation, it follows from Eq. (12) that the second-order complex potential  $w^{(2)}(z)$ , must satisfy the conditions

$$1. \quad \nabla^2 w^{(2)} = 0, \quad (19a)$$

$$2. \quad \text{Re} \left[ w_z^{(2)} - i\nu w^{(2)} \right] = \frac{1}{\rho U} f(x) \quad \text{at} \quad z = x + ib, \quad (19b)$$

where

$$\frac{1}{\rho} f(x) = \frac{1}{2} \left| w_z^{(1)} \right|^2 - \text{Im} \left[ w^{(1)} \right] \text{Re} \left[ i w_{zz}^{(1)} - \nu w_z^{(1)} \right],$$

$$3. \quad \lim \left| w_z^{(2)} \right| = 0 \quad \text{as} \quad \text{Im } z \rightarrow -\infty, \quad (19c)$$

$$4. \quad \lim \left| w_z^{(2)} \right| = 0 \quad \text{as} \quad \text{Re } z \rightarrow -\infty. \quad (19d)$$

This is exactly the same differential system as for the linear problem of a fixed pressure distribution  $f(x)$  on the free surface of a uniform stream of velocity  $U$ . Thus  $f(x)$  may be interpreted as a pressure distribution due to the first-order wave system.

The solution to the differential system of Eqs. (19) is therefore the same as the well-known solution to the moving pressure distribution problem, given by Wehausen and Laitone (6b), namely,

$$w^{(2)}(z) = \frac{i}{\pi U} \int_{-\infty}^{\infty} ds f(s) I[i\nu(z - ib - s)], \quad (20)$$

where the function  $f(s)$  is defined in Eqs. (19) and can be written by virtue of Eq. (17) as

$$\begin{aligned} f(s) = \rho U^2 & \left\{ 2\nu^2 \left[ \sum_{j=0}^{10} \frac{m_j}{2\pi U} \operatorname{Im} I_j \right]^2 + 2 \left[ \sum_{j=0}^{10} \frac{m_j}{2\pi U} \left( \frac{b}{s_j^2 + b^2} + \nu \operatorname{Re} I_j \right) \right]^2 \right. \\ & \left. + 4 \left[ \sum_{j=0}^{10} \frac{m_j}{2\pi U} \operatorname{Im} I_j \right] \left[ \sum_{j=0}^{10} \frac{m_j}{2\pi U} \left( \frac{2bs_j}{(s_j^2 + b^2)^2} + \frac{\nu s_j}{s_j^2 + b^2} \right) \right] \right\}, \end{aligned} \quad (21)$$

with

$$I_j = I\left[i\nu\left(s - \frac{j}{10} - ib\right)\right] \quad \text{and} \quad s_j = s - \frac{j}{10}.$$

The wave height can now be obtained easily. The first-order wave elevation is, by Eqs. (13a) and (17),

$$\eta^{(1)}(x) = \sum_{j=0}^{10} \frac{m_j}{\pi U} \operatorname{Im} I\left[i\nu\left(x - \frac{j}{10} - ib\right)\right], \quad (22)$$

and the second-order wave is

$$\eta^{(2)}(x) = \frac{1}{\pi U^2} \int_{-\infty}^{\infty} ds f(s) \operatorname{Re} I[i\nu(x - s + i0)] + \nu \left(\eta^{(1)}\right)^2, \quad (23)$$

where  $f(s)$  is given by Eq. (21). Note that

$$\left(\frac{p\nu}{\pi g P}\right) \operatorname{Re} I[i\nu(x - s + i0)]$$

is the wave height created by an integral pressure  $P$  concentrated on an infinitely narrow band of the free surface at  $x = s$ . A detailed discussion of this is given by Lamb (7). A plot of  $\operatorname{Re} I[i\nu(x - s + i0)]$  is seen in Fig. 2.



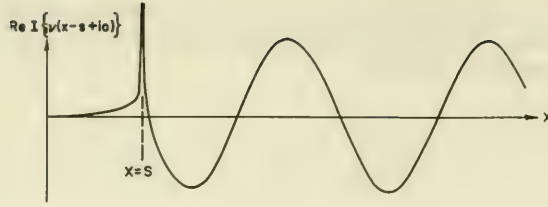


Fig. 2 - Plot of the function  
 $\text{Re I}[i\nu(x-s+i0)]$

For later reference we shall write the two terms in Eq. (23) as

$$\eta_I^{(2)}(x) = \frac{1}{\pi U^2} \int_{-\infty}^{\infty} ds f(s) \text{Re I}[i\nu(x-s+i0)] , \quad (24)$$

$$\eta_{II}^{(2)}(x) = \nu(\eta_I^{(1)})^2 .$$

We now turn to the numerical computation of the first-order and second-order waves. A MAD computer program was prepared for this purpose. Before we discuss the final results of these computations, let us look at some of the intermediate steps. It is interesting to note that the "pressure" function  $f(s)$ , defined by Eq. (21), is not sinusoidal far downstream but tends to the constant value  $\rho g \nu a^2/2$ , where  $a$  is the first-order wave amplitude, as can be seen from Fig. 3. We also note (Fig. 4) that due to this constant "pressure" value the first part of the second-order wave  $\eta_{II}^{(2)}(x)$  far downstream is sinusoidal but with the mean line a distance  $\nu a^2/2$  below the undisturbed free-surface level. The important fact is that this constant dislocation of the mean line is of exactly the right magnitude to balance the constant part of the term  $\eta_{II}^{(2)}(x) = \nu(\eta_I^{(1)})^2$ , leaving only harmonic wave components far downstream.

Plots of the final results, the surface elevation according to the first-order and the second-order theory, are given in Fig. 5 for the case of 1.25-foot body

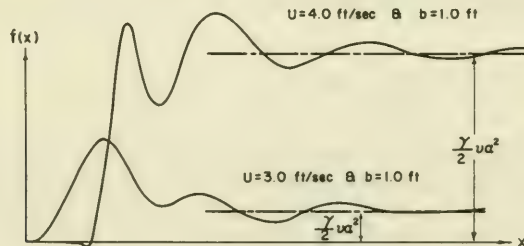


Fig. 3 - Plot of the "pressure" function  $f(x)$

# Submerged Two-Dimensional Bodies

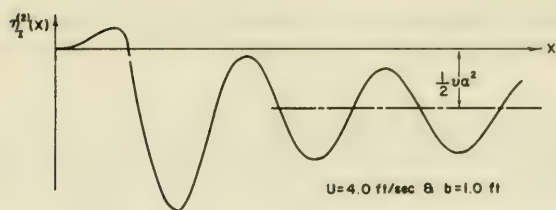


Fig. 4 - Plot of the second-order wave function  $\eta_I^{(2)}(x)$

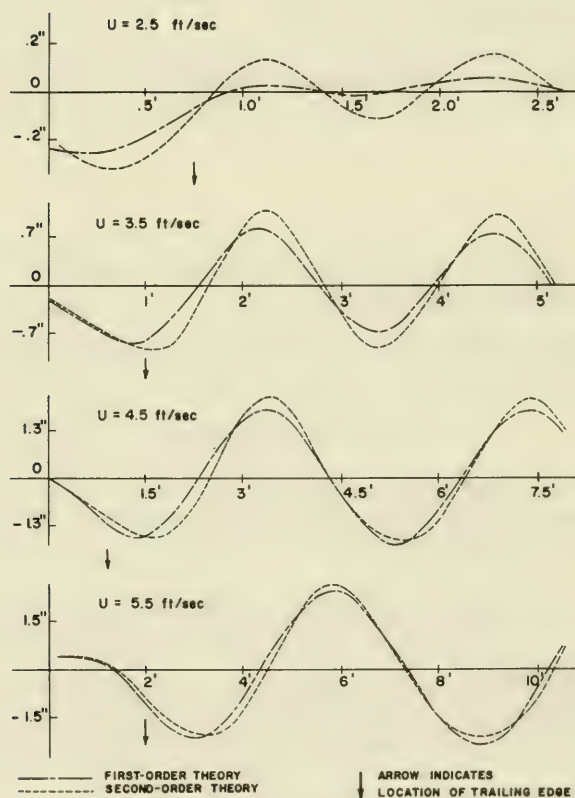


Fig. 5 - First-order and second-order wave elevation curves for 1.25-ft submergence

submergence and for four different speeds. This is, to the author's knowledge, the first time that waves created by a body have been computed to the second order.

Looking at these wave profiles, we immediately notice the extremely large difference between first-order and second-order waves for the lowest speed of 2.5 ft/sec. This seems to violate the original assumption of expansions of the form

$$\eta(x) = \epsilon \eta^{(1)}(x) + \epsilon^2 \eta^{(2)}(x) + \dots, \quad (25)$$

where the second-order term  $\epsilon^2 \eta^{(2)}(x)$  was assumed to be one order smaller than the first term  $\epsilon \eta^{(1)}(x)$ . The experimental results to be discussed will, however, show that the second-order waves agree surprisingly well with the measured waves for these lower speeds.

It should also be recognized that the shape of the second-order waves downstream are quite the same as given by second-order Stokes waves, as was expected.

### Wave Resistance

The wave resistance will be obtained from the asymptotic form of the velocity potential and the wave elevation.

It is well known that the first-order wave height, Eq. (22), far downstream takes the simple form

$$\eta^{(1)}(\bar{x}) = \alpha \cos \nu \bar{x}, \quad (26)$$

where  $\alpha$  is the first-order wave amplitude, and  $\bar{x}$  is an  $x$  coordinate so selected that the wave has no phase shift. The second-order part, Eq. (23), can be shown to have far downstream the form

$$\eta^{(2)}(\bar{x}) = \delta \cos \nu(\bar{x} - \tau) - \frac{1}{2} \nu \alpha^2 + \nu (\eta^{(1)})^2, \quad (27)$$

which by Eq. (26) becomes

$$\eta^{(2)}(\bar{x}) = \delta \cos \nu(\bar{x} - \tau) + \frac{1}{2} \nu \alpha^2 \cos 2\nu \bar{x}, \quad (28)$$

where  $\delta$  is of order  $\alpha^2$  and  $\tau$  is a phase shift.

From Eqs. (26) and (28) it can be realized that far downstream we have two Stokes waves. One has the form

$$\eta_1(\bar{x}) = \alpha \cos \nu \bar{x} + \frac{1}{2} \nu \alpha^2 \cos 2\nu \bar{x} + \dots, \quad (29)$$

and the other, which has a phase shift  $\tau$  and an amplitude  $\delta$  of second order in  $\alpha$ , has the form

$$\eta_2(\bar{x}) = \delta \cos \nu(\bar{x} - \tau) + \dots, \quad (30)$$

where both Eqs. (29) and (30) are correct to the second order in  $\alpha$ .

From the theory of Stokes waves (Appendix A) it follows that the velocity potential for these two Stokes waves can be written as

$$\phi_1(\bar{x}, y) = -\alpha U e^{\nu(y-b)} \sin \nu \bar{x}, \quad (31)$$

and

$$\phi_2(\bar{x}, y) = -\delta U e^{\nu(y-b)} \sin \nu(\bar{x} - \tau). \quad (32)$$

Assuming now that the potential and the wave elevation far downstream can be written as the sum of the two Stokes waves above, we have that

$$\phi = \phi_1 + \phi_2 = -U e^{\nu(y-b)} \left[ \alpha \sin \nu \bar{x} + \delta \sin \nu(\bar{x} - \tau) \right] \quad (33)$$

and

$$\eta = \eta_1 + \eta_2 = \alpha \cos \nu \bar{x} + \frac{1}{2} \nu \alpha^2 \cos 2\nu \bar{x} + \delta \cos \nu(\bar{x} - \tau). \quad (34)$$

Note that the potential, Eq. (33), and the wave elevation, Eq. (34), are valid far downstream, and that both are correct to the second order in  $\alpha$ .

The "exact" formula for the wave resistance derived by Havelock (8) and by Wehausen and Laitone (6c) using energy considerations and also derived in Appendix B by momentum considerations, is

$$R = \frac{1}{2} \rho \int_{-\infty}^{b+\eta(x_0)} \left[ -\phi_x^2(x_0, y) + \phi_y^2(x_0, y) \right] dy + \frac{1}{2} g \rho \eta^2(x_0), \quad (35)$$

where  $\phi(x, y)$  is the "exact" velocity potential,  $\eta(x)$  is the "exact" wave elevation, and  $x_0$  denotes any vertical plane behind the body.

Evaluating this expression far downstream using the velocity potential (Eq. (33)) and the wave profile (Eq. (34)) it can be shown that the wave resistance correct to the third order is

$$R = \frac{1}{4} \rho g [\alpha^2 + 2\alpha\delta \cos \nu\tau]. \quad (36)$$

Since by Eq. (34) the trough-to-crest wave height correct to order  $\alpha^2$  is

$$H \equiv \eta(0) - \eta(\pi) = 2\alpha + 2\delta \cos \nu\tau, \quad (37)$$

the resistance  $R$  can also be expressed in terms of the wave height



$$R = \frac{1}{4} \rho g \left( \frac{H}{2} \right)^2, \quad (38)$$

where  $H$  is the wave height obtained from the second-order wave theory. This is exactly the same result as obtained in Appendix A using a second-order Stokes wave theory.

A plot of the first-order and the second-order wave resistance is seen in Fig. 6, where the first-order resistance is obtained from the well-known relationship

$$R = \frac{1}{4} \rho g a^2, \quad (39)$$

with  $a$  denoting the first-order wave amplitude far downstream. Again we see the large difference between the first-order and the second-order theory for the lower speed range.

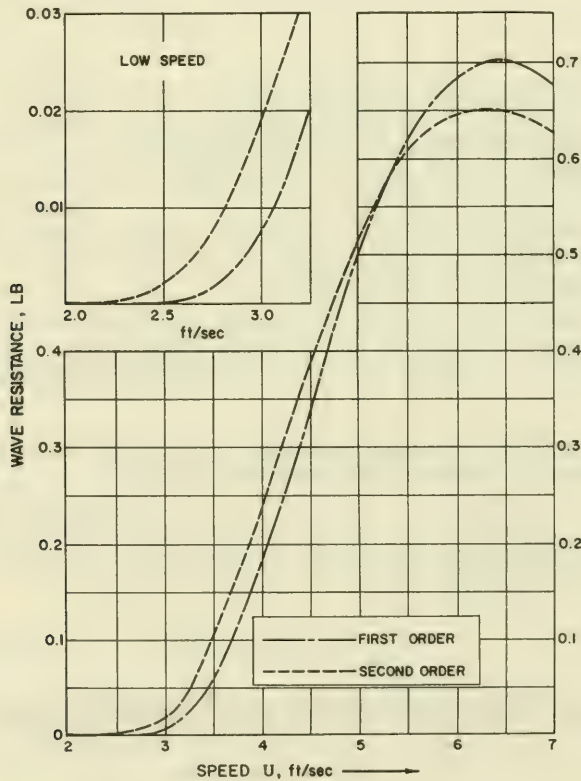


Fig. 6 - First-order and second-order wave resistance curves for 1.25-ft submergence

## Accuracy of the Mathematical Representation of the Body

We assumed in the mathematical formulation that the body could be represented by its singularities in a uniform flow without accounting for the free surface. The boundary condition on the cylinder wall is by this assumption correct only to the first order. To be correct to the second order, we should have modified these singularities such that the same body would be generated in a fluid with a linearized free surface. This is one of the critical assumptions in this work, and it was therefore felt that an investigation of its accuracy should be included.

A computer program was written which could trace the streamlines around the singularity distribution including the linearized free surface. This is a rather time consuming computation, especially because we do not know a priori the value of the stream function at the stagnation points. The streamlines were therefore only traced for two speeds,  $V = 4$  ft/sec and  $V = 6$  ft/sec, both with the same submergence  $b = 1.25$  ft. Each case took about 10 minutes on the IBM 7090 computer. The results are shown in Fig. 7.

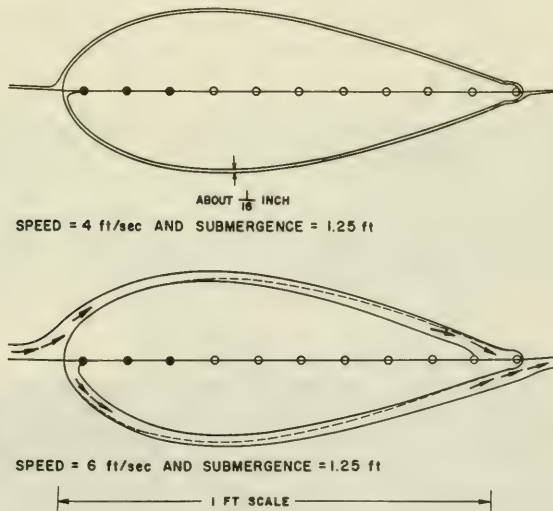


Fig. 7 - Effect of a linearized free surface on the flow around assumed singularities

It is seen clearly that when the linearized free-surface effect is included, the body is no longer closed, and that the forward and aft stagnation points are on different streamlines. This effect was first pointed out by Tuck (4). The author is of the opinion, however, that for lower speeds the given singularity distribution does represent the body fairly well and can be used with reasonable accuracy for a second-order theory. For larger speeds it is obvious that the

body representation cannot be used for a second-order theory. For a submergence of 1.25 feet the use of this body representation for speeds up to about 4.5 feet per second, and higher speeds for deeper submergences, seems justified, as can be seen from Fig. 7.

It is interesting to note that the body can be made closed simply by applying a small angle of attack and hence without introducing any new singularities. In other words, a closed body can be obtained by an appropriate rotation of the line on which the 11 sources are located. The amount of rotation necessary to close the body will obviously depend on the velocity and the submergence of the cylinder.

## EXPERIMENTAL RESULTS

An experiment was conducted to see how well this second-order theory can predict wave elevations and wave resistance for submerged two-dimensional foil-shaped bodies. The experiment was performed at the University of Michigan model tank (360 ft by 20 ft by 9 ft) on an 11-ft two-dimensional strut seen in Fig. 8. The model's cross-sectional offsets are given in Fig. 1. The instrumentation for measuring horizontal drag force and wave elevation is schematically shown in Fig. 9. The horizontal drag force was measured on the middle, 2-foot section of the model by waterproofed strain gages mounted on two small cantilever beams (Fig. 10). The wave elevation was measured by a 0.0016-in. capacitance wire attached to a small carriage which could move on a 10-ft rail as shown in Fig. 11. The change in capacitance of this wire as caused by changes in water elevation was registered by a highly sensitive capacitance bridge (Wayne Kerr B541) from which results were traced on an x-y recorder. The x coordinate was used for the position of the carriage, and the y coordinate was used for wave elevation. With this capacitance wire we were able to obtain the wave profile to a good accuracy, about  $\pm 0.03$  in. for speeds up to 4.5 ft/sec.

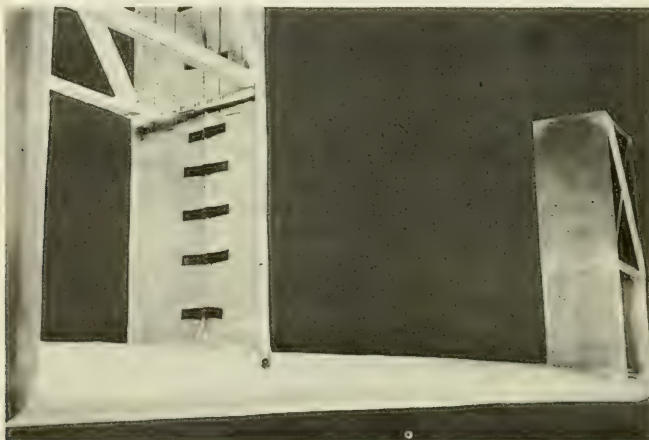


Fig. 8 - Foil with supports

## Submerged Two-Dimensional Bodies

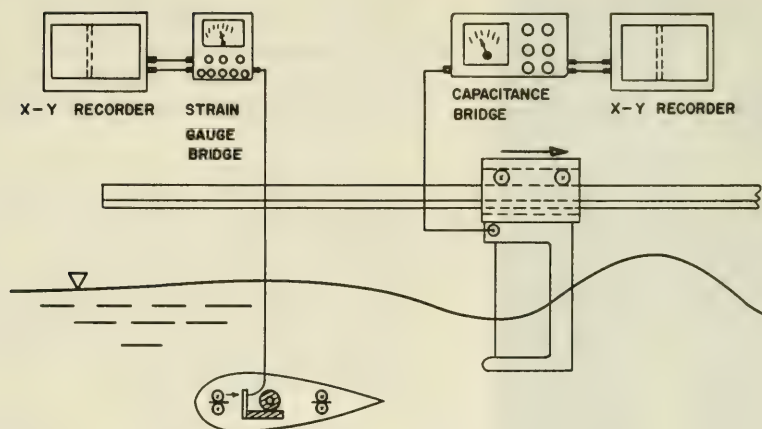


Fig. 9 - Instrumentation for measuring the horizontal drag force and the free-surface elevation

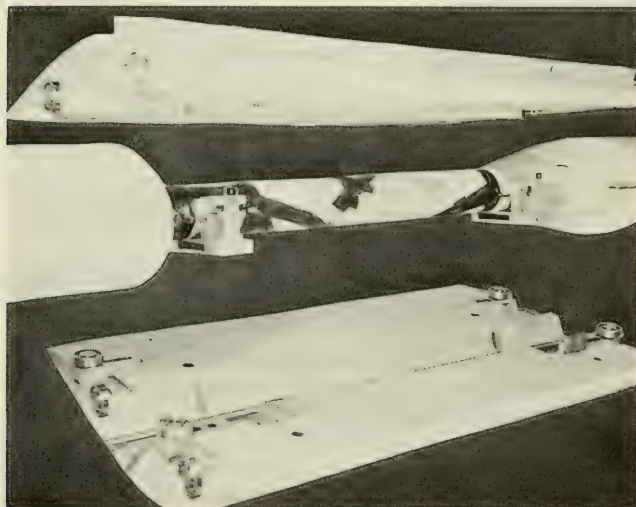


Fig. 10 - Test section of a foil with strain gages



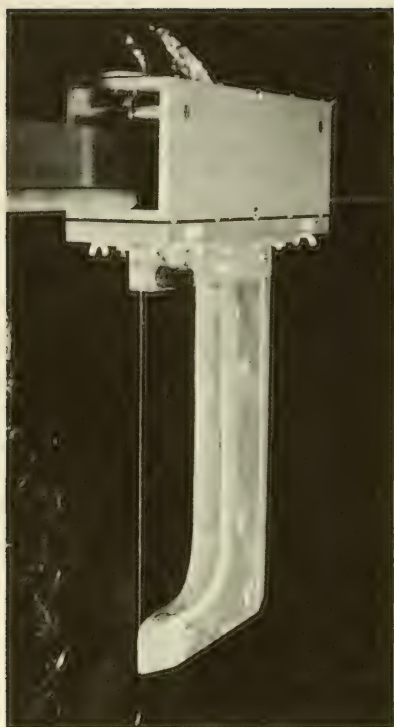


Fig. 11 - Carriage and capacitance wire on a 10-ft rail

No records could be taken at speeds higher than 6 ft/sec, however, due to an air cavity formed behind the wire. At speeds between 4.5 and 6 ft/sec the wire did not give as good accuracy as desired, so the wave heights were also checked mechanically.

Wave profiles were recorded for five submergences ( $b = 0.50, 0.75, 1.00, 1.25$ , and  $1.50$  ft) and for nine speeds for each submergence ( $U = 2.0, 2.5, 3.0, \dots, 6.0$  ft/sec). Every profile included in this report was obtained from at least two runs in order to check the results, and about 200 runs were performed.

The wave resistance was obtained from the model test data in two ways: (a) by subtracting the horizontal drag at 4.5-ft submergence from the total horizontal drag at the other submergences (assuming no wave resistance at the 4.5-ft submergence, that there is no interaction between wave and viscous resistance, and that the viscous drag is the same at a deep submergence as for a submergence where waves are created), and (b) from the derived equation

$$R = \frac{1}{4} \rho g \left( \frac{H}{2} \right)^2,$$

where  $H$  is the actual measured trough-to-crest wave height, this equation being correct to the third order in wave height.

The wave elevation curves for 1.25-ft submergence are shown in Fig. 12. Good correlation in wave height can be seen between the measured waves and second-order theory for speeds up to 4.5 ft/sec. The figure shows, on the other hand, some discrepancy between measured and theoretical wavelengths. We note especially the excellent agreement at the lowest speed, where the difference in elevation from first-order and second-order theory is extremely large. The wave resistance curves plotted in Fig. 13 show exactly the same trend. We do observe from these figures a poor agreement at speeds above 4.5 ft/sec, however. This is believed to be caused by the inaccurate mathematical representation of the body at higher speeds.

Presently the author is working on an extension of this problem, applying a consistent second-order theory satisfying the cylinder-wall condition to the same order of accuracy as the free-surface condition. The preliminary results

# Submerged Two-Dimensional Bodies

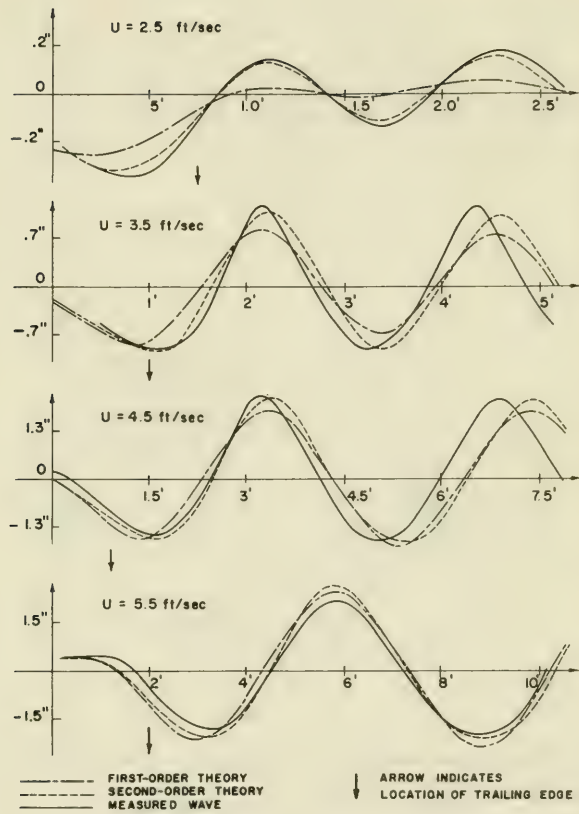


Fig. 12 - Wave elevations for 1.25-ft submergence

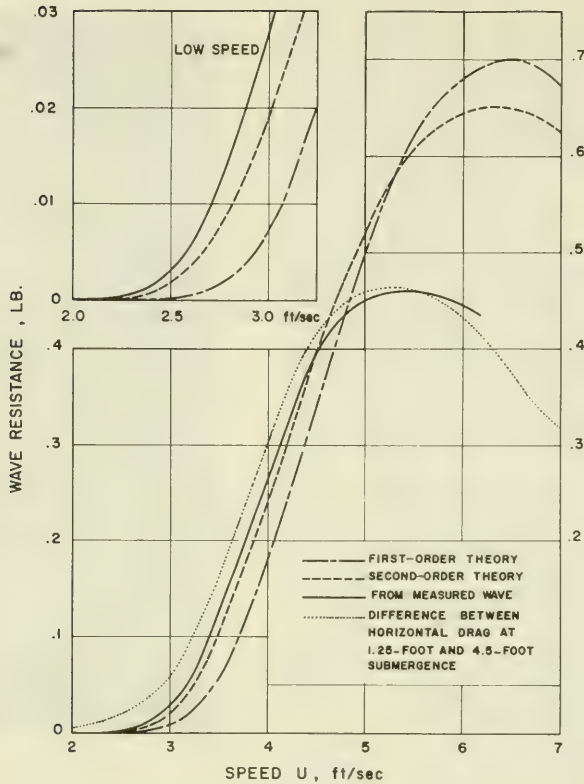


Fig. 13 - Wave resistances  
for 1.25-ft submergence

show that the rather large discrepancy at higher speeds between the theory presented here and the experiment is indeed due to the inaccurate body representation.

The wave resistance curve obtained from horizontal drag measurements seems to be rather high at the lower speeds, which seems to indicate that the viscous drag is not the same at 4.5-ft and 1.25-ft submergence but that it increases as the body gets closer to the free surface. This is most likely due to an increase in the velocity of the fluid particles next to the body as a result of the free surface. Furthermore, this proves how extremely difficult it is to determine the exact wave resistance from drag force measurements. The author, therefore, strongly believes that we should not compare wave resistance curves but rather wave profiles when checking the accuracy of theoretical work.

Figures 14 and 15 show the wave profiles and the wave resistance curves for 1.50-ft submergence, and as expected the agreement is even better at this larger submergence.

# Submerged Two-Dimensional Bodies

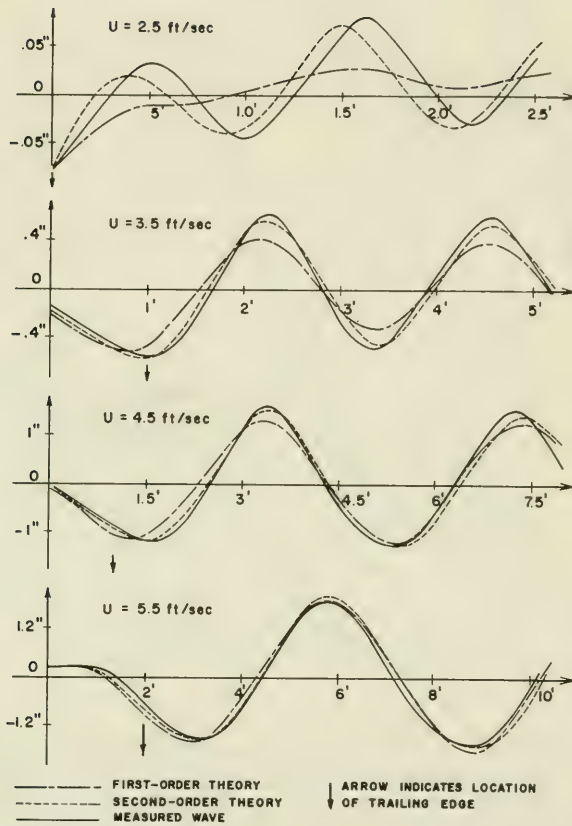


Fig. 14 - Wave elevations  
for 1.50-ft submergence



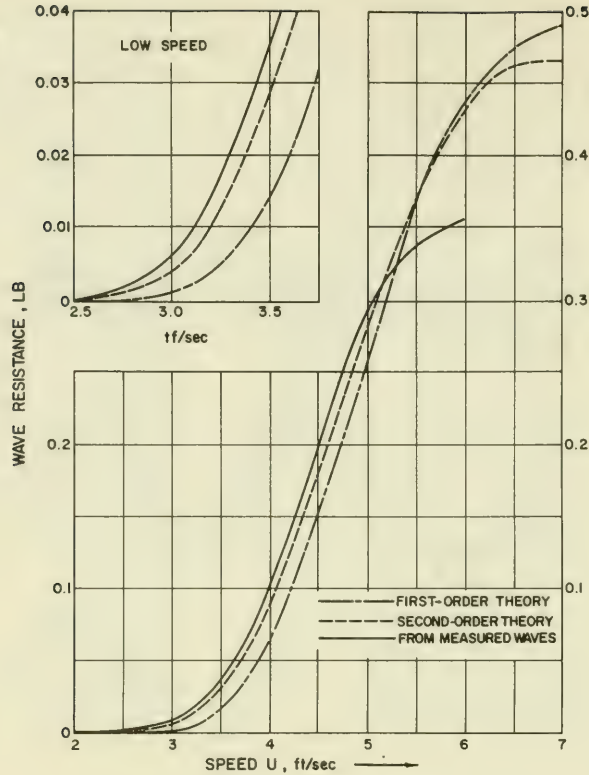


Fig. 15 - Wave resistances  
for 1.50-ft submergence

Another interesting result is that the waves start to break at submergences smaller than 1.25 ft. This can clearly be seen from the wave profiles plotted in Fig. 16 and from the pictures shown in Fig. 17, both for 1.0-ft submergence (about three times the thickness of the body). Many theorist investigators have assumed that submergences of this magnitude were large enough to apply small disturbance theory. Tuck (4) used twice the thickness as his largest submergence. This wave breaking, however, is such a highly nonlinear phenomenon that no perturbation theory can predict its occurrence.

We note that for the 1.0-ft submergence the waves break for speeds between about 2.5 and 5.5 ft/sec and that for lower speeds when the waves do not break, the second-order theory agrees fairly well with measurements.

Figure 18 is included to show that at even smaller submergences ( $b = 0.75$  ft) the waves break down completely, leaving no regular waves behind the body but only some kind of a "hydraulic jump." This has previously been observed by Laitone (9) and by Parkin, Perry, and Wu (10).

# Submerged Two-Dimensional Bodies

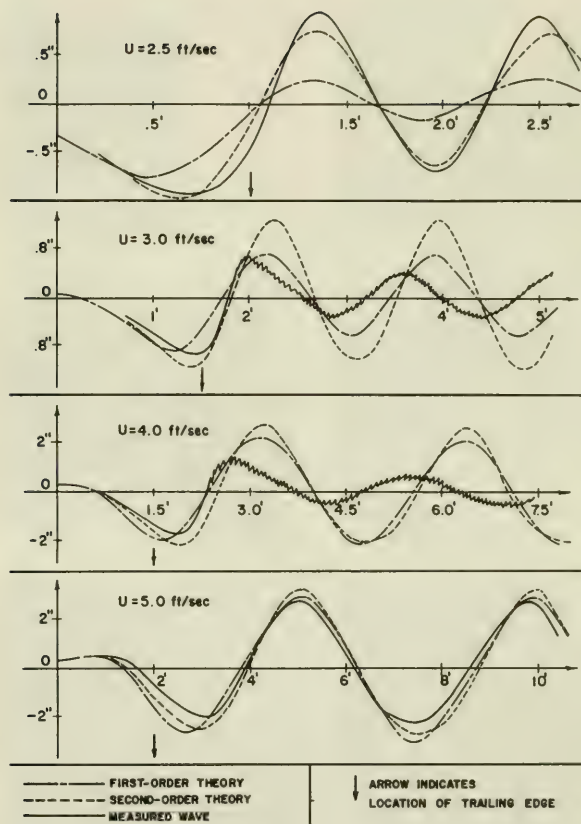


Fig. 16 - Wave elevations  
for 1.0-ft submergence

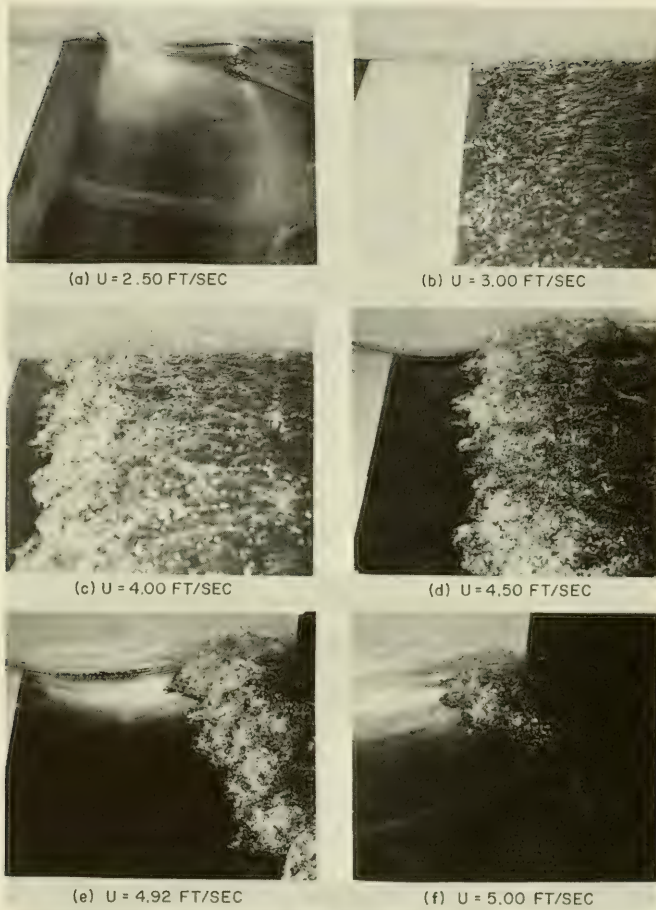
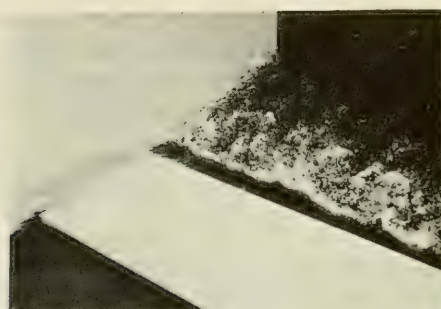
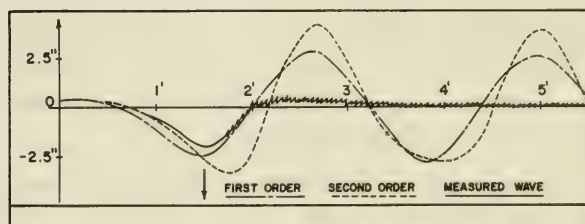


Fig. 17 - Breaking waves for 1.0-ft submergence

# Submerged Two-Dimensional Bodies



TOP VIEW OF  
FOIL AND  
BREAKING WAVE



Fig. 18 - Breaking wave for 0.75-ft  
submergence and  $U = 3.50$  ft/sec



## CONCLUSIONS

This work clearly proves the importance of the effect of nonlinearity at the free surface, and it shows how much better results can be achieved by the second-order theory than by the linear theory. The speed range and submergences are also indicated for which linear and for which second-order theory may be applied.

The following conclusions can be drawn for free-surface flow past submerged two-dimensional foil-shaped bodies:

1. The second-order theory gives very good results at moderate speeds ( $U/\sqrt{gb} < 0.72$ ) and submergences larger than 3-1/2 times the thickness of the body.
2. At very low speeds the second-order theory predicts wave heights several times the size given by the linear theory. The agreement between experiment and second-order theory is surprisingly good at these very low speeds.
3. For checking the accuracy of the theory a comparison between theoretical and measured wave profiles should be used rather than a comparison of wave resistance data.
4. Due to the wave breaking phenomenon at smaller submergences, the perturbation theory should not be used over the entire speed range if the submergence is smaller than three times the thickness of the body. The experiments show that at a submergence three times the thickness of the body the waves break in the speed range  $0.45 \leq U/\sqrt{gb} \leq 0.90$ ; and at smaller submergences the waves break in an even larger speed range.
5. Both the linear and the proposed second-order theory give rather poor results at higher speeds ( $U/\sqrt{gb} > 0.72$ ). This is believed to be due to the inaccurate mathematical representation of the body at these speeds.

We should be careful in extending any of these conclusions to the three-dimensional ship problem. It seems reasonable to conclude however, that non-linear free-surface effects are most likely more important for surface ships than previously believed.

Further research should be carried out in this area. Presently the author is applying a consistent second-order theory which seems to predict the wave resistance and the wave profiles very well also at higher speeds. In addition to this work the two following problems deserve attention:

1. A thorough investigation should be made of the viscous effects on free-surface flow past a cylinder by means of wake study and possibly also by applying boundary-layer suction control.
2. Work should be extended to include three-dimensional submerged bodies.

From this extended research, we may be able to draw conclusions of great importance to the surface-ship case.

## ADDENDUM

The author would like to mention briefly some additional work completed after the preparation of this paper. This additional work is included here in response to the discussion following this paper.

It is shown in this paper that at the free-surface and in the far-field the complex potential has the two following second-order terms:

$\epsilon^2_{wF_2}$ , the second-order free-surface term,

$\epsilon_{wB_1}$ , the second-order body correction term.

The author only included one of these second-order contributions, namely the free-surface term  $\epsilon^2_{wF_2}$ . The only justification for this was that Tuck (4) has shown that for a circular cylinder this is the most important higher-order contribution to the wave resistance. On the other hand, Giesing and Smith (5) have developed a method which neglects the second-order free-surface term  $\epsilon^2_{wF_2}$  but includes the second-order term from the body boundary condition  $\epsilon_{wB_1}$ .

The author has applied the method by Giesing and Smith and combined it with the method presented in this paper, resulting in an entirely consistent second-order theory. The wave resistance computed from the consistent second-order potential is shown in Fig. 19. Comparing Fig. 13 and Fig. 19 it is seen that both second-order terms are important and that neither should be neglected. It is especially interesting to note that at lower speeds ( $U/\sqrt{gb} < 0.65$ ), the main higher order contribution to the wave resistance comes from the free-surface term; however, at higher speeds ( $U/\sqrt{gb} > 0.75$ ) it is seen that the body-correction term gives the essential contribution. This is an interesting result, considering that in the case of the circular cylinder Tuck showed that the free-surface contribution was the most important for the entire speed range.

The reason that the body-condition term is so important for the wing-shaped body treated here is that in addition to the singularities introduced in closing the body a circulation term must also be introduced such that the Kutta condition is satisfied at the trailing edge. It can be shown that it is mainly this circulation term which gives rise to the large higher order effect at higher speeds ( $U/\sqrt{gb} > 0.75$ ).

## ACKNOWLEDGMENTS

To Professor Finn C. Michelsen and Professor Chia-Shun Yih I express my deep gratitude for their unending inspiration, encouragement, and guidance in

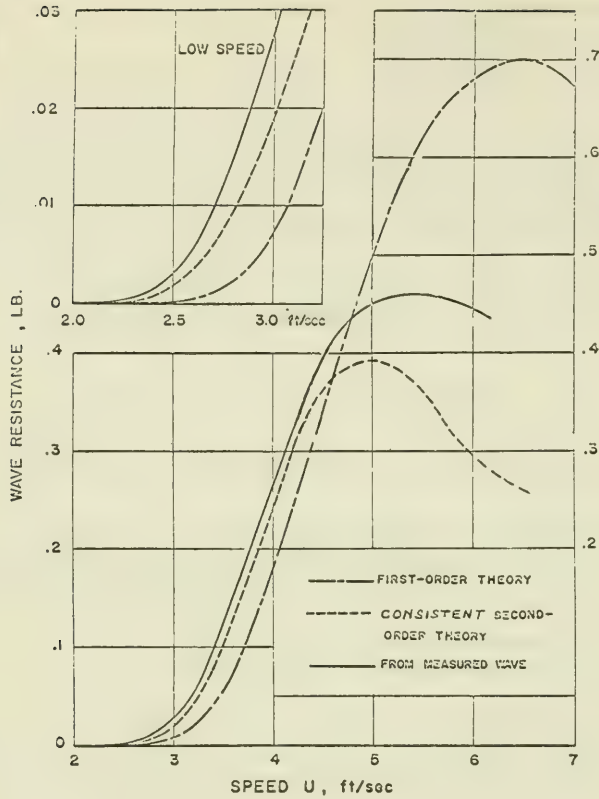


Fig. 19 - Wave resistances  
for 1.25-ft submergence

the course of this work. My appreciation is also extended to Professor R. B. Couch and other staff members of the Ship Hydrodynamics Laboratory at the University for their assistance and cooperation.

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## Appendix A

### STOKES WAVES AND THEIR APPLICATION TO WAVE RESISTANCE PROBLEMS

It was first shown by Stokes\* that the velocity potential and the stream function given by

$$\begin{aligned}\Phi &= Ux - U\beta e^{\nu y} \sin \nu x \\ \Psi &= Uy - U\beta e^{\nu y} \cos \nu x - C_2\end{aligned}\tag{A1}$$

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\*G. G. Stokes, "On the Theory of Oscillatory Waves," Trans. Cambridge Philosoph. Soc. 8:441-455 (1847).



is not only the linearized wave solution but also satisfies the free-surface conditions correctly to the second order, for the case of infinite depth. Lord Rayleigh\* pointed out that this solution can also be shown to be correct to the third order. In the present work, however, we will only be concerned with first-order and second-order terms.

The equation of the wave profile ( $\Psi = 0$ ) follows from Eq. (A1) by successive approximation:

$$\begin{aligned}\eta &= \frac{C_2}{U} + \beta e^{\nu\eta} \cos \nu x = \frac{C_2}{U} + \beta(1 + \nu\eta + \dots) \cos \nu x \\ &= \frac{C_2}{U} + \frac{\nu}{2} \beta^2 + \beta \cos \nu x + \frac{\nu}{2} \beta^2 \cos 2\nu x + \dots\end{aligned}\quad (\text{A2})$$

For convenience, let us set

$$C_2 = -\frac{\nu}{2} U \beta^2, \quad (\text{A3})$$

such that the wave profile becomes

$$\eta = \beta \cos \nu x + \frac{1}{2} \nu \beta^2 \cos 2\nu x \dots \quad (\text{A4})$$

We note that Eq. (A4) coincides with the equation for a trochoidal wave to the given order of accuracy.

The solution, Eq. (A1), with the wave profile, Eq. (A4), must, to be valid, also satisfy the condition of constant pressure ( $p = 0$ ) at the free surface. Applying the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2} |\text{grad } \Phi|^2 + gy = C_1 \quad (\text{A5})$$

we have that

$$\frac{1}{2} U^2 (1 - 2\nu\beta e^{\nu y} \cos \nu x + \nu^2 \beta^2 e^{2\nu y}) + gy = C_1. \quad (\text{A6})$$

At points on the line

$$\eta = -\frac{1}{2} \nu \beta^2 + \beta e^{\nu\eta} \cos \nu x$$

we therefore have

$$(g - \nu U^2) \eta + \frac{1}{2} U^2 = C_1, \quad (\text{A7})$$

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\*Lord Rayleigh, "On Waves," *Philosoph. Mag.*, Series 5, 1:257-279 (1876).

## Submerged Two-Dimensional Bodies

where only terms to the order  $\beta^2$  have been included. Hence, the pressure is constant ( $p = 0$ ) on the free surface if

$$\nu = \frac{U^2}{g} . \quad (\text{A8})$$

Let us now investigate if it is possible to have a Stokes wave behind a submerged two-dimensional body in a uniform stream. The two following conditions must be satisfied.

1. The pressure at the horizontal plane  $y = -h$  as  $h \rightarrow \infty$  must be the same far upstream and far downstream.

2. Conservation of mass: The inflow through a section upstream must be equal to the outflow through any section downstream.

The pressure at a depth  $y = -h$  far downstream is by the Bernoulli equation, Eq. (A5),

$$\frac{p}{\rho} + \frac{1}{2} U^2 (-2\nu\beta e^{-\nu h} \cos \nu x + \nu^2 \beta^2 e^{-2\nu h}) - gh = 0 , \quad (\text{A9})$$

and therefore when  $h$  is very large

$$p = \rho gh . \quad (\text{A10})$$

Hence, to satisfy the pressure condition 1, we have that the undisturbed free surface far upstream is given by  $y = 0$ .

The conservation of mass condition requires that

$$\lim_{h \rightarrow \infty} \frac{\int_{-h}^0 U dy}{h} = \lim_{h \rightarrow \infty} \frac{\int_{-h}^{\eta} \Phi_x dy}{h} , \quad (\text{A11})$$

where the left side is

$$\lim_{h \rightarrow \infty} \frac{\int_{-h}^0 U dy}{h} = U \quad (\text{A12})$$

and the right side is

$$\begin{aligned} \lim_{h \rightarrow \infty} \frac{\int_{-h}^{\eta} \phi_x dy}{h} &= \lim_{h \rightarrow \infty} \frac{1}{h} (U\eta - U\beta e^{\nu\eta} \cos \nu x + Uh) \\ &= \lim_{h \rightarrow \infty} \left( U - \frac{1}{2} \frac{\nu\beta^2}{h} \right) = U. \end{aligned} \quad (A13)$$

Hence, condition 2 is also satisfied, and a Stokes wave is therefore a possible wave behind a two-dimensional body.

Having concluded this we may now obtain the wave resistance for any two-dimensional body in terms of the Stokes wave far behind the body. The "exact" formula for the wave resistance, as derived in Appendix B, is

$$R = \frac{1}{2} \rho \int_{-\infty}^{\eta(x_0)} \left[ -\phi_x^2(x_0, y) + \phi_y^2(x_0, y) \right] dy + \frac{1}{2} g \rho \eta^2(x_0), \quad (A14)$$

where  $x_0$  denotes any vertical plane behind the body. Evaluating this expression far downstream, using the velocity potential, Eq. (A1) and the wave elevation, Eq. (A4), it is seen that the wave resistance correct to the third order in  $\beta$  is

$$R = \frac{1}{4} \rho g \beta^2 + O(\beta^4). \quad (A15)$$

Let us now compare the wave resistance, Eq. (A15), obtained by the second-order Stokes wave to the resistance obtained by the linear theory. The linearized velocity potential and wave elevation are

$$\Phi = Ux - Ua e^{\nu y} \sin \nu x \quad (A16a)$$

and

$$\eta = a \cos \nu x, \quad (A16b)$$

where  $a$  is the linearized wave amplitude. The well-known wave resistance for the linear theory is therefore

$$R = \frac{1}{4} \rho g a^2. \quad (A17)$$

It is important here to distinguish between the linearized wave amplitude  $a$  and the second-order Stokes wave amplitude  $\beta$ . If these amplitudes are mistakenly set equal, we have by Eqs. (A15) and (A17) the incorrect conclusion that the linear theory and the second-order Stokes wave both yield the same wave resistance.

In fact we have that

$$\beta = a + O(a^2), \quad (A18)$$

and in general the complete second-order problem must be solved in order to obtain  $\beta$  correct to the second order.

In other words, the Stokes wave theory gives us the form of the potential (Eq. (A1)) and the shape of the wave (Eq. (A4)) correct to the second order; however, the magnitude of the wave elevation cannot be obtained without solving the complete second-order problem.

A very useful result can be obtained from the theory of Stokes waves. By the wave profile, Eq. (A4), it follows that

$$\beta = \frac{H}{2},$$

where  $H$  is the wave height. Therefore the wave resistance, Eq. (A15), becomes

$$R = \frac{1}{4} \rho g \left( \frac{H}{2} \right)^2 + O(H^4). \quad (A19)$$

Applying this result to experimental work, we have that Eq. (A19) gives the wave resistance from measured waves *correct to the third order in wave height*.

## Appendix B

### DERIVATION OF THE "EXACT" WAVE RESISTANCE FORMULA AND A NOTE ON A VARIABLE RESISTANCE PARADOX

Consider a two-dimensional body in a uniform stream of velocity  $U$ , and let us apply the momentum theorem to the fluid region bounded by the plane  $x = a$  far ahead, another plane  $x = c$  behind the body, the bottom  $y = -h$ , the surface of the body, and the free surface. The horizontal force on the body is then given by

$$R = \int_{x=a} (\rho + \rho \Phi_x^2) dy - \int_{x=c} (\rho + \rho \Phi_x^2) dy, \quad (B1)$$

where  $\Phi = Ux + \phi$ . Introducing the Bernoulli equation, Eq. (3), and the continuity condition

$$\int_{x=a} \Phi_x dy = \int_{x=c} \Phi_x dy, \quad (B2)$$

we have that

$$R = \rho g \int_{x=a} -y dy + \rho g \int_{x=c} y dy + \frac{\rho}{2} \int_{x=a} (\phi_y^2 - \phi_x^2) dy - \frac{\rho}{2} \int_{x=c} (\phi_y^2 - \phi_x^2) dy. \quad (B3)$$



And as  $a \rightarrow -\infty$  it follows that the "exact" wave resistance formula under the assumption of ideal fluid, irrotational flow, and negligible surface tension is

$$R = \frac{\rho}{2} \int_{-h}^{\eta(c)} \left[ -\phi_x^2(c, y) + \phi_y^2(c, y) \right] dy + \frac{\rho}{2} g \eta^2(c). \quad (B4)$$

We note that the plane  $x = c$  may be taken at any distance behind the body and that  $\eta$  is the wave elevation measured relative to the undisturbed free surface far ahead of the body.

This expression is "exact" and therefore when applied to a linear problem will yield the wave resistance *correct to the second order* and when applied to a potential correct to the second order will give the wave resistance *correct to the third order*.

Sharma\* shows that if the continuity condition, Eq. (B2), is not applied, however, the wave resistance becomes

$$R = \frac{\rho}{2} \int_{x=c} (-\phi_x^2 + \phi_y^2) dy + \frac{\rho}{2} g \eta^2 + \rho U \int_{x=a} U dy - \rho U \int_{x=c} (U - \phi_x) dy. \quad (B5)$$

He then introduces the linearized velocity potential and wave profile and gets the "variable resistance paradox"

$$R = \frac{\rho g}{4} a^2 + \rho g a^2 \sin \nu x. \quad (B6)$$

This paradox shows the kind of erroneous conclusions which follow from a first-order theory if the equations are applied directly without a specific knowledge of the order of magnitude of each term involved. Equation (B6) was obtained using a linear theory; hence continuity is satisfied only to the first order. Therefore any second-order terms resulting from the continuity condition have no meaning and should have been disregarded by Sharma.

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\*S. D. Sharma, "A Comparison of the Calculated and Measured Free-Wave Spectrum of an Inued in Steady Motion," in "International Seminar on Theoretical Wave-Resistance," Vol. 1, University of Michigan, 1963, p. 255.

## DISCUSSION

J. P. Giesing  
*Douglas Aircraft Company, Inc.*  
*Long Beach, California*

This very good paper compares wave-resistance theory with an excellent set of experimental data obtained by the author. The comparison is designed to illustrate the importance of the higher order terms in the wave-resistance theory. There are two types of higher order terms (higher than first order), as pointed out by the author. One type is due to nonlinear free-surface effects, and one type is due to the satisfaction of the boundary condition on the body surface to higher than zeroth order. The author has included the second-order, non-linear free-surface term but has omitted the second-order term related to the body-surface boundary condition. This was done with the hope that the one omitted would be small compared to the one retained. His experiment indicates that, for the specific conditions considered, this is not the true case for speeds greater than 5 ft/sec. As an aid in explaining a possible cause of this failure of the theory, it may be helpful to inspect the terms in the wave-resistance expansion, especially those that were omitted.

In the author's notation,  $w_{B_q}$ ,  $q = 0, 1, 2, \dots$ , are the 0th, 1st, 2nd, ... order contributions to the complex potential due to the singularities on or within the body, and  $w_{F_q}$ ,  $q = 1, 2, \dots$ , are the 1st, 2nd, ... order contributions to the potential due to the free-surface disturbance. A multipole expansion plus a vortex may be used to represent  $w_{B_q}$  as

$$w_{B_q} = \frac{1}{2\pi} \left[ i\Gamma^{(q)} \ell_n z + \sum_{n=1}^{\infty} D_n^{(q)} z^{-n} \right], \quad q = 0, 1, 2, \dots,$$

where  $\Gamma^{(q)}$  are all real but  $D_n^{(q)}$  are complex. In the vicinity of the body surface the effects of the free surface may be expanded as a Taylor series as follows:

$$w_{F_q} = \frac{1}{2\pi} \sum_{n=1}^{\infty} C_n^{(q)} z^n, \quad q = 1, 2, \dots,$$

where the  $C_n^{(q)}$  are complex. With these definitions and the aid of the Blasius theorem, it can be shown that the wave resistance of a symmetrical body at zero incidence is

$$\begin{aligned}
 R = & \rho \left\{ 0 \right. & O(1) \\
 & + \operatorname{Re} \sum_{n=1}^{\infty} D_n^{(0)} C_{n+1}^{(1)} n(n+1) & O(\epsilon) \\
 & \left. \left[ - \operatorname{Re} \left[ i \Gamma^{(1)} C_1^{(1)} \right] + \operatorname{Re} \sum_{n=1}^{\infty} D_n^{(1)} C_{n+1}^{(1)} n(n+1) \right] + \operatorname{Re} \sum_{n=1}^{\infty} D_n^{(0)} C_{n+1}^{(2)} n(n+1) \right. & O(\epsilon^2) \\
 & \left. + \dots \right\} . & \dots
 \end{aligned}$$

The terms enclosed within the rectangle are the second-order terms arising from the boundary condition on the body surface. These terms are neglected by the author.

A conclusion reached by Tuck\* was that the second-order free-surface correction term is more important than the second-order body-surface correction term. This conclusion was arrived at by assuming, in part, that the body could not maintain circulation; i.e.,  $\Gamma^{(q)} = 0$ . With this assumption, Tuck's conclusion may be expressed as

$$\operatorname{Re} \sum_{n=1}^{\infty} D_n^{(0)} C_{n+1}^{(2)} n(n+1) > \operatorname{Re} \sum_{n=1}^{\infty} D_n^{(1)} C_{n+1}^{(1)} n(n+1) .$$

The body considered here by Salvesen cannot remain circulation-free, since it has the shape of a hydrofoil. Therefore, in this case, Tuck's conclusion may not be used. If the inequality given above is still valid, there may be justification for neglecting the term on the right-hand side; however, the term  $\operatorname{Re} [i \Gamma^{(1)} C_1^{(1)}]$  must still be considered. It may be that the omission of this term causes the major part of the difference between the theory and the experimental data presented.

The term  $\operatorname{Re} [i \Gamma^{(1)} C_1^{(1)}]$  is just the wave resistance of the first-order circulation. The zeroth-order circulation is zero, since the body is symmetrical and at zero incidence. The first-order wave system causes a nonuniform flow field that, in general, produces a first-order circulation, this circulation being required to maintain the Kutta condition at the trailing edge.

A method developed by Giesing and Smith† is based on a theory that is complementary, in terms of wave resistance, to the theory used by Salvesen. Specifically, it retains the terms in the rectangle in the expression for the

\*E. O. Tuck, "The Effect of Non-Linearity at the Free Surface of Flow Past a Submerged Cylinder," J. Fluid Mech. 22:401-414 (1965).

†J. P. Giesing and A. M. O. Smith, "Potential Flow about Two-Dimensional Hydrofoils," Douglas Aircraft Company Engineering Paper 3541, 1965.



second-order wave resistance and neglects the free-surface nonlinear term. It would be of interest to see results obtained by applying this method to the shape considered. An ideal situation, however, would result if this method were joined to the method used in this paper, since the result would then be an entirely consistent second-order theory.

\* \* \*

## DISCUSSION

E. O. Tuck  
*David Taylor Model Basin*  
*Washington, D.C.*

This is a most admirable piece of work, and is precisely the kind of more realistic study I hoped would be prompted by my work on the admittedly academic case of a circular cylinder. It is unfortunate that in the version presented the author has not carried the theory through to consistent second-order by satisfying the body boundary condition to this order, but since he has remedied this situation in more recent work, this objection is somewhat beside the point. It is important to note that in my work I did not recommend that the free-surface second-order effect be included while neglecting the less important body-condition second-order effect; I found a factor of 3 or 4 between the former and the latter for circular cylinders but do not consider this high enough to neglect the latter. Indeed, from the author's Fig. 13, it is quite clear that the body-induced second-order effect is vital for more streamlined bodies at higher speeds, this being presumably a consequence of the Kutta condition, which gives a second-order contribution at zero angle of attack.

I should comment on the author's statements in his introduction about my work. First, the second-order wave resistance for a circular cylinder does not tend to infinity as the Froude number tends to zero; it tends to zero, but more slowly than the linear resistance, so that the ratio between the two tends to infinity. Second, it is of course true that the computed results I presented are for a circular cylinder too close to the free surface for the validity of a perturbation series, and indeed I would expect gross nonlinearity including breaking waves under these conditions. While one can easily make the series converge by submerging the cylinder deeper and deeper, the relative order of magnitude of the two second-order effects is unchanged. It was therefore convenient to present for illustrative purposes a case where the second-order effects are rather too large for convergence but which shows more clearly the relative orders of magnitude. This argument applies with equal force to the present work; thus, Fig. 16 is valuable in spite of the fact that breaking was observed to occur at the intermediate speeds. The really remarkable feature of the present work is that in some cases (e.g., the lowest speed in Fig. 12) where the second-order wave effect is substantially greater than the first-order, the total nevertheless agrees well with the author's experiments.



Last, I should make it clear that my use of the word "true" corresponds to the author's use of "exact," as defined by his footnote. That is, by the "true wave resistance" of the circular cylinder is meant the  $x$  component of force on the body in the "exact" potential flow of an inviscid fluid past it, irrespective of whether this flow is a good representation of reality or not. For a circular cylinder there is no question that this flow does not represent reality, but in the present case of a streamlined body there is plenty of evidence to suggest that it is a good approximation.

It would seem from this fine paper that the mild warnings I gave at the end of my paper have been strengthened a little, and it is now clear that it was not the bluntness of the circular cylinder that caused free-surface nonlinearity to be so important. The question remains — was it the two-dimensionality? The analogous results for three-dimensional bodies now become even more interesting, and I understand some work is being done at Stevens Institute on a submerged sphere. If the nonlinear effect persists for submerged three-dimensional bodies, it can hardly be less important for nonsubmerged bodies.

\*     \*     \*

## DISCUSSION

F. C. Michelsen  
*University of Michigan*  
*Ann Arbor, Michigan*

It has become increasingly evident that research on second-order wave theories should be increased in magnitude and scope in an attempt to answer some of the basic questions raised when applying the linearized first-order theory to the study of wave resistance of ships. In the past the linear theory has required a major part of our attention. Besides, we may be guilty of some reluctance in tackling the second-order theory because of its complexity when applied to three-dimensional ship shapes. Furthermore, one should demand that basic theoretical results be verified experimentally, which in view of the difficulties in defining the measured wave-resistance of ship models is not particularly encouraging to the researcher contemplating investigations of second-order effects.

It can be concluded that a much simpler model should be studied, and this is exactly what the author has done. He selected a two-dimensional submerged strut which was shaped in such a way that a flow separation as well as viscous boundary layer would hopefully have a negligible influence on the results.

Because of close contact with the research described in this paper I find myself at a handicap in the role as a critical reviewer. On the other hand, I do not believe there is much I could have said anyway. Research objectives are well formulated and only in one or two cases does the author express ideas

which may be broadly construed as matters of opinion. Thus, there is very little room for argument and I must restrict myself to comments of general nature.

First, I submit that the author has presented a remarkable paper. It is not every day that one can study what may be referred to as a classical problem and be able to supply information not already available. This is exactly what he has been able to do, however. Theoretical findings which are here presented for the first time are of the greatest significance for the basic understanding of the physical and mathematical features of the second-order wave theories. This is particularly true when viewed against the background of experimental data which are here indispensable. One can only imagine the author's feeling of relief when measured data verified the large differences predicted between first-order and second-order theories. It is comforting indeed to observe the close correlation between second-order theory and experiments, although this leads to the nagging question of convergence of the perturbation expansion of the velocity potential. The less satisfactory agreement between experiments and theory in the higher speed range may be due to the fact that the sources and sinks are not suitable representation for the strut. I ask the author if he could comment on this.

It is stated in the paper that because the cylinder-wall condition is satisfied only to the first order of approximation it can be argued that the second-order theory presented here is not consistent. It is my opinion that for the depth of submergence used by the author only higher than second-order resistance terms will be influenced by the cylinder-wall conditions, provided the traced streamlines are made to form a closed body. This matter ought to be carefully investigated for at least one speed.

I have a slight objection to referring to Eq. (30) as a Stokes wave. Within the order of approximation the author is, of course, free to add a term of the form

$$\frac{1}{2} \nu \delta^2 \cos 2\nu(\bar{x} - \tau)$$

and make it represent a higher order Stokes wave if it is preferable to do so. To the order of approximation, however, it is not important whether the second-order wave is a Stokes wave or not.

In closing I wish to point out that the apparatus designed by the author will be used at the University of Michigan for further studies into some of the areas of continued research outlined by him. This work is being sponsored by the David Taylor Model Basin. I wish therefore to thank the author for his direct contribution to the education of new talent so vital to our profession.

\* \* \*

## DISCUSSION

L. van Wijngaarden  
*Netherlands Ship Model Basin*  
*Wageningen, The Netherlands*

Dr. Salvesen has calculated the drag by calculation of the wave energy. Another method to calculate the wave resistance is with help of the pressure distribution around the submerged body. Looking at the contours as given in Fig. 7, which shows open bodies, the question occurred to me whether there is also a vertical force on the body and of which order of magnitude.

\*       \*       \*

## DISCUSSION

W. D. Kim  
*Davidson Laboratory*  
*Stevens Institute of Technology*  
*Hoboken, New Jersey*

In line with Tuck's earlier endeavor, given as Ref. 4, this paper points out the nonlinearity effect from the free-surface condition which may partly be accountable for the disagreement between the analytic and measured values of the wave resistance of a body translating beneath the free surface. The author successfully presented his contention through an extensive experiment in which the addition of the nonlinear correction term clearly improves the prediction of the wave profile.

In Tuck's procedure, at outset, all the lengths of the problem were made dimensionless by dividing with a typical length  $U^2/g$ , which is a ratio of the forward speed squared to the gravitational acceleration. Then the solution is expressed in an asymptotic series with respect to the square of the dimensionless radius

$$a = \frac{g}{U^2} \bar{a},$$

$\bar{a}$  being the actual radius, as

$$f = \sum_{n=0}^{\infty} a^{2n} f_n. \quad (D1)$$

Therefore, it is essential to keep the perturbation parameter  $a$  small in order that the series should converge.



It appears that for the problem of a symmetrical foil the author chose a new parameter

$$\delta = \frac{g}{U^2} \bar{\delta}$$

in terms of one-half the maximum foil thickness  $\bar{\delta} = 0.187$  ft. It follows then for the lowest to the highest speeds tested,  $U = 2.5$  to  $5.5$  ft/sec, the corresponding values of the perturbation parameter  $\delta$  ranges from  $0.92$  to  $0.20$ , while the Froude numbers defined in a customary manner by use of the full chord length correspond from  $0.43$  to  $0.92$ .

In the asymptotic series given as Eq. (D1),  $f_0$  represents a uniform stream;  $f_1$  represents the image of the uniform flow within a circle and its reflection with respect to the free surface;  $f_2$  consists of the image of the first-order free-surface disturbance within the circle, its reflection with respect to the free surface, and the nonlinear correction term corresponding to the particular solution of the inhomogeneous equation describing the second-order free-surface condition, and so on.

To complete a true second-order solution of the problem, the author indicated that missing parts of the solution in  $f_2$  are now being sought by a process similar to that employed by Giesing and Smith for the two-dimensional problem, as cited in Giesing's discussion, by the preceding footnote. For the configuration of a circle or a sphere, the disturbed kinematic condition due to the introduction of the external singularities can readily be adjusted by invoking the circle or sphere theorem. However, for a general geometry, the violation of the kinematic condition can be arrested by the solution of the Neumann's exterior problem via an integral equation in which the density of the surface-distributed sources appears as the unknown, with the induced normal velocity due to the external singularity being given over the surface of the body.

Application of Tuck's procedure to the three-dimensional problem of a sphere, which is under progress at Davidson Laboratory, is found to be rather complicated due to the appearance of the wave-angle integral in the analysis, in which case the velocity potential can be expanded in an asymptotic series of the form

$$\varphi = \sum_0^{\infty} a^{3n} \varphi_n,$$

here  $a$  being the perturbation parameter, the actual radius of the sphere  $\bar{a}$  divided by the typical length  $U^2/g$  and where, for example, one finds the first-order potential as

$$\varphi_1 = \frac{1}{2} \operatorname{Re} \left[ \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \theta d\theta}{U^2} + \frac{i}{2\pi} \int_{-\pi}^{\pi} (q_1)_1 u \cos \theta d\theta + 2 \int_{-\pi/2}^{\pi/2} (q_2)_1 u \cos \theta d\theta \right] + O(a^3 u^2)$$

with  $u = y + i(x \cos \theta + z \sin \theta)$ ,



$$(q_1)_1 = \frac{2!}{(2h)^3} - \frac{1}{2} \frac{d^2}{dh^2} e^{-2h \sec^2 \theta} E_i^+ (2h \sec^2 \theta) \sec^2 \theta,$$

and

$$(q_2)_1 = e^{-2h \sec^2 \theta} (\sec^2 \theta)^3.$$

\* \* \*

## REPLY TO DISCUSSION

Nils Salvesen

The author is grateful to the discussers for their contributions to this work.

Mr. Giesing, Dr. Tuck, and Professor Michelsen questioned the assumption of omitting the second-order contributions related to the body-surface boundary condition. These very important second-order contributions have recently been investigated by the author, and the results have now been included in an addendum to this paper.

It is interesting to note that Mr. Giesing predicted in his discussion the importance of the circulation term as mentioned in the author's addendum.

Dr. van Wijngaarden raises the question of "whether there is also a vertical force on the body and of which order of magnitude." For this symmetrical body at zero incidence the first-order theory will predict no vertical force, while the second-order theory clearly will give a vertical force due to the nonuniform flow caused by the first-order wave system. In this work only the surface waves and the wave resistance were investigated; however, the vertical force can easily be determined from the consistent second-order potential.

In closing the author would like to express how pleased he was to hear that Dr. Kim at the Davidson Laboratory is now applying Tuck's procedure to the three-dimensional problem of a sphere and that at the University of Michigan one of Professor Michelsen's students is studying boundary layer effects for the same two-dimensional body treated by the author.

\* \* \*

# ADDED RESISTANCE OF SHIPS IN WAVES

W. P. A. Joosen  
*Netherlands Ship Model Basin*  
*Wageningen, The Netherlands*

## ABSTRACT

An expression is derived for the drift force acting on a slender ship in waves. The speed effect in the added resistance is taken into account by using the frequency of encounter. Numerical calculations based on a two-dimensional strip theory are carried out for various ship forms and the results are compared with experimental data.

## INTRODUCTION

The problem of a ship heaving and pitching in the presence of incident waves can be treated by a linear hydrodynamical theory. The forces acting on the ship are purely harmonic and have a linear relation with the motion amplitude or the wave height. Within the framework of the linear theory it is possible, however, also to determine higher order steady forces in the horizontal plane which are due to various nonlinear effects.

After the work of Havelock (1) and Hanaoka (2) the first complete analysis of the drift forces was obtained by Maruo (3). He derived expressions in terms of the Kochin function. The drift force appeared to be composed of two effects: the motion of the ship and the reflection of the waves. Maruo carried out calculations for simple ship shapes.

The work of Maruo was extended by Newman (4), who included the determination of the vertical moment. He further obtained approximate expressions for slender ships in waves of moderate or large lengths.

In the present paper a slender body approximation will be developed for short waves. It is well known that in the theory of ship motions this procedure leads to a two-dimensional strip theory, the resulting values being in good agreement with the experimental data. It is expected that a strip theory will give useful results for the drift force as well. Therefore Maruo's expression will be expanded into an asymptotic series with respect to the slenderness parameter, taking into account the first order term only.

## DRIFT FORCE

First we shall consider a ship at zero forward speed in the presence of incident plane progressive waves.

The fluid is supposed to be incompressible, irrotational, and inviscid; therefore a potential theory can be applied. The  $x$  coordinate is taken in the longitudinal direction of the ship, and the  $z$  coordinate is taken positive upward. The  $x$ - $y$  plane coincides with the position of the undisturbed free surface of the fluid.

The ship is supposed to be slender, i.e., the beam-to-length ratio  $\epsilon = B/L$  is small. Moreover, the wave amplitude  $\bar{h}$  and the amplitude of the ship motion  $\bar{a}$  are assumed to be small; consequently a linearized theory can be used. The motion is assumed to be harmonic in time with frequency  $\omega/2\pi$ . The fluid velocity can be represented by the gradient of a velocity potential  $\Phi$ .

The following dimensionless quantities are introduced:

$$x = 2\bar{x}/L, \quad \epsilon y = 2\bar{y}/L, \quad \epsilon z = 2\bar{z}/L, \quad h = 2\bar{h}/L, \quad a = \bar{a}/\bar{h},$$

$$k = \omega^2 B/2g, \quad \Phi = gL/2\omega \cdot h\phi e^{-i\omega t}.$$

In this coordinate system the transverse coordinates are stretched in the ratio  $\epsilon^{-1}$ ; therefore the longitudinal and transverse dimensions of the ship are of the same order of magnitude. The cross section curves  $C$  are given by the equation  $y = f(x, z) \operatorname{sgn} y$ . The frequency parameter is assumed to be of order unity.

Let the incident wave be characterized by the potential function

$$\phi_0 = \exp \left[ k \left( z + i \frac{x}{\epsilon} \cos \beta + iy \sin \beta \right) \right],$$

where  $\beta$  is the angle of wave incidence relative to the  $x$  axis. Let  $\phi_1$  be the disturbance potential due to the presence of the body.  $\phi_1$  must satisfy the Laplace equation, the free surface condition, the radiation condition, and a boundary condition at the ship's hull. This last condition reads in the slender body approximation

$$\frac{\partial \phi_1}{\partial \nu} = - \frac{\partial \phi_0}{\partial \nu} - ikf_z a (1 - f_z^2)^{-1/2},$$

where  $\nu$  is the direction normal to the ship's hull  $S$ . Since the problem is linear  $\phi_1$  can be split up in two other potential functions

$$\phi_1 = \phi_d + a\phi_r,$$

with

$$\frac{\partial \phi_d}{\partial \nu} = - \frac{\partial \phi_0}{\partial \nu}$$

and

$$\frac{\partial \phi_r}{\partial \nu} = -ikf_z (1 + f_z^2)^{-1/2}$$

at S.  $\phi_d$  is the diffraction potential due to the incident wave  $\phi_0$ , and  $\phi_r$  is the radiation potential due to the oscillatory motion of the body for the case of amplitude unity.

These functions can be determined by solving the boundary value problems. We shall restrict ourselves to slender bodies and to frequencies for which  $k$  is of order unity. It is well known that these restrictions lead to the results of the two-dimensional strip theory (5).

The mean force and moment acting horizontally on the ship can be related to the rate of change of linear momentum within the fluid domain bounded by the ship's hull, the free surface, and a control surface at infinity. The result is derived by Maruo (3) and Newman (4) and reads for the two force components

$$F_x = \frac{\bar{F}_x}{\frac{1}{4} \rho g h^2 B^2 L} = \frac{k}{16\pi\epsilon^2} \int_0^{2\pi} (\cos \theta + \cos \beta) |H(\theta)|^2 d\theta, \quad (1)$$

and

$$F_y = \frac{\bar{F}_y}{\frac{1}{4} \rho g h^2 B^2 L} = \frac{k}{16\pi\epsilon^2} \int_0^{2\pi} (\sin \theta + \sin \beta) |H(\theta)|^2 d\theta, \quad (2)$$

where  $H(\theta)$  is the Kochin function

$$H(\theta) = \int_{-1}^1 dx \int_{C(x)} \left( \frac{\partial \phi_1}{\partial \nu} - \phi_1 \frac{\partial}{\partial \nu} \right) e^{k(z+i(x/\epsilon) \cos \theta + iy \sin \theta)} (1 + f_z^2 + \epsilon^2 f_x^2)^{1/2} dz.$$

These expressions are still valid for bodies of arbitrary shape.

We shall apply this result to a slender ship for the case of  $k$  being of order unity.

The first order term in the series expansion of Eq. (1) and Eq. (2) with respect to  $\epsilon$  will be derived in the next section.

## ASYMPTOTIC EXPANSION

The expression for  $H(\theta)$  can be written as



$$H(\theta) = -k \int_{-1}^1 dx \int_{C(x)} a G_1(x, z, \theta) e^{ik(x/\epsilon) \cos \theta} dz \\ - k \int_{-1}^1 dx \int_{C(x)} G_2(x, z, \theta) e^{ik(x/\epsilon)(\cos \theta + \cos \beta)} dz ,$$

where

$$G_1 = \left[ if_z + (i \sin \theta - f_z) \phi_r \right] e^{k(z+iy \sin \theta)} ,$$

and

$$G_2 = \left[ (i \sin \beta - f_z) e^{k(z+iy \sin \beta)} + (i \sin \theta - f_z) \phi_d e^{-ik(x/\epsilon) \cos \beta} \right] e^{k(z+iy \sin \theta)} .$$

The expressions for the force components become

$$F = \frac{k^3}{16\pi\epsilon^3} \int_{-1}^1 dx \int_{C(x)} dz \int_{-1}^1 d\xi \int_{C(\xi)} d\zeta \int_0^{2\pi} g(\theta) d\theta \left\{ A_1 e^{(ik/\epsilon)(x-\xi) \cos \theta} \right. \\ + A_2 e^{(ik/\epsilon)(x-\xi)(\cos \theta + \cos \beta)} + A_3 e^{(ik/\epsilon)[(x-\xi) \cos \theta - \xi \cos \beta]} \\ \left. + A_4 e^{(ik/\epsilon)[(x-\xi) \cos \theta + x \cos \beta]} \right\} ,$$

where

$$g(\theta) = \cos \theta + \cos \beta$$

for the  $x$  component;

$$g(\theta) = \sin \theta + \sin \beta$$

for the  $y$  component, and further

$$A_1(\xi, \theta) = G_1 \bar{G}_1 , \quad A_2(\xi, \theta) = G_2 \bar{G}_2 , \quad A_3(\xi, \theta) = G_3 \bar{G}_3 , \quad A_4(\xi, \theta) = G_4 \bar{G}_4 .$$

The bar means that the complex conjugate has to be taken.

Applying the method of stationary phase on the inner integrals of  $F$  (5,6) we obtain the result

$$F_x = \frac{k^2}{16\epsilon^2} \cos \beta \int_{-1}^1 dx \int_{C(x)} \left[ A_1 \left( x, \frac{\pi}{2} \right) + A_1 \left( x, \frac{3\pi}{2} \right) \right] dz ,$$

and

$$F_y = \frac{k^2}{16\epsilon^2} \int_{-1}^1 dx \int_{C(x)} \left[ (1 - \sin \beta) A_1 \left( x, \frac{\pi}{2} \right) + (-1 + \sin \beta) A_1 \left( x, \frac{3\pi}{2} \right) + 2A_2(x, \pi - \beta) \right] dz.$$

It can immediately be concluded that in this approximation the longitudinal force component depends on the radiation potential  $\phi_r$  only; in contrast with the transverse component, which also depends on the diffraction potential  $\phi_d$ . It is consistent with the theory developed here to use for  $\phi_r$  the values which can be calculated with the two-dimensional strip theory.

In the next part of the paper we shall restrict ourselves to the longitudinal force component, since this component is important for the computation of the added resistance of a ship in waves.

After substitution of the various expressions the formula for  $F_x$  reads

$$F_x = -\frac{k^2}{8\epsilon^2} \cos \beta \int_{-1}^1 |a|^2 |P(x)|^2 dx, \quad (3)$$

where

$$P(x) = \int_{C(x)} \left[ i f_z + (i - f_z) \phi_r \right] e^{k(z+iy)} dz$$

and  $a$  is the dimensionless amplitude of the ship motion at the point  $x$ . Let the heave amplitude be given by  $\bar{z} e^{i\epsilon_1}$  and the pitch angle amplitude by  $\bar{\psi} e^{i\epsilon_2}$ . Introducing the dimensionless quantities

$$z = \bar{z}/\bar{h}, \quad \psi = \lambda \bar{\psi}/2\pi \bar{h}$$

we obtain

$$|a|^2 = z^2 + \frac{\pi^2 L^2}{\lambda^2} x^2 \psi^2 + \frac{2\pi L}{\lambda} x \psi z \cos(\epsilon_1 - \epsilon_2).$$

The expression for  $|P(x)|^2$  can be reduced to a more simple form

$$P(x) = \frac{1}{k} \int_{C(x)} \left( \frac{\partial \phi_r}{\partial \nu} - \phi_r \frac{\partial}{\partial \nu} \right) e^{k(z+iy)} d\ell.$$

The behaviour of  $\phi_r$  at infinite distance of the two-dimensional body can be written as

$$\phi_r = \bar{A} e^{k(z+iy)} + i\delta.$$

Applying Green's theorem in the  $y$ - $z$  plane leads to

$$P(x) = i \frac{\bar{A}}{k} e^{i\delta}$$

and

$$|P(x)|^2 = \frac{\bar{A}^2}{k^2}.$$

The final expression for  $F_x$  becomes then

$$F_x = \frac{\cos \beta}{8\epsilon^2} \int_{-1}^1 |a|^2 \bar{A}^2(k, x) dx, \quad (4)$$

where  $\bar{A}$  is related to the two-dimensional damping by

$$n = \bar{A}^2 \frac{\rho g^2}{\omega^3}. \quad (5)$$

The result obtained by Havelock is

$$\bar{F}_x = \frac{1}{2} \frac{\omega^2}{g} (B_0 \zeta_0 \sin \gamma + P_0 \theta_0 \sin \gamma')$$

where  $B_0$  is the amplitude of the exciting force for heave,  $\zeta_0$  the amplitude of the heave motion, and  $\gamma$  the phase lag.  $P_0$ ,  $\theta_0$ , and  $\gamma'$  are the same quantities corresponding to the pitch motion. Using Havelock's uncoupled equations of motion the result can also be written as

$$\bar{F}_x = \frac{1}{2} \frac{\omega^3}{g} (N \zeta_0^2 + N' \theta_0^2), \quad (6)$$

where  $N$  and  $N'$  are the damping values. This expression is exactly the same as Eq. (4) except for the coupling term of heave and pitch.

## ADDED RESISTANCE

In order to obtain the added resistance of a ship moving in waves, it seems necessary to derive a theory in which the forward speed effect is included.

Let us consider, however, the case of a slender, sharply pointed body which oscillates in short waves, the frequency parameter  $k$  being of order unity. If the body is moving forward at speed  $V$ , we have to assume the order of magnitude of the Froude number in regard to the slenderness parameter in order to develop a correct asymptotic expansion with respect to  $\epsilon$ .

The speed  $V$  appears in the first order term of the velocity potential if we assume the Froude number to be of order unity or the parameter  $\omega V/g$  to be close to 0.25. When considering practical values, however, it seems reasonable to suppose the Froude number to be of order  $\sqrt{\epsilon}$ . From (5) we know that this

assumption leads to a first order term in the potential which does not contain forward speed effects. We may expect, therefore, that the expression for the added resistance is also independent of the speed.

Therefore we shall apply the expression for the drift force at zero forward speed to the problem of the moving body as well:  $\Delta R = F_x$ . For the determination of the frequency parameter  $k$ , we shall use the frequency of encounter:

$$\omega = \left( \frac{2\pi g}{\lambda} \right)^{1/2} - \frac{2\pi}{\lambda} V \cos \beta.$$

Although not being consistent with the mathematical theory we shall apply this procedure, since we know that it gives very reliable results in determining the ship motions.

## THEORETICAL AND EXPERIMENTAL RESULTS

To determine the drift force and added resistance for an actual ship, a computer program has been written based on Eq. (3). The values of the potential  $\phi_r$ , the motion amplitudes, and the phase difference are calculated by a method of Tasai (7) for a two-parameter family of cross sections.

Numerical results are obtained for a "Series 60" model ( $C_B = 0.65$ ) in head waves at zero forward speed. To compare the results with experimental data, the model was tested in the sea-keeping laboratory of the Netherlands Ship Model Basin. The results are plotted against the wavelength ratio  $\lambda/L$  in Fig. 1. The agreement between theory and experiment is satisfactory for wavelengths

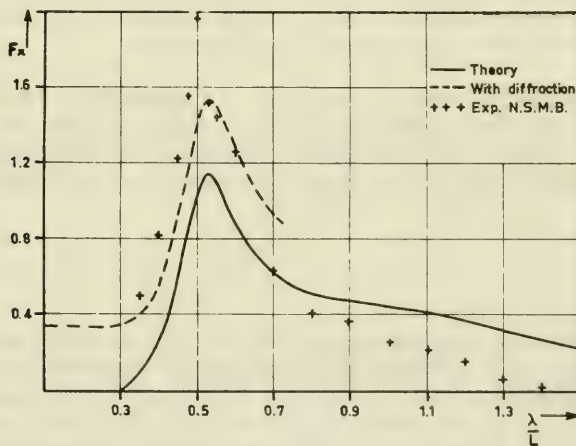


Fig. 1 - Longitudinal force component  $F_x$  vs wavelength ratio  $\lambda/L$  for "Series 60" model ( $C_B = 0.65$ ) ( $F_r = 0$ ) in head waves at zero forward speed



of the order of the ship length. For short waves, however, a rather serious discrepancy occurs. This can be explained by a simple theoretical consideration.

The expression for the longitudinal drift force depends on the radiation potential only. It is proportional to the square of the motion amplitude. The diffraction effect of the waves is of negligible order of magnitude. This is correct provided that the motion amplitude is of the same order of magnitude for the whole frequency range. For short waves, however, the motion amplitudes appear to be much smaller than in the case of larger wavelengths. Therefore it is to be expected that for higher frequencies the diffraction effect becomes more and more important in relation to the radiation effect. In the limiting case for infinite small wavelength the added resistance is caused by the diffraction effect only. To get an impression about the diffraction effect a correction term is added to the theoretical curve in Fig. 1. For this correction term an expression is taken for the drift force in very short waves, obtained by Havelock (8) depending on the wave diffraction only:

$$F_x = 2\epsilon^{-1} \int_0^1 \sin^2 \alpha \, dy,$$

where  $\alpha$  is the angle between the tangent of the water plane curve and the  $x$  axis. The agreement between theoretical values and experimental data then becomes much better.

As a second example the added resistance is calculated for a fast cargo ship with a block coefficient  $C_B = 0.62$ . The values for Froude number  $F_r = 0.225$  are represented in Fig. 2 together with the experimental results (9). Especially the peaks show an excellent agreement. It is plausible that the discrepancy in the high frequency range is again due to the diffraction effect, while the discrepancy in the low-frequency range can be explained by a failure of the theory, since it is derived for rather high frequencies.

Finally, experimental data of Sibul (10) are compared with theoretical calculations. For three "Series 60" models ( $C_B = 0.6, 0.7, 0.8$ ) the results are plotted in Fig. 3. The model with block coefficient 0.8 shows a remarkable discrepancy. In Fig. 4 the peak values are represented as a function of the Froude number.

## CONCLUSIONS

The first-order term of the longitudinal component of the drift force is proportional to the square of the motion amplitude. The diffraction effect is found to be of negligible order. Except for a coupling term the expression obtained is equivalent to the result of Havelock. From a calculatory point of view the formula is more useful, since it is easier to calculate the damping coefficient than the phase difference between exciting force and ship motion.

# Added Resistance of Ships in Waves

Fig. 2 - Added resistance for a fast cargo ship with a block coefficient  $C_B = 0.62$  and a Froude number  $F_r = 0.225$

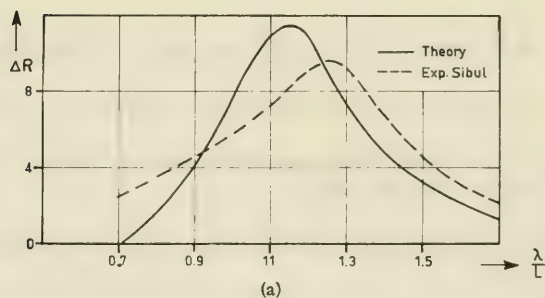
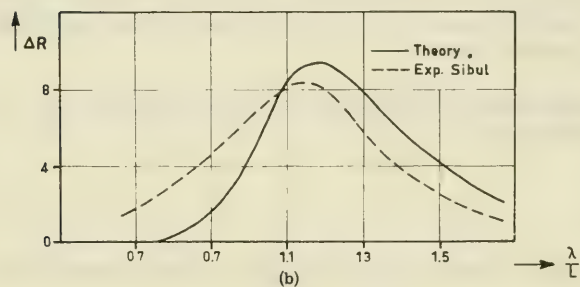
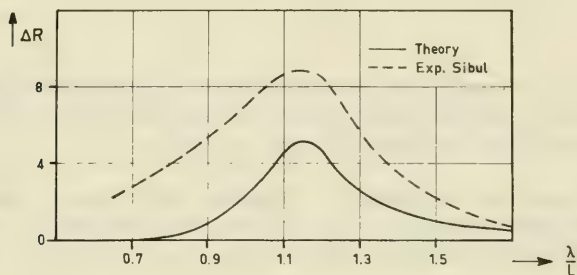
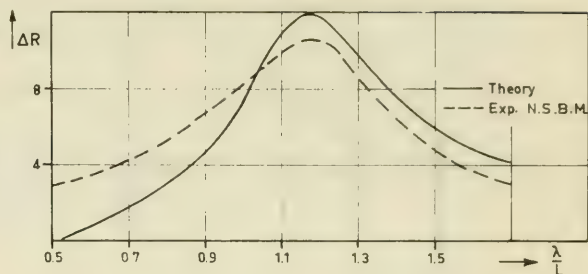


Fig. 3 - Added resistance for "Series 60" models ( $F_r = 0.2$ ); (a)  $C_B = 0.6$ , (b)  $C_B = 0.7$ , and (c)  $C_B = 0.8$

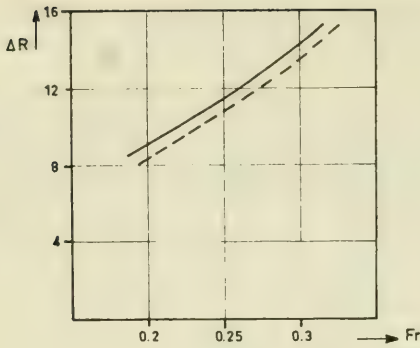


Fig. 4 - Peak resistance values as a function of Froude number

The theoretical result is in agreement with experimental data in the range where the wavelength has the same order of magnitude as the ship length. For shorter waves the diffraction effect is not negligible.

Ships which are moving forward at constant speed have a peak of added resistance in the neighbourhood of  $\lambda/L = 1$ . Using the frequency of encounter, the theory gives very useful results in this range.

From the theoretical calculations it can be concluded that the added resistance decreases with increasing block coefficient. This cannot be verified by the available experimental data.

Further experiments are necessary to check the theory especially for larger block coefficients.

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# SECOND-ORDER CONTRIBUTIONS TO SHIP WAVES AND WAVE RESISTANCE

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The ONR-NSF Symposium on Wave Resistance Theory held in Ann Arbor in 1963 made clear that current research is focused around (a) determination of quantities from the wave pattern representative for wave resistance, (b) formal and semiempirical corrections to the classical linearized theory, and (c) more refined techniques for optimizing ship forms within linear theory. In the present paper I shall report on work done since then which might provide material to reinforce progress in any of these directions. The "pièce de résistance" of this contribution is the gradual evolution of a computer program which in a rationalized way gives the basic information of flow and wave components due to typical singularities (as discrete doublets, doublet struts, continuous parabolic distributions on submerged lines, infinite and truncated vertical planes) all within linearized approach. This information lends itself readily for application to item (a). Any method proposed for determination of energy flow from characteristics of the flow, in particular from the geometry of wave pattern, can be tested for accuracy and for consistency on such a theoretical wave field available numerically before entering into expensive experimental work which provides in general too little reliable information on optimal choice of region where to perform measurements. The overwhelming part of the methods proposed for (a) is implicitly based on validity of certain asymptotic representations for the wave pattern. Only numerical calculations can tell what distances are already large enough, especially regarding decay of the so-called "local flow components" in order that such representations may be applied.

For (b), the theories of wave resistance used nowadays are of second order, based on linearized flow models. For calculation of wave resistance, only a far-field component of this flow has to be known explicitly; for any consistent approach to third-order resistance contributions, however, the knowledge of the entire first-order flow is essential. Aside from some semiempirical approaches to alternate formulations of linear theory, which we shall submit to some critical examination, and aside from an indirect approach as successfully carried out by Kajitani recently, the tool for a systematic perturbation attack to the higher order flow components has been provided by Wehausen (1-3) in a series of papers starting with that read before this audience in 1956 up to his contribution to the Ann Arbor conference. As, however, the step to formulate

resistance expressions was not performed, credit is generally given to Sisov (4) for first dealing with these. We should, nevertheless, be aware that expressions given by Sisov so far essentially contain divergent integrals due to the selection of an improper radiation condition for Green's function of the pressure point. In our present investigation, we will rederive some of Sisov's results from a Green's theorem approach essentially following Wehausen. We will, in particular, show some simplifications which make calculations straightforward once a Fourier representation of first-order flow components is given. It will become evident that integration over the undisturbed free surface has to be performed only in a small domain where local flow is significant; third-order wave resistance is, therefore, much more tractable to numerical evaluation than is apparent from what was formulated by Sisov, provided we decide on an appropriate definition of wave resistance.

We decided to deviate from Wehausen's approach by some simplifications regarding the actual flow boundaries. However, the resulting expressions found for third-order resistance depend in a simple manner only on the ship's offsets and on first-order velocity components. We, therefore, feel that these deviations at least have not introduced artificial complications against results still to be found from more refined analysis.

Regarding the third problem, i.e., ships of minimum resistance within lowest order theory, our investigation should throw light on the question as to what degree third-order contributions might counteract the tendencies predicted. At the present stage, however, our calculations are limited to a two-parameter class of hull forms having parabolic waterlines. This is mainly due to the fact that we preferred analytical evaluation of integrals over the geometry of the ship. An extension of our program for local flow, to include contributions from empirical surface elements is feasible, but loss of closed integration would probably increase time for computation and weaken control of accuracy. Moreover, the necessary degree of hull subdivision will in general depend on the Froude number and is not known beforehand. Even for analytical ship forms, the development of formal expressions for closed integration cannot be done by the computer and provides many opportunities for errors in evaluation of singular regions of integrands for local flow.

#### DESCRIPTION OF ANALYSIS TO DERIVE POTENTIALS AND WAVE RESISTANCE

We shall essentially follow the approach of Wehausen (3), but modify it for flow in a tank of rectangular cross section. This will simplify the formulation of radiation conditions for the flow and allows the use of a Green's function in a Fourier series representation regarding the coordinate  $y$  chosen in the direction perpendicular to the vertical tank walls. The ship's motion is in the  $+x$  direction with speed  $c$ ; the  $z$  coordinate is taken vertically upward to conform with earlier work (6). As far as possible we otherwise use notation consistent with Ref. 3. However, the direction of normal vectors is reversed resulting from our definition of Green's function with an opposite sign. Extension of the results to unrestricted water is straightforward.



# Derivation of the Second-Order Potential

We introduce the dimensionless coordinates  $X = 2x/L$ ,  $Y = 2y/L$ , and  $Z = 2z/L$ , where  $L$  is the ship's length. The velocity potential is nondimensionalized as  $\psi = 2\phi/Lc$ . As speed parameter we use  $\gamma_0 = gL/2c^2$ .

Let  $Y = \pm \epsilon F(X, Z)$  be the dimensionless representation of the hull geometry, where  $B$  is the ship's breadth.  $\epsilon = B/L$  will serve as a perturbation parameter and is considered as a small quantity. Let  $X = X_a$  and  $X = X_e$  be the equations of two vertical control planes  $S_a$  and  $S_e$  ahead of and behind the ship (Fig. 1). Let  $S_b$  stand for the tank bottom plane  $Z = -H$ . Let  $Y = \pm T$  be the equations of the vertical tank walls  $S_r$  and  $S_l$ , where  $T = b/L$  and  $b$  = tank width. Let  $S_f$  stand for the free surface  $Z = \zeta(X, Y)$  for  $X_e < X < X_a$  and  $T < Y < T$ , and let  $S_f^0$  stand for the undisturbed free surface  $Z = 0$  with the water-plane area of the ship excluded. Let  $S_w$  stand for the wetted surface of the ship, and let  $S_w^0$  stand for the part of the surface up to  $Z = 0$ . Let  $D$  stand for the domain of the complete flow, bounded by  $S_w$ ,  $S_f$ ,  $S_a$ ,  $S_e$ ,  $S_r$ ,  $S_l$  and  $S_b$ , and let  $D^0$  describe the corresponding domain if  $S_w$  and  $S_f$  are replaced by  $S_f^0$  and  $S_w^0$ . Let  $\psi^{(1)}$  stand for the Michell type first approximation to the exact potential  $\psi$ . Let  $P$  stand for a point in  $D$  or  $D^0$  with coordinates  $X$ ,  $Y$ , and  $Z$ , and let  $P'$  represent a point on a boundary surface with coordinates  $\xi$ ,  $\eta$ , and  $\zeta$ . Let  $G(P, P')$  stand for the potential of a source of output  $4\pi$  as defined in the appendix.

The functions  $\psi$ ,  $\psi^{(1)}$ , and  $G$  of the variables  $X$ ,  $Y$ , and  $Z$  are subject to the following set of conditions:

A. Laplace equation:  $\Delta\psi = 0$  in  $D$ ,  $\Delta\psi^{(1)} = 0$  in  $D^0$ , and  $\Delta G = 4\pi\delta(P - P')$  in  $D^0$ , where  $\Delta$  stands for  $\delta^2/\delta X^2 + \delta^2/\delta Y^2 + \delta^2/\delta Z^2$  and  $\delta$  means the Dirac delta function, which is zero if  $P$  is unequal  $P'$ . (Condition A implies that  $G$  becomes singular at  $-1/|P - P'|$ .)

B. On  $S_f^0$  we have (linearized free surface condition):

$$\gamma_0 \psi_Z^{(1)} + \psi_{XX}^{(1)} = 0; \quad \gamma_0 G_Z + G_{XX} = 0.$$

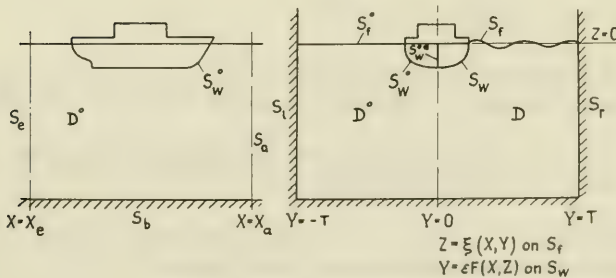


Fig. 1 - Nomenclature



For the exact potential  $\psi$ , no such condition holds. But we define a function  $\delta(X, Y)$  by

$$\gamma_0 \psi_Z + \psi_{XX} = \delta(X, Y).$$

C. On  $S_\ell$  and  $S_r$  we have  $\psi_Y = 0$ ,  $\psi_Y^{(1)} = 0$ , and  $G_Y = 0$ .

D. On  $S_b$  we have  $\psi_Z = 0$ ,  $\psi_Z^{(1)} = 0$ , and  $G_Z = 0$ .

E. On  $S_w$  we have  $\psi_n = \pm \epsilon F_X / \sqrt{\epsilon^2 F_X^2 + \epsilon^2 F_Z^2 + 1}$ , where  $\pm$  stands for  $\eta$  positive or negative and the index  $n$  stands for derivation in normal direction out of the fluid's domain  $D$ . On  $S_w^0$ , the projection of  $S_w^0$  on the plane  $Y = 0$ , we have  $\psi_Y^{(1)} = \pm \epsilon F_X$  for the first-order potential.

F. For fixed  $P'$  we have

$$G = O(1) \text{ as } X \rightarrow -\infty; \quad G = O(X^{-1}) \text{ as } X \rightarrow +\infty,$$

$$G_X = O(1) \text{ as } X \rightarrow -\infty; \quad G_X = O(X^{-1}) \text{ as } X \rightarrow +\infty.$$

Application of Green's theorem shows that  $\psi^{(1)}$  and  $\psi$  can be defined subject to the same modes of asymptotic decay, provided the quantity  $\delta(X, Y)$  will then turn out to be well behaved. The symmetry of functions  $\psi$  and  $\psi^{(1)}$  with regard to the lateral coordinate  $Y$  will be taken for granted by the symmetry of the ship sections and the tank profile.

We should bear in mind that (3) for the function  $\psi$  existence as a harmonic function is — if at all — guaranteed only in domain  $D$  but not necessarily in  $D^0$ . However, low-order approximations have been derived (3) which are found to exist in the whole interior of  $D^0$ . For the moment we seek terms up to second degree only; we shall in the following formulate the problem for the domain  $D^0$  with boundaries known a priori in favor of a less intricate analysis, and derive approximate solutions to this auxiliary problem by perturbation techniques. Then, for point  $P$  within  $D^0$ , we may apply Green's theorem to functions  $\psi$  and  $G$  to find a representation of  $\psi(P)$  as

$$\psi(P) = \frac{1}{4\pi} \int \left[ \psi_n(P') G(P, P') - \psi(P') G_n(P, P') \right] dS', \quad (1)$$

where the closed boundary  $S'$  is composed of  $S_w^0$ ,  $S_f^0$ ,  $S_r$ ,  $S_\ell$ ,  $S_b$ ,  $S_a$ , and  $S_e$  and the subscript  $n$  stands for normal derivative outward in  $P'$  space.

From conditions C and D we may conclude that the contributions of  $S_\ell$ ,  $S_r$ , and  $S_b$  can be omitted on the right-hand side.

The integral over  $S_f^0$ , where the normal derivative is in  $Z$  direction, may be transformed by integration regarding  $\xi$  and use of B:

$$\begin{aligned} \frac{1}{4\pi} \int_{S_f^0} (\psi_Z G - \psi G_Z) dS &= \frac{1}{4\pi} \frac{1}{\gamma_0} \int_{-T}^T \left| (\psi_X G - \psi G_X) \right|_{\xi=x_e}^{\xi=x_a} d\eta \\ &+ \frac{1}{4\pi\gamma_0} \int_{S_f^0} \delta(\xi, \zeta) G dS - \frac{1}{4\pi\gamma_0} \oint_{L_P} (\psi_X G - \psi G_X) d\eta, \quad (2) \end{aligned}$$

where the line integral around the ship's load waterline  $L_P$  has to be understood in counterclockwise sense when viewed from above (compare Eq. (19) of Ref. 3).

The line integral has been thoroughly investigated by Yim (5). We shall find, however, that it is pertinent to merge it with a similar term from the wetted surface  $S_w^0$ . If now we assume the functions  $\psi$  and  $\psi_X$  uniformly bounded for  $X < X_e$ , then, due to the finite size of  $S_e$  with conditions F, we may infer that the contribution of  $S_e$  becomes insignificant as we let  $X_e$  tend to  $-\infty$ . Similarly, if  $\psi$  and  $\psi_X$  tend to zero as  $X \rightarrow \infty$ , then, due to boundedness of  $G$  and  $G_X$ , the contribution of  $S_a$  may be neglected with  $X_a$  becoming large. But the contributions of  $S_a$  and  $S_b$  must be independent of position  $X_a$  and  $X_e$  inasmuch as the contribution of the defect  $\delta(X, Y)$  may be neglected. Considering higher order terms, however, we will see that independence from  $X_e$  cannot be assumed in general.

For the first integral in Eq. (2) over the wetted surface  $S_w^0$ , we shall make the assumption that condition E for  $\psi$  holds even for parts of the hull not included in  $S_w$  up to the undisturbed free surface, so that we may substitute

$$\psi_n dS = \epsilon F_X / \sqrt{\epsilon^2 F_X^2 + \epsilon^2 F_Z^2 + 1} dS_w^0, \quad (3)$$

and observe that

$$dS_w^0 = \sqrt{\epsilon^2 F_X^2 + \epsilon^2 F_Z^2 + 1} dS_w^{0*}. \quad (4)$$

For the second integral over  $S_w^0$  we substitute the actual components of the normal vector to obtain (compare Eq. (18) of Ref. 3)

$$(-F_X, \pm 1, -F_Z) / \sqrt{1 + \epsilon^2 F_X^2 + \epsilon^2 F_Z^2}$$

and thereby have

$$\begin{aligned} \frac{-1}{4\pi} \int_{S_w^0} \psi G_n dS' &= \frac{-1}{4\pi} \iint_{S_x^{0+}} \psi \left[ \xi, \pm \epsilon F(\xi, \zeta), \zeta \right] \left[ \epsilon F_X (G_\xi^+ + G_\xi^-) \right. \\ &\quad \left. + \epsilon F_Z (G_\zeta^+ + G_\zeta^-) - G_\eta^+ + G_\eta^- \right] d\xi d\zeta \quad (5) \end{aligned}$$

where  $\pm$  stands for  $\eta$  positive or negative. By partial integration regarding  $\xi$  and  $\zeta$ , observing the Laplace equation for  $G$  as stated in A and making use of the fact that  $F = 0$  at the integration limits if  $\zeta < 0$ , we then have

$$\begin{aligned} -\frac{1}{4\pi} \int_{S_w^0} \psi G_n dS' &= \frac{\epsilon}{4\pi} \iint_{S_w^0} F(\xi, \zeta) \left[ \psi_X (G_\xi^+ + G_\xi^-) + \psi_Z (G_\zeta^+ + G_\zeta^-) \right] d\xi d\zeta \\ &- \frac{\epsilon}{4\pi} \iint_{S_w^0} F(\xi, \zeta) \psi (G_{YY}^+ + G_{YY}^-) d\xi d\zeta + \frac{1}{4\pi} \iint_{S_w^0} \psi (G_\eta^+ - G_\eta^-) d\xi d\zeta \\ &- \frac{\epsilon}{4\pi} \int_{\xi=-1}^1 \psi F(\xi, 0) (G_\zeta^+ + G_\zeta^-) d\xi. \end{aligned} \quad (6)$$

We can now transform the line integral, obtained previously, as

$$\begin{aligned} -\frac{1}{4\pi\gamma_0} \oint_{L_P} (\psi_X G - G_\xi \psi) d\eta &= -\frac{1}{4\pi\gamma_0} \int_{-1}^1 \left[ \psi_X (G^+ + G^-) - \psi (G_\xi^+ + G_\xi^-) \right] F_X(\xi, 0) d\xi \\ &= -\frac{1}{4\pi\gamma_0} \int F(\xi, 0) \left[ \psi_{XX} (G^+ + G^-) - \psi (G_{\xi\xi}^+ + G_{\xi\xi}^-) \right] d\xi \end{aligned} \quad (7)$$

and then combine all components, observing B for  $G$ , as

$$\begin{aligned} \psi &= -\frac{\epsilon}{4\pi} \iint_{S_w^0} F(\xi, \zeta) (G_\xi^+ + G_\xi^-) d\xi d\zeta + \frac{1}{4\pi\gamma_0} \iint_{S_f^0} \delta(\xi, \eta) G d\xi d\eta \\ &+ \frac{\epsilon}{4\pi\gamma_0} \int_{-1}^1 F(\xi, 0) \psi_{XX} (G^+ + G^-) d\xi - \frac{\epsilon}{4\pi} \int_{S_w^0} F(\xi, \zeta) \left[ \psi_X (G_\xi^+ + G_\xi^-) + \psi_Z (G_\zeta^+ + G_\zeta^-) \right] d\xi d\eta \\ &+ \left| \frac{1}{\gamma_0} \int_{-T}^T \left\{ \psi_X(\xi, \eta, 0) G - \psi(\xi, \eta, 0) G_\xi \right. \right. \\ &\quad \left. \left. + \int_{-H}^0 \left[ \psi_{XX}(\xi, \eta, \zeta) G - \psi_X(\xi, \eta, \zeta) G_\xi \right] d\zeta \right\} d\eta \right|_{\xi=x_a}^{\xi=x_e} \\ &+ \frac{1}{4\pi} \iint_{S_w^0} \psi \left[ \xi, \epsilon F(\xi, \zeta), \zeta \right] \left[ (G_\eta^+ - \epsilon F G_{\eta\eta}^+) - (G_\eta^- + \epsilon F G_{\eta\eta}^-) \right] d\xi d\zeta. \end{aligned} \quad (8)$$

If we here neglect the contributions of  $S_a$  and  $S_e$ , the remaining expression, proper behavior of  $\delta(X, Y)$  assumed, really makes this omission legitimate due to the properties stated under  $F$  for the function  $G$ .

So far, we have not used any considerations regarding the smallness of  $\epsilon = B/L$ . We should note that  $G$  is defined even for  $\eta = 0$ , i.e., for  $P'$  within the ship's hull, and is well behaved there, if  $P$  is not too close to  $P'$ . By development in a Taylor series regarding  $\epsilon F$  we therefore may infer that

$$G_{\eta}^{+} = -\epsilon F G_{\eta\eta}^{+} + O(\epsilon^2) \quad (9a)$$

and

$$G_{\eta}^{-} = +\epsilon F G_{\eta\eta}^{-} + O(\epsilon^2), \quad (9b)$$

which shows that the factor of  $\psi$  in the last integral is small at least of order  $\epsilon^2$ . In general we have

$$G_{\zeta}^{+} + G_{\zeta}^{-} = G_{\zeta}(\zeta, 0, \zeta) + O(\epsilon^2) \quad (10a)$$

and

$$G_{\xi}^{+} + G_{\xi}^{-} = G_{\xi}(\xi, 0, \zeta) + O(\epsilon^2). \quad (10b)$$

If we now assume an expansion

$$\varphi = \epsilon \varphi^{(1)} + \epsilon^2 \varphi^{(2)} + O(\epsilon^3), \quad (11a)$$

$$\delta = \epsilon \delta^{(1)} + \epsilon^2 \delta^{(2)} + O(\epsilon^3), \quad (11b)$$

inserting into Green's formula and collecting terms of equal order in  $\epsilon$ , we find that  $\psi^{(1)}$  is just the expression from Michell's theory with  $\delta^{(1)} = 0$  and no contributions from  $S_a$  and  $S_e$ .

For examination of  $\psi^{(2)}$  we must go into the nature of  $\delta^{(2)}(X, Y)$ . From Ref. 2, page 464, Eq. (10.12), we find with  $p = \text{const.}$  and replacing  $\partial/\partial_t$  by  $-\partial/\partial_x$  and  $g$  by  $\gamma_0$  for our nondimensional representation,

$$\gamma_0 \psi_Z^{(2)} + \psi_{XX}^{(2)} = \delta^{(2)}(X, Y) = \left[ \text{grad } \psi^{(1)} \right]_X^2 - \frac{1}{\gamma_0} \psi_X^{(1)} \left[ \gamma_0 \psi_Z^{(1)} + \psi_{XX}^{(1)} \right]_Z. \quad (12)$$

(It should be observed that only the local component of  $\psi^{(1)}$  contributes to the expression brackets in the second term due to structure of  $G$  — see the appendix.) Sisov's expression corresponding to Eq. (12) is incorrect.

From the decay of  $G$  and its derivatives as  $O(X^{-1})$  for  $X \rightarrow +\infty$ , causing the same mode of decay for  $\psi^{(1)}$ , it may be seen that  $\delta^{(2)} = O(X^{-2})$  for  $X \rightarrow +\infty$ , and this means that  $\psi^{(2)} = O(X^{-1})$  ahead of the ship and that the contribution of  $S_a$  to the Green's formula expression may be neglected. If now we can assume that the potential

$$\psi_{\delta} = \frac{1}{4\pi} \int_{S_f^0} \delta G \, dS'$$



and its  $x$  derivative are uniformly bounded for  $x < x_e$  — and to prove this for not too peculiar  $\psi^{(1)}$  should be possible with moderate effort — then we may drop the contribution of  $S_e$  as well as  $x_e$  tending to infinity and may finally write  $\psi^{(2)} = \psi_1^{(2)} + \psi_2^{(2)}$ ,

$$\begin{aligned} \psi_1^{(2)}(X, Y, Z) = & \frac{2}{4\pi} \iint_{S_w^0} F(\xi, \zeta) \left[ \psi_X^{(1)}(\xi, \epsilon F, \zeta) G_\xi(\xi, 0, \zeta) + \psi_Z^{(1)}(\xi, \epsilon F, \zeta) G_\zeta(\xi, 0, \zeta) \right] d\xi d\zeta \\ & + \frac{2}{4\pi\gamma_0} \int_{-1}^1 F(\xi, 0) \psi_{XX}^{(1)} G(\xi, 0, 0) d\xi, \end{aligned} \quad (13)$$

and

$$\psi_2^{(2)}(X, Y, Z) = \frac{1}{4\pi\gamma_0} \iint_{S_f^0} \delta^{(2)}(\xi, \eta) G d\xi d\eta. \quad (14)$$

The second-order potential is thus produced by:

1. A distribution of doublets with moments corresponding to the deviation of the local first-order flow relative to the ship from a uniform parallel flow (giving rise to the Michell distribution) times the local volume of the ship.
2. A distribution of sources over the plane  $Z = 0$  whose density is essentially the time derivative in an inertial system of dynamic pressure (save a contribution of the local flow components in the vicinity of the ship).
3. A line distribution of sources around the ship's load waterline of output corresponding to the local breadth times the wave slope in the  $x$  direction along the ship's contour according to linear theory.

One should observe that the potential appears only in derivative form. On the other hand, no differentiability of the hull surface function  $F$  is required to make the expression for  $\psi^{(2)}$  meaningful.

For numerical evaluation, the following approximations are made: (a) calculating the potential from a distribution  $\delta(X, Y)$  extending over the entire undisturbed free surface  $Z = 0$  including the waterplane area, and (b) inserting the flow components calculated for the plane  $\eta = 0$  rather than on the hull surface. Both these steps require continuation of the flow potential and some of its derivatives into the domain occupied by the ship. This is achieved by extension of the corresponding Fourier series in  $\eta$ . Insofar as these series would not converge for  $\eta = 0$  in the usual sense (for example, the series for  $\psi_Y$ ) we treat them as generalized functions. The second term of  $\delta(X, Y)$  in Eq. (12) will in general become singular at bow and stern; however, for a symmetric hull we will find that it can be left out for calculation of wave resistance.

The error involved with these modifications will in general be of higher order in  $\epsilon$  than the terms to be determined; nevertheless, it should be checked for components not uniformly bounded in the extended domain. It should be

noted that Eq. (13) for  $\psi^{(2)}$  can be retransformed by partial integration in  $x$  and  $z$  to a representation by a source distribution over  $S_w^{0*}$ , eliminating the line integral, so that

$$\psi^{(2)} = \frac{-1}{2\pi} \iint_{S_w^{0*}} G \left\{ \left[ F\psi_X^{(1)} \right]_X + \left[ F\psi_Z^{(1)} \right]_Z \right\} d\xi d\zeta + \frac{1}{4\pi\gamma_0} \int_{-T}^T \int_{-\infty}^{\infty} \delta(\xi, \eta) G d\xi d\eta. \quad (15)$$

We should observe that the source density corresponds to the change of internal flow with coordinate rather than to the normal velocity.

Equation (15) may be compared with Eq. (43) of Ref. 3, in which the influence of trim and sinkage is included. We see that all line integrals presented there can be eliminated under validity of our assumptions.

### Determination of Wave Resistance

Having thus selected a model for the approximation of the flow, there is a decision to be made for definition of wave resistance to the corresponding degree of approximation. Three different approaches may be considered: (a) integrate pressure components over the wetted part of the hull bounded by the calculated wave profile, retaining only terms up to third order, (b) start with expressions for the energy flow through a vertical plane behind the ship, as given in Eq. (8.6), page 460 of Ref. 2, and evaluate these for approximate flow, using the wave contour from this approximate flow, or (c) consider the approximate second-order flow to be physically real in the domain  $D^0$ , that is, consider a closed surface, part of which is the wetted hull, and from the fact that momentum in the enclosed volume  $D$  should not change with time, infer that action of pressure on the hull can be expressed through flow of generalized momentum across the rest of the surface.

It can be shown easily that for the linearized flow model one and the same expression for the resistance can be derived by either approach (see however the objections raised by Sharma (9)). Up to third order, however, (a) and (c) should only give equivalent expressions  $R_a$  and  $R_c$ , if the boundary condition on the hull is already met exactly by the approximate flow, as otherwise we may have substantial flux of momentum into the ship's interior. The formula for (b) was derived under assumption of a free surface under constant pressure and composed of streamlines. For a second-order flow, it is unrealistic to maintain this assumption. We should therefore expect that resistance  $R_b$ , calculated by this formula applied to the approximate flow, could, even in a nonmonotonic way depend on the location  $X_e$  of the vertical control plane where data are taken.

But  $R_c$ , derived by approach (c), should be independent of choice of domain  $D$ . We shall select  $D = D^0$ , the domain bounded above by the undisturbed free surface as described before. Due to conservation of momentum we have for surface integrals enclosing any domain  $D$  of the flow (compare Refs. 2 and 7)

$$\rho \int \{ (\mathbf{v} \cdot \mathbf{v})/2n - (\mathbf{v} \cdot \mathbf{n})\mathbf{v} \} ds = 0, \quad (16)$$

where  $\mathbf{v}$  is the flow vector in any system of reference either at rest or in uniform translatory motion and  $\mathbf{n}$  is the unit normal vector directed outward. If we now select

$$\mathbf{v} = (\varphi_X, \varphi_Y, \varphi_Z) = (\psi_X, \psi_Y, \psi_Z) |c|$$

and define  $R_c$  as the  $X$  component of

$$R = \rho \int_{S_w^0} [(\mathbf{v} \cdot \mathbf{n})\mathbf{v} - (\mathbf{v} \cdot \mathbf{v})/2\mathbf{n}] dS, \quad (17)$$

where integration has to be performed over  $S_w^0$ , the hull surface up to  $Z = 0$ , then we have from Eq. (15), returning to nondimensional quantities, with  $\mathbf{e}_X$  as the unit vector in the  $X$  direction,

$$\begin{aligned} R_c = \frac{R_c \cdot \mathbf{e}_X}{\rho c^2 (L/2)^2 \gamma_0} &= \gamma_0 \iint_{S_f^0} \psi_X \psi_Z d\xi d\eta - \gamma_0 \left\{ \iint_{S_r} - \iint_{S_l} \right\} \psi_X \psi_Y d\xi d\zeta \\ &- \gamma_0 \iint_{S_b} \psi_X \psi_Z d\xi d\eta + \gamma_0 \left( \iint_{S_e} - \iint_{S_a} \right) \frac{\psi_Y^2 + \psi_Z^2 - \psi_X^2}{2} d\eta d\zeta. \end{aligned} \quad (18)$$

Reference to conditions C and D shows that the surface  $S_r$ ,  $S_l$ , and  $S_b$  may be left out. The integral over  $S_f^0$  may be transformed to line integrals along the boundaries and an integral containing the function  $\phi(X, Y)$  in a similar way as was done for the potential given by Eq. (2). The contribution from  $S_a$ , including the line integral from  $S_f^0$ , tends to zero as  $X_a \rightarrow \infty$  due to F. Thus, we are left with

$$\begin{aligned} R_c = & - \iint_{S_f^0} \delta(\xi, \eta) \psi_X(\xi, \eta, 0) d\xi d\eta - \oint_{L_P} \frac{\psi_X^2[\xi, \epsilon F(\xi, 0), 0] d\eta}{2} \\ & + \int_{-T}^T \left[ \frac{\psi_X^2(X_e, \eta, 0)}{2} + \gamma_0 \int_{-H}^0 \frac{\psi_Y^2(X_e, \eta, \zeta) + \psi_Z^2(X_e, \eta, \zeta) - \psi_X^2(X_e, \eta, \zeta)}{2} d\zeta \right] d\eta, \end{aligned} \quad (19)$$

where the line integral over the load waterline  $L_P$  is again in counterclockwise direction when viewed from above.

There is no reason to assume that the contribution from  $S_f^0$ , though bounded in magnitude, should tend to a definite limit as  $X_e \rightarrow -\infty$ , nor can we postulate this for the contribution of the region  $X = X_c$ . It is only by some property of the Green's function  $G$  involved that we shall be able to evaluate the contribution of  $\phi(X, Y)$  to the resistance by an integration on  $S_f^0$  in the vicinity of the ship only. We shall now look for a relation between the quantities defined as  $R_c$  and  $R_a$ . To achieve this, we add an expression to the integrand of Eq. (17) which has no component in the direction of  $\mathbf{e}_X$ . We set

$$R_c = R_e + \left\{ \mathbf{e}_1 \rho \left[ (\mathbf{v} \times \mathbf{n}) \times \mathbf{c} \right] dS \right\} = R_e + \rho \left\{ \mathbf{e}_1 \int \left[ (\mathbf{v} \cdot \mathbf{c}) \mathbf{n} - (\mathbf{c} \cdot \mathbf{n}) \mathbf{v} \right] dS \right\}. \quad (20)$$

Therefore, with  $(\mathbf{c} \cdot \mathbf{n}) = (\mathbf{v} \cdot \mathbf{n})$  assumed even for the first-order flow,

$$R_c = \int \rho (\mathbf{e}_1 \cdot \mathbf{n}) \left[ (\mathbf{c} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{v})/2 \right] dS. \quad (21)$$

But this is just  $R_a$ , the resistance defined from pressure integration over the hull, because the nonstatic pressure is  $\rho[(\mathbf{v} \cdot \mathbf{c}) - (\mathbf{v} \cdot \mathbf{v})/2]$  and we thus have from the contribution of  $S_w^0$

$$R_a = R_c + \rho \int_{S_w^0} \varphi_X [(\mathbf{c} \cdot \mathbf{n}) - (\mathbf{v} \cdot \mathbf{n})] dS. \quad (22)$$

To  $R_c$  as defined above we now add an appropriate correction for the influence of the wave profile along the waterline, as only the wetted part of the hull can experience pressure from the fluid, and then define the quantity obtained as "third-order wave resistance." Up to second order, pressure is atmospheric pressure plus hydrostatic pressure due to the wave elevation  $\zeta$ . Integrating the last quantity over  $dZdY$ , the projection of the surface element on the plane vertical to  $X$  axis, we find a correction as

$$\Delta R = \rho g \iint_{L_P} (Z - \zeta) dZd\eta = \rho g \int_{L_P} \frac{\zeta^2(\xi, \epsilon F)}{2} d\eta. \quad (23)$$

Now the perturbation procedure gives the first-order wave elevation (2) as

$$\zeta^{(1)} = \epsilon \frac{c}{g} \varphi_X, \text{ i.e., } \frac{\zeta^{(1)}}{L/2} = \frac{1}{\gamma_0} \psi_X^{(1)}(X, Y, 0), \quad (24)$$

and adding this in nondimensional form to Eq. (19), we see that this correction just cancels the line integral around  $L_P$ , which we therefore can happily discard (8).

A further simplification will be made by extending the integration of  $\delta$  over the whole plane  $\zeta = 0$ ,  $-T < \eta < T$ , and  $\xi < X_e$ , which means an error of order  $\epsilon^4$ , as the waterplane area is of order  $\epsilon$ . Inserting now  $\psi = \epsilon \psi^{(1)} + \epsilon^2 \psi^{(2)}$ ,  $\delta = \epsilon^2 \delta^{(2)}$ , and  $R = \epsilon^2 R^{(2)} + \epsilon^3 R^{(3)}$  we have

$$R^{(2)} = \int_{-T}^T \left[ \frac{1}{2\gamma_0} \psi_X^{(1)2}(X_e, \eta, 0) + \int_{-H}^0 \frac{\psi_Y^{(1)2} + \psi_Z^{(1)2} - \psi_X^{(1)2}}{2} d\zeta \right] d\eta \quad (25)$$

and  $R^{(3)} = R_1^{(3)} + R_2^{(3)}$  (corresponding to the partition  $\psi^{(2)} = \psi_1^{(2)} + \psi_2^{(2)}$ ), with



$$R_1^{(3)} = \int_{-T}^T \left\{ \frac{1}{\gamma_0} \psi_X^{(1)} \psi_X^{(2)} + \int_{-H}^0 \left[ \psi_Y^{(1)} \psi_{1Y}^{(2)} + \psi_Z^{(1)} \psi_{1Z}^{(2)} - \psi_X^{(1)} \psi_{1X}^{(2)} \right] d\zeta \right\} d\eta \quad (26)$$

and

$$R_2^{(3)} = -\frac{1}{\gamma_0} \int_{-T}^T \int_{X_e}^{\infty} \delta^{(2)}(\xi, \eta) \psi_X^{(1)} d\xi d\eta \\ + \int_{-T}^T \left\{ \frac{1}{\gamma_0} \psi_X^{(1)} \psi_{2X}^{(2)} + \int_{-H}^0 \left[ \psi_X^{(1)} \psi_{2Y}^{(2)} + \psi_Z^{(1)} \psi_{2Z}^{(2)} - \psi_X^{(1)} \psi_{2X}^{(2)} \right] d\zeta \right\} d\eta. \quad (27)$$

### Resistance Due to Additional Singularities Within the Hull

Let us now, only to save labor in writing down formulas, assume that the depth of the tank is large enough that we may put  $H = \infty$ . If  $\psi_X^{(1)}$  can at  $X = X_e$  be represented by a system of free waves (we omit terms nonsymmetric in  $Y$  for reasons of simplicity) as

$$\psi_X^{(1) \text{ free}} = \sum_{\nu=-\infty}^{\infty} \left[ A_{\nu}^{(1)} \cos(W_{\nu} \gamma_0 X) + B_{\nu}^{(1)} \sin(W_{\nu} \gamma_0 X) \right] e^{K_{\nu} \gamma_0 Z} \cos(U_{\nu} \gamma_0 Y), \quad (28)$$

where  $\Delta U = \pi/(\gamma_0 T)$  and  $U_{\nu} = \nu \Delta U = \sec^2 \theta_{\nu} \sin \theta_{\nu}$ ,  $M_{\nu} = 1 + 4U_{\nu}^2$ ,  $K_{\nu} = (1 + M_{\nu})/2 = \sec^2 \theta_{\nu}$ ,  $W_{\nu} = \sqrt{K_{\nu}} = \sec \theta_{\nu}$ ,  $A_{\nu}^{(1)} = A_{-\nu}^{(1)}$ , and  $B_{\nu}^{(1)} = B_{-\nu}^{(1)}$  (where  $\theta_{\nu}$  stands for the angle of wave propagation against the  $X$  axis), then (6) we can evaluate the integrals in closed form as

$$R^{(2)} = T \sum_{\nu=-\infty}^{\infty} \frac{2 - \cos^2 \theta_{\nu}}{2} \left[ A_{\nu}^{(1)2} + B_{\nu}^{(1)2} \right] / \gamma_0. \quad (29)$$

(This formula reflects the fact that the resistance is essentially equal to the average energy in the wave components times the difference between the ship's speed  $c$  and the  $X$  component of the group-velocity, divided by  $c$ .)

If  $\psi_{1X}^{(1)}$  has a corresponding far-field representation

$$\psi_{1X}^{(2) \text{ free}} = \sum_{\nu=-\infty}^{\infty} \left[ A_{1\nu}^{(2)} \cos(W_{\nu} \gamma_0 X) + B_{1\nu}^{(2)} \sin(W_{\nu} \gamma_0 X) \right] e^{K_{\nu} \gamma_0 Z} \cos(U_{\nu} \gamma_0 Y), \quad (30)$$

then  $R_1^{(3)}$  as interference between both systems can be written down directly as

$$R_1^{(3)} = T \sum_{\nu=-\infty}^{\infty} (2 - \cos^2 \theta_{\nu}) \left( A_{1\nu}^{(2)} A_{\nu}^{(1)} + B_{1\nu}^{(2)} B_{\nu}^{(1)} \right) / \gamma_0. \quad (31)$$

For evaluation of Eqs. (29) and (31) we have to keep in mind that for  $\eta = 0$  the Green's function  $G$  has a representation for  $\xi \gg X$  as a system of free waves like

$$G_X \approx \sum_{\nu=-\infty}^{\infty} g_{\nu} \cos \left[ W_{\nu} \gamma_0 (X - \xi) \right] e^{K_{\nu} \gamma_0 (Z + \zeta)} \cos (U_{\nu} \gamma_0 Y) \quad (32)$$

with  $g_{\nu} = g_{-\nu} = -8\pi K_{\nu} \gamma_0 / (M_{\nu} T)$  (see the appendix) and that we have

$$\psi^{(1)} = -\frac{2}{4\pi} \iint_{S_w^{0*}} F(\xi, \zeta) G_{\xi} d\xi d\zeta \quad (33)$$

and

$$\begin{aligned} \psi_{1X}^{(2)} &= \frac{2}{4\pi} \iint_{S_w^{0*}} F(\xi, \zeta) \left[ \psi_X^{(1)} G_{\xi X} + \psi_Z^{(1)} G_{\zeta X} \right] d\xi d\zeta + \frac{2}{4\pi \gamma_0} \int_{-1}^1 F(\xi, 0) \psi_{XX}^{(1)} G_X d\xi \\ &= -\frac{1}{2\pi} \iint_{S_w^{0*}} \left\{ \left[ F\psi_X^{(1)} \right]_X + \left[ F\psi_Z^{(1)} \right]_Z \right\} G_X d\xi d\zeta. \end{aligned} \quad (34)$$

Therefore

$$A_{\nu}^{(1)} = \frac{g_{\nu}}{2\pi} \iint_{S_w^{0*}} F(\xi, \zeta) \sin (W_{\nu} \gamma_0 \xi) e^{K_{\nu} \gamma_0 \zeta} W_{\nu} \gamma_0 d\xi d\zeta, \quad (35)$$

$$B_{\nu}^{(1)} = -\frac{g_{\nu}}{2\pi} \iint_{S_w^{0*}} F(\xi, \zeta) \cos (W_{\nu} \gamma_0 \xi) e^{K_{\nu} \gamma_0 \zeta} W_{\nu} \gamma_0 d\xi d\zeta, \quad (36)$$

$$\begin{aligned} A_{1\nu}^{(2)} &= \frac{g_{\nu}}{2\pi} \iint_{S_w^{0*}} F(\xi, \zeta) \left[ \psi_X^{(1)} W_{\nu} \gamma_0 \sin (W_{\nu} \gamma_0 \xi) + \psi_Z^{(1)} K_{\nu} \gamma_0 \cos (W_{\nu} \gamma_0 \xi) \right] d\xi d\zeta \\ &+ \frac{g_{\nu}}{2\pi \gamma_0} \int_{-1}^1 F(\xi, 0) \psi_{XX}^{(1)} \sin (W_{\nu} \gamma_0 \xi) d\xi, \end{aligned} \quad (37)$$

and

$$\begin{aligned} B_{1\nu}^{(2)} &= -\frac{g_{\nu}}{2\pi} \iint_{S_w^{0*}} F(\xi, \zeta) \left[ \psi_X^{(1)} W_{\nu} \gamma_0 \cos (W_{\nu} \gamma_0 \xi) - \psi_Z^{(1)} K_{\nu} \gamma_0 \sin (W_{\nu} \gamma_0 \xi) \right] d\xi d\zeta \\ &- \frac{g_{\nu}}{2\pi \gamma_0} \int_{-1}^1 F(\xi, 0) \psi_{XX}^{(1)} \cos (W_{\nu} \gamma_0 \xi) d\xi. \end{aligned} \quad (38)$$

The above expressions can in general be evaluated in closed form for mathematical elementary hulls, save the contributions of the local flow  $\psi^{(1)}$  to the integrands, where however the  $v$  integration may be interchanged with closed-form  $\xi, \zeta$  integration.

### Resistance Due to Additional Singularities at the Undisturbed Free Surface

Consider a strip of width  $d\xi$  extending from  $\eta = -T$  to  $\eta = T$  at  $\zeta = 0$  with ordinate  $X = \xi$ . Assume that a Fourier expansion for  $\delta(\xi, \eta)$  holds as

$$\delta^{(2)}(\xi, \eta) = \sum_{\nu=-\infty}^{\infty} \delta_{\nu}(\xi) \cos(U_{\nu}\gamma_0\eta). \quad (39)$$

If  $X_e \ll \xi$ , this strip will contribute to  $\psi_{2X}^{(2)}$  by

$$d\psi_{2X}^{(2)} = \int_{-T}^T \frac{\delta}{4\pi\gamma_0} G_X^{\text{free}}(\xi, \eta, 0, X, Y, Z) d\eta d\xi \quad (40)$$

with

$$G_X^{\text{free}} = - \sum_{\nu=-\infty}^{\infty} g_{\nu} e^{K_{\nu}\gamma_0 Z} \cos[W_{\nu}\gamma_0(X-\xi)] \cos(U_{\nu}\gamma_0 Y) \cos(U_{\nu}\gamma_0 \eta) \quad (41)$$

(+ terms odd in  $\eta$  not needed here), where

$$g_{\nu} = g_{-\nu} = 4\pi K_{\nu}\gamma_0^2/M_{\nu}T \quad (42)$$

(compare Eq. (A1) in the appendix). We then have

$$d\psi_{2X}^{(2)} = -2T \sum_{\nu=-\infty}^{\infty} (\delta_{\nu}K_{\nu}/M_{\nu}T) e^{K_{\nu}\gamma_0 Z} [\cos(W_{\nu}\gamma_0\xi) \cos(W_{\nu}\gamma_0 X) + \sin(W_{\nu}\gamma_0\xi) \sin(W_{\nu}\gamma_0 X)] \cos(U_{\nu}\gamma_0 Y) d\xi. \quad (43)$$

If now for  $\psi_X^{(1)}$  as well, only free waves are significant at  $X = X_e$ , i.e., if we have

$$\psi_X^{(1)} = \psi_X^{(1)\text{free}} = \sum_{\nu=-\infty}^{\infty} [A_{\nu}^{(1)} \cos(W_{\nu}\gamma_0 X) + B_{\nu}^{(1)} \sin(W_{\nu}\gamma_0 X)] e^{K_{\nu}\gamma_0 Z} \cos(U_{\nu}\gamma_0 Y), \quad (44)$$

then  $d\psi_{2X}^{(2)}$  will make up a contribution to  $R_2^{(3)}$  as

$$dR_2^{(3)} = -T \sum_{\nu=-\infty}^{\infty} \delta_{\nu} (2 - \cos^2 \theta_{\nu}) \frac{K_{\nu}}{M_{\nu}} \left[ A_{\nu}^{(1)} \cos (W_{\nu} \gamma_0 \xi) - B_{\nu}^{(1)} \sin (W_{\nu} \gamma_0 \xi) \right] d\xi. \quad (45)$$

But from Eq. (28) we may derive that

$$K_{\nu}/M_{\nu} = K_{\nu}/(2K_{\nu} - 1) = 1/(2 - \cos^2 \theta_{\nu}). \quad (46)$$

Thus

$$\begin{aligned} dR_2^{(3)} &= -2T \sum_{\nu=-\infty}^{\infty} \delta_{\nu} \left[ A_{\nu}^{(1)} \cos (W_{\nu} \gamma_0 \xi) + B_{\nu}^{(1)} \sin (W_{\nu} \gamma_0 \xi) \right] \\ &= - \int_{\eta=-T}^T \delta(\xi, \eta) \psi_X^{(1) \text{ free}}(\xi, \eta, 0) d\eta d\xi. \end{aligned} \quad (47)$$

This is not yet the whole contribution of the strip to  $R_2^{(3)}$ , however; from Eq. (27) we have to add

$$dR_2^{(3)} = \int_{\eta=-T}^T \delta^{(2)}(\xi, \eta) \psi_X^{(1)}(\xi, \eta, 0) d\eta d\xi. \quad (48)$$

This leads to the simple result

$$R_2^{(3)} = \int_{-\infty}^{\infty} \int_{-T}^T \delta^{(2)}(\xi, \eta, 0) \tilde{\psi}_X(\xi, \eta, 0) d\eta d\xi \quad (49)$$

with

$$\tilde{\psi}_X = \psi_X^{(1)} - \psi_X^{(1) \text{ free}} \quad (50)$$

The potential  $\tilde{\psi}$  would correspond to the solution of the first-order boundary value problem if we had postulated waves traveling ahead of the ship instead of behind the ship. For a ship symmetrical to the midship section we may insert  $\tilde{\psi}(X) = \psi^{(1)}(-X)$ .

The above integral, Eq. (49), will have significant contributions to resistance only from the vicinity of the ship, as  $\delta$  has strong decay ahead and the factor  $\psi_X$  shows a decay aft. The overwhelming contribution should therefore come from the rhombe-shaped region bounded by a Kelvin angle drawn from the bow and an opposite angle from the stern.

The expression for  $R_2^{(3)}$  could have been derived directly as the Lagally force of the wave field due to the surface disturbance  $\delta(\xi, \eta)$  acting on the singularities creating the first-order flow field of the ship. It would, therefore, have been found by Sisov under use of the proper radiation condition. For the case of a nonsubmerged body, we felt that formal application of Lagally's law even for higher order contributions deserved caution. Inserting Eq. (13) we have



$$R_2^{(3)} = \int_{-T}^T \int_{-\infty}^{\infty} \left\{ \left[ \text{grad } \psi^{(1)} \right]_X^2 - \psi_X^{(1)} \left[ \gamma_0 \psi_Z^{(1)} + \psi_{XX}^{(1)} \right]_Z / \gamma_0 \right\} \left( \psi_X^{(1)} - \psi_X^{(1)\text{free}} \right) dXdY. \quad (51)$$

In particular for a symmetrical hull, where  $\gamma_0 \psi_Z^{(1)} + \psi_{XX}^{(1)}$  is odd and  $\psi_X^{(1)} \tilde{\psi}_X$  is even, we have

$$\begin{aligned} R_2^{(3)} &= \int_{-T}^T \int_{-\infty}^{\infty} \left[ \text{grad } \psi^{(1)} \right]_X^2 \psi_X^{(1)}(-X) dXdY = - \int_{-T}^T \int_{-\infty}^{\infty} \left[ \text{grad } \psi^{(1)} \right]^2 \psi_{XX}^{(1)}(-X) dXdY \\ &= \frac{1}{\gamma_0} \int_{-T}^T \int_{-\infty}^{\infty} \left[ \text{grad } \psi^{(1)} \right]^2 \psi_Z^{(1)}(-X) dXdY. \end{aligned} \quad (52)$$

If now for a symmetrical hull we have

$$\psi_X^{(1)}(X, Y, 0) = \sum_{\nu=-\infty}^{\infty} \alpha_{\nu}(X) \cos(U_{\nu} \gamma_0 Y), \quad \text{with } \alpha_{\nu} = \alpha_{-\nu} \quad (53a)$$

$$\psi_Y^{(1)}(X, Y, 0) = \sum_{\nu=-\infty}^{\infty} \beta_{\nu}(X) \sin(U_{\nu} \gamma_0 Y), \quad \text{with } \beta_{\nu} = \beta_{-\nu}, \quad (53b)$$

and

$$\psi_Z^{(1)}(X, Y, 0) = \sum_{\nu=-\infty}^{\infty} \gamma_{\nu}(X) \cos(U_{\nu} \gamma_0 Y), \quad \text{with } \gamma_{\nu} = \gamma_{-\nu}, \quad (53c)$$

where according to Eqs. (33) and (A1) the coefficients  $\alpha_{\nu}$ ,  $\beta_{\nu}$ , and  $\gamma_{\nu}$  depend on hull geometry given by  $Y = \pm \epsilon F(X, Z)$  through the relations

$$\alpha_{\nu} = \partial H_{\nu} / \partial X, \quad \beta_{\nu} = U_{\nu} \gamma_0 H_{\nu}, \quad \gamma_{\nu} = \partial H_{\nu} / \partial Z, \quad (54)$$

with the function  $H_{\nu}(X, Y, Z)$  given by

$$\begin{aligned} H_{\nu}(X, Y, Z) &= \frac{2K_{\nu} \gamma_0}{\Gamma M_{\nu}} \iint_{S_w^{0*}} [\text{sign}(X - \xi) - 1] e^{K_{\nu} \gamma_0 \xi} F(\xi, \zeta) \cos[W_{\nu} \gamma_0 (X - \xi)] d\xi d\zeta \\ &\quad - \frac{2\gamma_0}{\pi T} \iint_{S_w^{0*}} \int_{V=0}^{\infty} \text{sign}(X - \xi) F(\xi, \zeta) e^{-U \gamma_0 |X - \xi|} \left[ V \cos(V \gamma_0 \zeta) \right. \\ &\quad \left. - U^2 \sin(V \gamma_0 \zeta) \right] \frac{U \cdot dV}{U^4 + V^2} d\xi d\zeta \end{aligned} \quad (55)$$

and  $U_{\nu}$ ,  $M_{\nu}$ ,  $W_{\nu}$ ,  $K_{\nu}$ , and  $U$  as given by Eq. (A4b); then we can express

$$R_2^{(3)} = R_2^{(3)} \gamma_0 4L / (\rho B^3 c^2)$$

as

$$R_2^{(3)} = \frac{T}{\gamma_0} \int_{-\infty}^{\infty} \sum_{\mu=-\infty}^{\infty} \sum_{\lambda=-\infty}^{\infty} \left\{ \left[ \alpha_{\mu}(X) \alpha_{\lambda}(X) - \beta_{\mu}(X) \beta_{\lambda}(X) + \gamma_{\mu}(X) \gamma_{\lambda}(X) \right] \gamma_{\lambda+\mu}(-X) \right. \\ \left. + \left[ \alpha_{\mu}(X) \alpha_{\lambda}(X) + \beta_{\mu}(X) \beta_{\lambda}(X) + \gamma_{\mu}(X) \gamma_{\lambda}(X) \right] \gamma_{|\mu-\lambda|}(-X) \right\} dX \quad (56)$$

due to the Fourier orthogonality relations for  $Y$  integration, where the  $X$  integral may be truncated soon after  $X$  exceeds 1 in absolute value.

The above formula can easily be extended to the case of infinite tank width as  $T \rightarrow \infty$ ; however, for practical evaluation it is recommended to consider  $T$  as inverse of spacing in integration by trapezoidal rule and let  $T$  be just large enough, dependent on  $X$ , that for  $\xi = \pm X$  the actual wave pattern is well within  $|Y| < T$ , i.e., that no tank effect can be felt.

For actual calculations we have to reintroduce dimensions; we have

$$\gamma_0 R^{(3)} = (B/L)^3 \rho c^2 (L/2)^2 \quad R^{(3)} = \rho c^2 B^3 / (4L) R^{(3)} \quad (57a)$$

and

$$\gamma_0 R^{(2)} = (B/L)^2 \rho c^2 (L/2)^2 \quad R^{(2)} = \rho c^2 B^2 / 4L R^{(2)} \quad (57b)$$

(compare Eqs. (18), (25), (26), and (27)), where  $R^{(3)}$  and  $R^{(2)}$  are the actual third-order and second-order resistance components.

## NUMERICAL RESULTS

For most of the computations done so far, at the present stage the author's caution predominates over the temptation to have them presented prematurely. From the field of wave pattern analysis, however, two typical examples shall be shown.

For a ship with parabolic waterlines and a draft/length ratio equal to 1/20, in Fig. 2 we have plotted our calculations for

$$\frac{R_{XY}(X, Y)}{R^{(2)}} = \frac{\int_{-X}^X \int_{-\infty}^0 \varphi_{XX}(\xi, Y, \zeta) d\zeta \int_{-\infty}^0 \varphi_{XY}(\xi, Y, \zeta) d\zeta d\xi}{R^{(2)} \gamma_0 (B/L)^2}.$$

It has been shown by Ward (14) that this quantity, measured by him as an integral over the product of  $X$  force and  $Y$  force experienced by a vertical cylinder placed at distance  $Y$  from the ship's path, is representative for the portion of the total wave resistance  $R^{(2)}$  which is manifested in the energy transport of

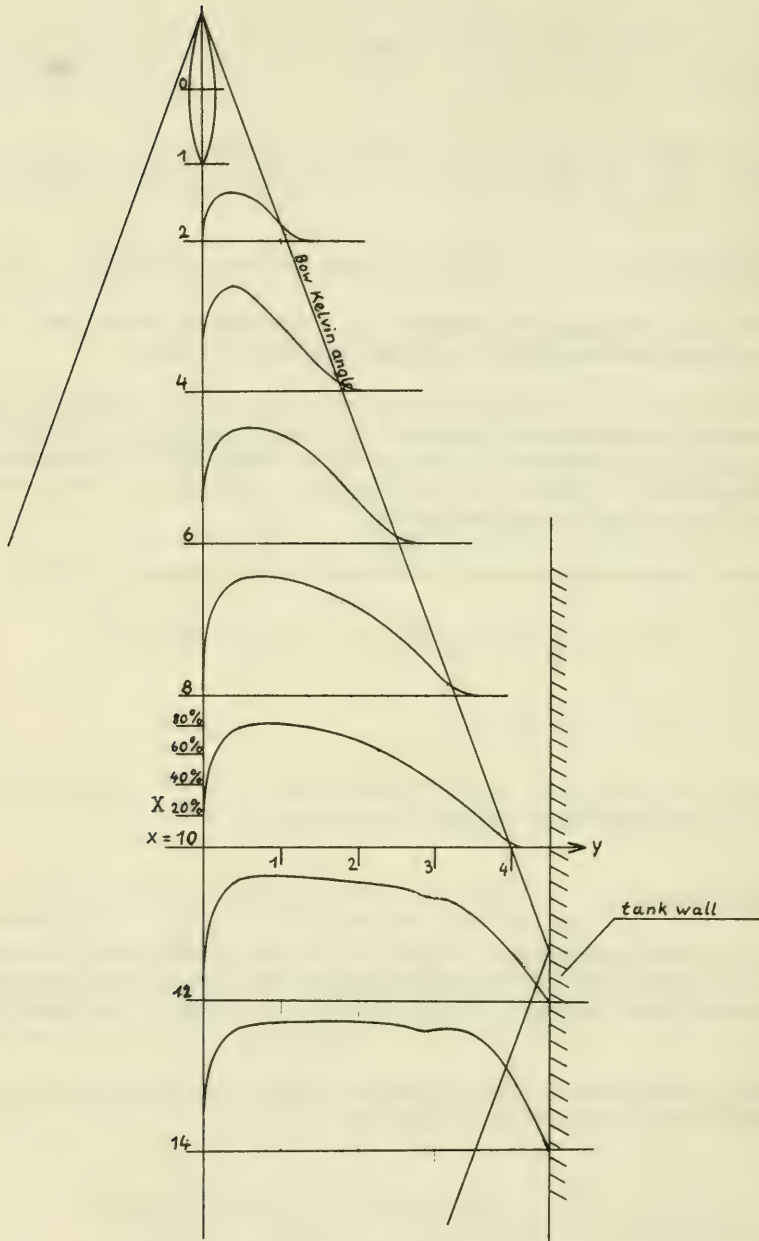


Fig. 2 - Portion of resistance obtainable from cylinder force measurements by Ward's X-Y method: ship with parabolic waterlines, draft/length =  $1/20$ ,  $\gamma_0 = 0.5$ , and  $F_n^{(2)} = 10$ . The curve with the percentages labeled is that of the wave resistance from the various cuts  $Y = \eta$  truncated at  $X = 10$ .

waves passing through the plane  $\eta = Y$  between  $x = -X$  and  $x = X$ . Roughly speaking, the diagram shows that at least a record of five times the ship's length within the Kelvin pattern is necessary to attain 80 percent of the resistance. Our calculation showed that omission of the local wave components did not significantly change the result even for  $Y$  less than 1.

In Fig. 3, we have plotted the ratio of resistance derived from transverse cut analysis to exact second-order resistance as function of distance  $x$  from midship section. The diagram essentially reflects the decay of the local waves in relation to the residual free wave system. The data for this calculation were selected in correspondence with investigations of Kobus in Ref. 13; the influence of a tankwidth equal to  $5/3$  the ship's length is included.

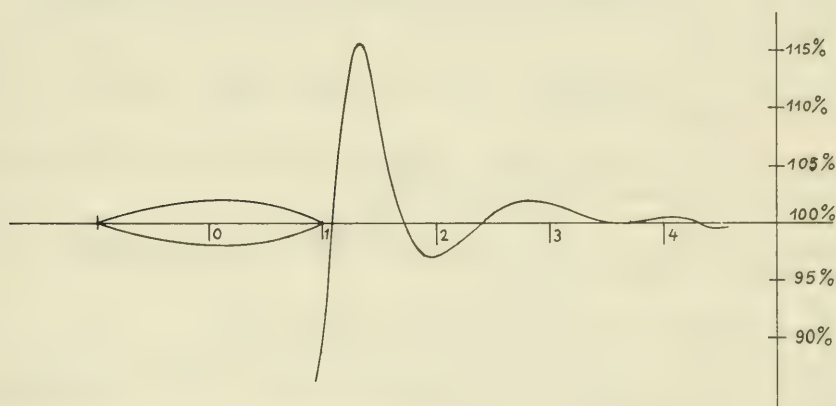


Fig. 3 - Ratio of the calculated resistance  $R_t^{(2)}(x)$  from the transverse cut analysis (6) (wave elevation and  $x$  slope) to the asymptotic value  $R^{(2)}$  (tank width equal to  $5/6$  the ship length,  $\gamma_0 = 3.86$ , and infinite strut)

## SUMMARY

With the above analytical considerations, an attempt was made to coordinate the intuitive approach of Sisov with the rigorous procedure of Wehausen. Allowing some simplifications, we found that even the latter leads to a representation of the second-order wave potential by sources only, located on the undisturbed free surface and on the longitudinal centerplane of the ship; in particular, all line integrals can be eliminated. Additional resistance can be expressed in terms of first-order flow components which determine these singularities. Only a region of the free surface close to the ship need be considered.



## ACKNOWLEDGMENT

Development of the fundamental program for computations was initiated during the author's National Science Foundation Fellowship at the University of Notre Dame and continued under support of Deutsche Forschungsgemeinschaft. The author wishes to express gratitude to Deutsches Electronen Synchrotron, Hamburg, for generosity in making available their IBM 7044 computer facilities. He also thanks the H-5 Panel of the Society of Naval Architects and Marine Engineers for their continued interest in his work.

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## APPENDIX

A FOURIER SERIES REPRESENTATION OF THE SOURCE  
POTENTIAL IN A TANK OF FINITE WIDTH

The expression to be presented here has essentially been derived in Ref. 6. We shall confine ourselves to show certain properties which are needed in the foregoing applications.

The expression describes a wave potential of a source of output  $+4\pi$ , i.e., a singularity like negative inverse distance, in coordinates made dimensionless by ship's half length as introduced under the first subheading of the text. The expression is

$$G = 2\gamma_0 \sum_{\nu=-\infty}^{\infty} \left[ g_{\nu}^{\text{free}}(X, \xi, Z, \zeta) + g_{\nu}^{\text{local}}(X, \xi, Z, \zeta) \right] \left\{ \cos(U_{\nu}\gamma_0 Y) \cos(U_{\nu}\gamma_0 \eta) [1 + (-1)^{\nu}] \right. \\ \left. + \sin(U_{\nu}\gamma_0 Y) \sin(U_{\nu}\gamma_0 \eta) [1 - (-1)^{\nu}] \right\} \Delta U \quad (\text{A1})$$

with

$$\Delta U = \pi/(\gamma_0 T),$$

$$U_{\nu} = \nu \Delta U = \sec^2 \theta_{\nu} \sin \theta_{\nu},$$

$$g_{\nu}^{\text{free}} = g_{-\nu}^{\text{free}} = [\text{sign}(X - \xi) - 1] \frac{W_{\nu}}{M_{\nu}} e^{K_{\nu}\gamma_0(Z+\zeta)} \sin W_{\nu}\gamma_0(X - \xi),$$

$$M_{\nu} = \sqrt{1 + 4U_{\nu}^2},$$

$$W_{\nu} = \sqrt{K_{\nu}} = \sec \theta_{\nu}$$

$$K_{\nu} = (1 + M_{\nu})/2 = \sec^2 \theta_{\nu},$$

and

$$g_{\nu}^{\text{local}} = g_{-\nu}^{\text{local}} = -\frac{1}{\pi} \int_{\nu=0}^{\infty} \left( e^{-U\gamma_0 |X-\xi|} \left\{ \left[ V \cos(V\gamma_0 Z) - U^2 \sin(V\gamma_0 Z) \right] \left[ V \cos(V\gamma_0 \zeta) \right. \right. \right. \\ \left. \left. \left. - U^2 \sin(V\gamma_0 \zeta) \right] \right\} - \delta_{\nu}^0(U) V^2 \right) / [U(U^4 + V^2)] dV \quad (\text{A2})$$

with  $U = +\sqrt{V^2 + U_{\nu}^2}$  and  $\delta_{\nu}^0(U) = 1$  for  $\nu = 0$  and  $\delta_{\nu}^0 = 0$  elsewhere or only the Hadamard "part fini" should be taken for  $\nu = 0$ . As shown in Ref. 6 it is easy to find out by investigation of single terms of the series that the function  $G$  is subject to the following conditions:

A.  $G_{XX} + G_{YY} + G_{ZZ} = 0$  provided the corresponding  $v$  integrals exist, which is guaranteed for  $|X - \xi| > 0$ ;

B.  $\gamma_0 G_Z + G_{XX} = 0$  for  $Z = 0$ ;

C.  $G_Y = 0$  for  $Y = \pm T$ ;

D.  $G_Z \rightarrow 0$  as  $Z \rightarrow -\infty$ ;

F.  $G = O(X - \xi)^{-1}$  as  $X \rightarrow +\infty$ ,  $G = O(1)$  as  $X \rightarrow -\infty$ ,  $G_X = O(X - \xi)^{-1}$  as  $X \rightarrow +\infty$ ,  $G_X = O(1)$  as  $X \rightarrow -\infty$ .

Since the structure of  $G$  is symmetric, corresponding relations can be obtained under exchange of  $X$ ,  $Y$ , and  $Z$  with  $\xi$ ,  $\eta$ , and  $\zeta$ .

It remains to be shown that (a) the expressions for  $G$  and  $G_X$  match in a continuous way at  $X = \xi$  and (b) for  $|Y - \eta| \leq T$ ,  $Z \leq 0$ , and  $\zeta \leq 0$ ,  $G$  and  $G_X$  become singular only for  $X = \xi$ ,  $Y = \eta$ , and  $Z = \zeta$ , and that the functions

$$G^* = G + 1/r \quad (\text{A3a})$$

and

$$G_X^* = G_X + (1/r)_X, \quad (\text{A3b})$$

with

$$r = [(X - \xi)^2 + (Y - \eta)^2 + (Z - \zeta)^2]^{-1/2}$$

remain finite here (convergence of the series for  $\zeta \neq Z$  is shown in Ref. 6).

Statement (a) is evident for the function  $G$ . Assume for simplicity  $\xi = 0$ . Then it is sufficient to show that

$$\begin{aligned} \frac{K_\nu}{M_\nu} e^{K_\nu \gamma_0 (Z + \zeta)} &= \lim_{X \rightarrow 0} \frac{1}{\pi} \int_{v=0}^{\infty} e^{-K \gamma_0 |X|} \left[ V \cos(V \gamma_0 Z) - U^2 \sin(V \gamma_0 Z) \right] \left[ V \cos(V \gamma_0 \zeta) \right. \\ &\quad \left. - U^2 \sin(V \gamma_0 \zeta) \right] \frac{dV}{U^4 + V^2} \end{aligned} \quad (\text{A4a})$$

for arbitrary  $U_\nu \geq 0$  and  $Z \neq \zeta$  with

$$U = \sqrt{V^2 + U_\nu^2}, \quad M_\nu = \sqrt{1 + 4U_\nu^2}, \quad K_\nu = (1 + M_\nu)/2. \quad (\text{A4b})$$

On the right-hand side we may substitute

$$\begin{aligned}
 &= \frac{1}{2\pi} \lim_{X \rightarrow 0} \left[ \int_{V=0}^{\infty} \left( V^2 \left\{ \cos \left[ V \gamma_0 (Z + \gamma) \right] + \cos V \gamma_0 (Z - \gamma) \right\} - U^4 \left\{ \cos \left[ V \gamma_0 (Z + \gamma) \right] - \cos \left[ V \gamma_0 (Z - \gamma) \right] \right\} \right. \right. \\
 &\quad \left. \left. + 2U^2 V \sin V \gamma_0 (Z + \zeta) \right) \frac{e^{-U \gamma_0 X} dV}{U^4 + V^2} \right] \\
 &= \frac{1}{4\pi} \lim_{X \rightarrow 0} \operatorname{Re} \left[ \int_{V=-\infty}^{\infty} \frac{V + iU^2}{V - iU^2} e^{iV \gamma_0 (Z + \zeta) - V \gamma_0 |X|} dV + \int_{V=-\infty}^{\infty} e^{-U \gamma_0 X + iV \gamma_0 (Z - \zeta)} dV \right. \\
 &\quad \left. + \int_{V=-\infty}^{\infty} \frac{\left( e^{-U \gamma_0 |X|} - e^{-V \gamma_0 |X|} \right) (V + iU^2) e^{iV \gamma_0 (Z + \zeta)}}{V - iU^2} dV \right]. \quad (\text{A5})
 \end{aligned}$$

The second integral term in the last expression is zero for any finite  $X$ . The last integral term is  $O(X)$ . The first integral term may be written

$$= \frac{1}{4\pi} \lim_{X \rightarrow 0} \operatorname{Re} \left\{ \int_{V=-\infty}^{\infty} \frac{(V - iK_\nu)[V + i(M_\nu + 1)/2]}{(V + iK_\nu)[V - i(M_\nu + 1)/2]} e^{iV \gamma_0 (Z + \zeta) - V \gamma_0 |X|} dV \right\}. \quad (\text{A6})$$

which shows poles of the integrand for  $V = i(M_\nu - 1)/2$  and  $V = -i(M_\nu + 1)/2 = -iK_\nu$ . By shifting the path of integration downward in the complex plane we can make the integral arbitrarily small after splitting off the residuum at  $V = -iK_\nu$ ; thus we finally get

$$= 2\pi i \operatorname{Res}_{V = -iK_\nu} \left\{ \frac{(V - iK_\nu)[V + i(M_\nu + 1)/2]}{(V + iK_\nu)[V - i(M_\nu + 1)/2]} e^{iV \gamma_0 (Z + \zeta)} \right\} \frac{1}{4\pi} = \frac{K_\nu e^{K_\nu \gamma_0 (Z + \zeta)}}{M_\nu}. \quad (\text{A7})$$

To prove statement (b) we start with the representation

$$1/r_1 = 1/(2\pi) \int_{U=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-u|X-\xi|} \cos u[(Y-\eta) \cos \theta + (Z+\zeta) \sin \theta] du d\theta \quad (\text{A8a})$$

and

$$1/r = 1/(2\pi) \int_{U=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-u|X-\xi|} \cos u[(Y-\eta) \cos \theta + (Z-\zeta) \sin \theta] du d\theta, \quad (\text{A8b})$$

where  $r_1$  corresponds to  $r$  with  $\zeta$  under negative sign. Introducing new variables of integration  $U$ ,  $V$ , and  $\tilde{U}$  by

$$U = u/\gamma_0, \quad \tilde{U} = u \cos \theta/\gamma_0, \quad V = u \sin \theta/\gamma_0, \quad (\text{A9})$$



we have

$$1/r - 1/r_1 = \gamma_0/(2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-U\gamma_0|X-\xi|} 2 \sin(V\gamma_0 Z) \sin(V\gamma_0 \zeta) \cos[\tilde{U}\gamma_0(Y-\eta)]/U \, d\tilde{U} dV \quad (A10)$$

and

$$(1/r - 1/r_1)_X = -\gamma_0/(2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-U\gamma_0|X-\xi|} 2 \sin(V\gamma_0 Z) \sin(V\gamma_0 \zeta) \times \cos[\tilde{U}\gamma_0(Y-\eta)] \, d\tilde{U} dV \, \text{sign}(X-\xi). \quad (A11)$$

A general law in the theory of Fourier transforms (see Ref. 11 of the preceding list) — essentially known as Poisson's summation rule — states that if the function  $F(y)$  has a representation

$$F(y) = \int_{-\infty}^{\infty} G(\tilde{U}) e^{i\tilde{U}y} \, d\tilde{U}, \quad (A12)$$

then

$$F(\delta, y) = \sum_{\nu=-\infty}^{\infty} F(y + \nu\delta),$$

provided this series converges, has a representation

$$F^*(\delta, y) = \sum_{\nu=-\infty}^{\infty} G(U_\nu) e^{iU_\nu y} \Delta\tilde{U} \quad (A13)$$

with  $\Delta\tilde{U} = 1/\delta$  and  $U_\nu = \nu\Delta\tilde{U}$ . With  $\delta = T/\pi$ , we therefore have the representation

$$\sum_{\nu=-\infty}^{\infty} \left\{ [(X-\xi)^2 + (Y+\nu T-\eta)^2 + (Z+\zeta)^2]^{-1/2} - [(X-\xi)^2 + (Y+\nu T-\eta)^2 + (Z-\zeta)^2]^{-1/2} \right\} \\ = -2/\pi \int_{V=-\infty}^{\infty} \sum_{V=-\infty}^{\infty} e^{-U\gamma_0|X-\xi|} \sin(V\gamma_0 Z) \sin(V\gamma_0 \zeta)/U \, dV \cos(U_\nu\gamma_0(Y-\eta)) \quad (A14)$$

with  $U_\nu = 2\pi\nu/(\gamma_0, T)$  and  $U = \sqrt{V^2 + U_\nu^2}$ . But the terms under summation are equivalent to corresponding terms in the series for  $G$  given by Eq. (A1), and it can be seen that after subtraction of these terms the integrals for the coefficients  $g_\nu^{\text{local}}$  in Eq. (A1) converge even in the case  $X = \xi$ , and  $Z = \zeta$ .

The argument for the function  $G_X$  is analogous.

\* \* \*

## DISCUSSION

John V. Wehausen  
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I am not convinced by, and probably do not understand correctly, Dr. Eggers' reason for not using his approach (b) to determine the wave resistance (second subsection of his analysis section). His discarding it seems especially puzzling in view of the fruitful use that he has made in other contexts of the associated formula

$$R = \frac{1}{2} \rho g \int_{-T}^T \zeta^2(x_e, y) dy + \frac{1}{2} \rho \int_{-T}^T dy \int_{-H}^{\zeta(x_e, y)} \left[ -\varphi_x^2(x_e, y, z) + \varphi_y^2 + \varphi_z^2 \right] dz.$$

If one is granted the assumption of irrotational flow of an inviscid fluid, and Dr. Eggers makes this assumption, the above formula is exact; i.e., no further approximations are required to derive it. This being so, an approximation to  $\varphi$  and  $\zeta$  of any particular sort should at the same time also yield an approximation to  $R$  of an appropriate sort. If we suppose then, following Dr. Eggers, that

$$\varphi = \epsilon \varphi^{(1)} + \epsilon^2 \varphi^{(2)} + \dots,$$

$$\zeta = \epsilon \zeta^{(1)} + \epsilon^2 \zeta^{(2)} + \dots$$

$$= \epsilon \frac{c}{g} \varphi_x^{(1)}(x, y, 0)$$

$$+ \epsilon^2 \left\{ \frac{c}{g} \varphi_x^{(2)} - \frac{1}{2g} \left[ \varphi_x^{(1)2} + \varphi_y^{(1)2} + \varphi_z^{(1)2} \right] + \frac{c^2}{g^2} \varphi_x^{(1)} \varphi_{xz}^{(1)} \right\} + \dots$$

(see e.g., Eq. (14) of Ref. 1 or Eq. (27.6) of Ref. 2), substitution into the formula for  $R$  and some easy manipulations yield

$$\begin{aligned} R = & \epsilon^2 \left\{ \frac{\rho c^2}{2g} \int_{-T}^T \varphi_x^{(1)2}(x_e, y, 0) dy + \frac{1}{2} \rho \int_{-T}^T dy \int_{-H}^0 \left[ -\varphi_x^{(1)2}(x_e, y, z) + \varphi_y^{(1)2} + \varphi_z^{(1)2} \right] dz \right\} \\ & + \epsilon^3 \left\{ \frac{\rho c^2}{g} \int_{-T}^T \varphi_x^{(1)}(x_e, y, 0) \varphi_x^{(2)}(x_e, y, 0) dy \right. \\ & - \frac{\rho c}{g} \int_{-T}^T \left[ \varphi_x^{(1)3}(x_e, y, 0) - \frac{c^2}{g} \varphi_x^{(1)2} \varphi_{xz}^{(1)} \right] dy \\ & \left. + \rho \int_{-T}^T dy \int_{-H}^0 \left[ -\varphi_x^{(1)}(x_e, y, z) \varphi_x^{(2)} + \varphi_y^{(1)} \varphi_y^{(2)} + \varphi_z^{(1)} \varphi_z^{(2)} \right] dz \right\} + \dots \end{aligned}$$

This agrees with Dr. Eggers' results given by Eqs. (25), (26), and (27) except that his integral containing  $\phi^{(2)}$  is replaced by the integral above with the integrand

$$\phi_x^{(1)^3} - \frac{c^2}{g} \phi_x^{(1)^2} \phi_{xy}^{(1)}.$$

It is not immediately clear that these integrals are equal, although this may be true. If they are not identical, the difference should be  $O(\epsilon)$ , provided neither of us has made an error. In any case, the formula above seems to be well founded, perhaps easier to evaluate than the other by using asymptotic expressions as  $x_e \rightarrow \infty$ , and to lend itself to Dr. Eggers' later analysis. However, Dr. Eggers evidently has special insights into this problem as a result of his experience with the computation, and I would appreciate his clarifying further the reasons for choosing his method for computing  $R$ .

Finally, I should like to express my pleasure in learning that my paper at the Ann Arbor symposium has played some part in the genesis of Dr. Eggers' interesting work.

\* \* \*

## DISCUSSION

B. Yim  
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I respect Dr. Eggers' courage of committing himself in such a complicated and difficult task of evaluating the second-order wave resistance due to a finite-draft ship. It is so complicated that I would even be glad at this stage to see just the partial numerical results, for example, those related to the higher order free surface effect or to the ship bottom effect. In the formulation of the second-order theory I think he very nicely combined the line integral with the ship surface integral. However, he omitted the influence on the potential due to the change of ship singularity domain caused by wave height, namely, the change of the wet area of a ship surface. This was considered by Wehausen (3) and Sisov (4) as another line integral on the free surface. However, as long as we consider a Green's function defined under the mean free surface, it is meaningless to put a ship singularity over the mean free surface so that the free surface condition is satisfied on the mean free surface. Thus, it is not easy to deal with this problem in a conventional way.

Recently I considered this (Hydronautics, Incorporated, Technical Report 503-1, "Higher Order Wave Theory for Slender Ships," 1966) with a distorted coordinate system  $(\xi, \eta, \zeta)$ ,  $x = \xi$ ,  $y = \eta$ ,  $z = \zeta + f(x, y)$ , and  $f(x, y)$  = free surface wave height, so that  $\zeta = 0$  in  $(\xi, \eta, \zeta)$  space corresponds exactly to the free

surface. The Jacobian is equal to 1, and the one-to-one correspondence holds. Since  $f(x, y)$  is small, the first-order potential in  $(\xi, \eta, \zeta)$  space again satisfies the Laplace equation. Thus, the first-order problem is essentially the same as that of Michell's solution except the interpretation of the free surface. The higher order potential, in general, satisfies the Poisson's equation in the transformed space. Thus, the effect due to the change of ship singularity domain comes from near the ship bottom rather than the free surface.

If we investigate the influence of the Froude number (or the wavelength) on the higher order potential, we find that we have to be careful to analyze the order of magnitudes. The inverse of the wavenumber is equal to the square of the Froude number, and this is a smaller quantity than the beam/length ratio for the usual merchant ship. However, it does not seem to affect the consistency of the development except in the case of a high-speed, small-draft ship where the second-order potential is simpler. This case was also discussed by Wehausen (3).

I certainly believe that Dr. Eggers' contribution will be a very significant one in the development of more accurate ship wave theory.

\* \* \*

## DISCUSSION

Lawrence W. Ward  
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It is a great pleasure for me to have the opportunity to comment on this paper, as I had the rewarding experience of spending the past academic year at the Institut für Schiffbau der Universität Hamburg under a National Science Foundation Fellowship, and I therefore know something of the ingenuity and effort which my friend and colleague Dr. Eggers expended to produce these results. Development of the computer program for the linearized case including the total wave field for shiplike bodies in an ideal fluid provides us with a minimum test for our various wave analysis methods, which are based on the same assumptions, that must be met more-or-less exactly, before we can expect any hope of even approximate validity of these methods for an actual model in a real fluid. I will comment further on this point in respect to the "X-Y" method in a moment. The carrying on of a consistent next approximation beyond the linearized theory, following the lead of Wehausen, promises to be even more interesting. It is not obvious to me that the effect of viscosity alone must explain the major portion of the discrepancy between the Michell theory and experiment — this would be equivalent to saying that the linearized theory would agree with tests run on the actual model in an ideal fluid; we must however await numerical results from the author to clarify this picture.



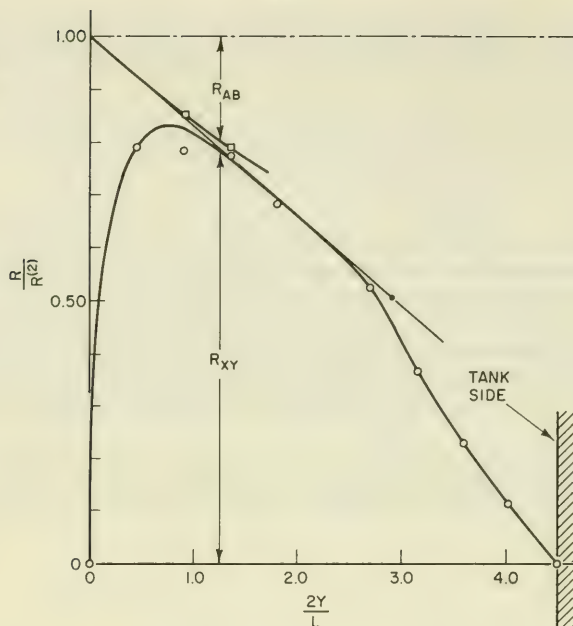


Fig. D1 - Total "X-Y" method prediction from Eggers' computer results (ship with parabolic waterlines,  $\gamma_0 = 10.0$ , and  $2X/L = 10.0$ )

I thank the author for selecting the very specialized "X-Y" function to use as an illustration of his computer results, and I take this opportunity to point out that it is not necessary for this quantity to equal 100% of the theoretical wave resistance for validity of the method to be established. This is because the "X-Y" method also includes a "centerline wave term,"  $R_{AB}$ , which is not included in the author's Fig. 2. Using wave elevation data from the same computer program, kindly supplied to me by the author, I have made an estimate of this term and have added it to the "X-Y" term for the location  $2X/L = 10.0$  as shown in Fig. D1. The centerline wave term  $R_{AB}$  is drawn for convenience from the top down, and is roughly linear with the distance  $2Y/L$  off the centerline. Similar lines can be drawn for the other locations. It can be seen that the addition of this term brings the method into almost exact agreement with the theory at this section, assuming measurements to be taken at the distance  $1.0 \leq 2Y/L \leq 2.5$ , the former limit being required by the need to exclude the local effect from the measurements and the latter because the centerline wave term as calculated is an approximation based on a plane wave which becomes inexact too far from the centerline, and also because of tank wall effects. This dimension in the case of all previous tests using this method was selected within these limits, but I am of course happy to have a more rational basis for making this selection in the future.

I again thank the author for his significant contribution to a subject of great interest.

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## REPLY TO DISCUSSION

K. W. H. Eggers

### REPLY TO THE DISCUSSION BY WEHAUSEN

I am pleased to have Prof. Wehausen raise his question. In fact, I tried approach (b) first. In my notation, I had to evaluate

$$R_2^{(3)} = I_1 + I_2$$

where

$$I_1 = \int_{-T}^T \left\{ \psi_X^{(1)} \psi_X^{(2)} / \gamma_0 + \int_{-H}^0 \left[ \psi_Y^{(1)} \psi_Y^{(2)} + \psi_Z^{(1)} \psi_Z^{(2)} - \psi_X^{(1)} \psi_X^{(2)} \right] d\zeta \right\} d\eta$$

and

$$I_2 = \int_{-T}^T \psi_X^{(1)} \left\{ \psi_X^{(1)} \psi_{XZ}^{(1)} / \gamma_0 - [\text{grad } \psi^{(1)}]^2 \right\} d\eta / \gamma_0$$

$$+ \int_{-T}^T \psi_X^{(1)} \left[ \psi_Y^{(1)2} + \psi_Z^{(1)2} - \psi_X^{(1)2} \right] d\eta / 2\gamma_0 = \int_{-T}^T \psi_X^{(1)2} \left[ \psi_{XZ}^{(1)} / \gamma_0 - \psi_X^{(1)} \right] d\eta.$$

For finite  $X_e$ , both  $I_1$  and  $I_2$  are difficult to evaluate. Due to tank wall reflection, there is no decay of first-order flow components. This implies that  $\delta(X, Y)$  has no decay as well; thus  $\psi^{(2)}$  everywhere includes local components from nearby singularities. Furthermore  $I_2$ , after performing the  $\eta$  integration on triple products of Fourier terms, leaves us with expressions depending on  $X_e$  with an infinite variety of nonrelated periods.

I met the first difficulty, which arises under approach (c) as well, by transforming  $I_1$  to an integral over the undisturbed free surface for  $-X_e \rightarrow \infty$  (see Eq. (47)). This was possible by use of a peculiar property of the Green's function involved, which appears even in the case of finite depth: If we represent the free wave system of a point source as a Fourier integral in  $Y$ , then the amplitude function (in our case  $K_\nu / M_\nu T$ , see Eq. (42)) is inverse to the weighting factor of spectral energy (in our case  $T(2 - \cos^2 \theta_\nu)$ , see Eq. (29)) of the expression for wave resistance.

The difficulties with the term  $I_2$  made me resign and decide on approach (c), for which I was certain that the sum of  $I_1$  and the integral of  $\psi_X^{(1)} \epsilon(X, Y)$  must be independent of  $X_e$  by Gauss' theorem.

Following now Prof. Wehausen's suggestion, using conditions A and B, I can finally prove that

$$\psi_X^{(1)} \delta(X, Y) = \frac{d}{dX} \left[ \psi_X^{(1)^2} \psi_{XZ} / \gamma_0 - \psi_X^{(1)} \right] + \frac{d}{dY} \left[ \psi_X^{(1)^2} \psi_Y^{(1)} \right].$$

The second term vanishes under  $Y$  integration due to condition C, if we may disregard a contribution from the ship's load waterline. Thus definitely approach (b) gives one and the same result for any value of  $X_e$  behind the ship, as it should, and our result, Eq. (49), can be obtained by approach (b) as well.

A critical review of our analysis showed that this equivalence even holds if we do not disregard the influence of the waterline. We had, however, to acknowledge that extending the integration over  $\delta \psi_X^{(1)}$  in Eq. (19) to the waterplane area meant a change of order  $\epsilon^3$ , which, after a correction to be made in Eq. (19), again made all line integrals cancel in the final expression for  $R^{(3)}$ , Eq. (27).

For the case of infinite tank width, it may be argued that  $I_2$  tends to zero as  $X_e \rightarrow -\infty$ , though for the contribution of the cusp region this is not necessarily evident. But even in this case it is only the addition of the expression for  $I_2$  to  $I_1$  which enables us to introduce the reversed potential  $\tilde{\psi}$  into the integrand and thus obtain the convenient expression (Eq. (49)) amenable to numerical evaluation.

#### REPLY TO THE DISCUSSION BY YIM

There is at least an equal amount of respect on my side for Dr. Yim's courage to attack the problem for the domain bounded by the first-order free surface, where the analysis is definitely less elegant to perform. I sympathize with his idea of using coordinates adjusted to the wave profile, and I wonder if to first order this is just equivalent to distorting the ship's geometry as was done by Hanaoka.

We can hardly expect one-to-one correspondence between our components of potential and resistance; however, if Dr. Yim finds one of his line integral terms missing in my analysis, perhaps a hint may help: A recent study of Bessho reveals that if only the boundary condition on the body surface below  $Z = 0$  is prescribed, different second-order potentials may exist for a floating body of finite beam. Within his scheme, my potential appears as the limiting case of that due to a flat-topped body, and so does Sisov's.

For a wall-sided ship, the second summand of my double integral term in Eq. (13) could be transformed to line integrals representing contributions from the bottom and from the load waterline. But I am not convinced that the difference of these line integrals should dominate against the remaining double integral over the vertical surface, even if the ship has shallow draft.

I share Dr. Yim's cautions regarding a study of the asymptotic influence of the speed parameter. The potential is certainly more intricately dependent on the Froude number than is apparent from explicit factors. But I shall try to extend my investigations in this direction.

Fortunately, I can meet Dr. Yim's request for some numerical results; these are selected for a parabolic ship with a draft/length ratio equal to  $1/20$  and  $\gamma_0 = 10$ . Additional resistance due to the free-surface singularity  $\phi(X, Y)$  is  $R_2^{(3)} = -0.04(B/L)R^{(2)}$ , additional resistance due to second-order singularities to represent the ship is  $R_1^{(3)} = 0.15(B/L)R^{(2)}$  with 10% accuracy, where  $R^{(2)}$  is the resistance from linear theory. Unfortunately, only the free-wave part of the first-order flow on the hull was accounted for when calculating  $R_1^{(3)}$ , though the local component is of equal magnitude. The local component gave contributions obviously inconsistent and too large. Very probably the Fourier series representation in  $Y$  is not appropriate at the center plane for the local part of the potential due to the discontinuity in the  $Y$  derivative. We shall try either to evaluate the integrals on the actual ship surface or to split off the singular part and integrate it in some closed form.

The effect of  $\psi_1^{(2)\text{free}}$  on the free wave spectrum is a uniform phase shift of all wave components toward the bow.

#### REPLY TO THE DISCUSSION BY WARD

I thank Prof. Ward for his well meaning comments, which need no additional clarification. Prof. Ward is certainly right in pointing out that my result on the X-Y method is incomplete; thus I welcome his elucidating remarks.

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# ANALYSES ON BOW WAVES AND STERN WAVES AND SOME SMALL-WAVE-SHIP SINGULARITY SYSTEMS

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## ABSTRACT

The qualities of the bow and stern waves of a ship are discussed. The local disturbance of ships with zero wave resistance is formulated. Thus the image systems of simple no-wave singularities are obtained. The force distribution of the zero wave ship is analyzed. The application of these theories to the case of practical ships is discussed.

## INTRODUCTION

The mechanism of the reduction in wave resistance due to a bulb appendage at a ship bow has been explained by the reduction in the wave height of each elementary wave starting from the bow (1). This explanation has been further verified through the existence of an ideal bulb which cancels all the elementary bow waves (2). However, complete understanding of ship wave phenomenon even just in connection with bulbs is yet far from complete. For the time being, we may ask ourselves two questions: First, what happens to the local disturbance when we cancel all the regular bow waves? This is directly concerned with higher order effects, and, in addition, this question leads us to find the exact and simple image system which satisfies the linearized free surface boundary condition. Second, what is the relation between the wave height on the ship hull and the total wave resistance, especially when the regular bow waves are canceled? The answer to this question may explain the sizable wave resistance for a ship which appears to have small wave heights on the hull. These questions are of fundamental importance not only in connection with the design of a bulb through the observation of wave heights on the ship's hull but also for a more profound understanding of the mechanism of ship waves.

Before investigating these questions we will re-examine the physical and mathematical meanings of bow waves and stern waves, and their relation to the waves due to a bulb.

It turns out that the local disturbance due to the small-wave ship treated by bulbs is in low Froude numbers, approximately the same as the local disturbance

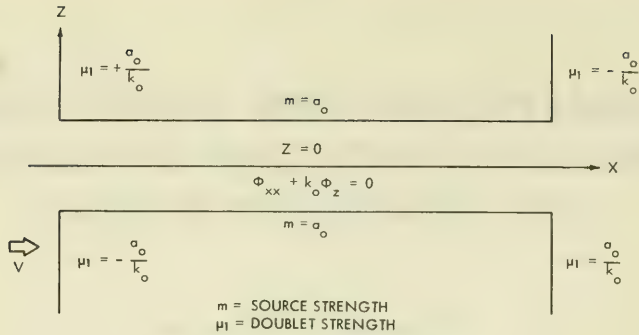


Fig. 1 - Image system of a simple zero-wave ship

proportional to the perturbation pressure on the mean free surface caused by a double model of the same ship. It is found that the image system of the simple zero-wave ship is very simple (Fig. 1). This image system can be applied to build ships by tracing streamlines from the optimum singularity distributions for bulbous ships.

Through Lagally's theorem, it is found that the force equivalent to the stern wave resistance of the symmetric ship is exerted on the doublet distribution which is distributed along the stem line forming the bow bulb and which cancels the regular bow waves. Thus the regular wave along the surface waterline of the hull of an optimum bulbous ship may be very small. Yet, with the expense of the force exerted on the bow bulb the stern wave energy can be propagated far away.

## ASSUMPTIONS AND BASIC EQUATIONS

We consider a Michell thin ship in a steady, incompressible, inviscid flow with a free surface and with an infinitely deep bottom. We locate the origin of the coordinate at the bow and on the mean free surface which forms the  $xy$  plane. The flow at infinity is in the positive  $x$  direction. The  $z$  axis is positive upward.

We use the solution (see Ref. 3) for the Laplace equation

$$\nabla^2 \phi = 0, \quad (1)$$

where  $\phi$  is a perturbation potential, with the conventional linearized boundary condition

$$\phi_{xx} + k_0 \phi_z + \mu \phi_x = 0 \quad (2)$$

on the mean free surface  $z = 0$ , where  $k_0 = g/V^2$ ,  $g$  is the acceleration of gravity, and  $V$  is the flow velocity at  $\infty$ . The wave height  $\zeta$  is

$$\zeta = \frac{\phi_{\mathbf{x}}}{k_0} . \quad (3)$$

The wave height due to a simple point source of strength  $m$  at  $(x_1, 0, -z_1)$ , with  $z_1 > 0$ , is given by

$$\zeta = -\frac{m}{2\pi V} \operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{i 2k \sec \theta e^{k(i\omega - z_1)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta , \quad (4)$$

where  $\mu$  is the fictitious frictional force and

$$\omega = (x - x_1) \cos \theta + y \sin \theta \quad (5)$$

or, letting  $\zeta = \zeta_r + \zeta_l$ , where  $\zeta_r$  refers to an integral representing the regular wave pattern and  $\zeta_l$  refers to a double integral representing the local disturbance,

$$\begin{aligned} \zeta_r &= 4 \frac{m}{V} k_0 \int_{-(\pi/2)+\delta}^{\pi/2} \exp(-k_0 z_1 \sec^2 \theta) \sec^3 \theta \cos(k_0 \omega \sec^2 \theta) d\theta \\ &- \frac{2m}{\pi V} \int_{-(\pi/2)+\delta}^{(\pi/2)+\delta} \int_0^{\infty} \frac{\exp(-t\omega) t \sec \theta}{k_0^2 \sec^4 \theta + t^2} \left[ k_0 \sec^2 \theta \sin(tz_1) - t \cos(tz_1) \right] dt d\theta , \end{aligned} \quad (6)$$

where

$$\delta = \arctan \frac{y}{x - x_1} . \quad (7)$$

The wave height due to a point doublet of strength  $-\mu_1$  located at  $(0, 0, -z_1)$  is

$$\zeta_d = \frac{\mu_1}{2\pi V} \operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{2k^2 e^{k(i\omega - z_1)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta \quad (8)$$

or

$$\begin{aligned} \zeta_d &= -4 \frac{\mu_1}{V} k_0^2 \int_{-(\pi/2)+\delta}^{\pi/2} \exp(-k_0 z_1 \sec^2 \theta) \sec^4 \theta \sin(k_0 \omega \sec^2 \theta) d\theta \\ &+ \frac{2\mu_1}{\pi V} \int_{-(\pi/2)+\delta}^{(\pi/2)+\delta} \frac{\exp(-m\omega) m^2}{k_0^2 \sec^4 \theta + m^2} \left[ k_0 \sec^2 \theta \sin(mz_1) - m \cos(mz_1) \right] dm d\theta . \end{aligned} \quad (9)$$



## BOW WAVES AND STERN WAVES

We consider a source distribution  $m(x_1, z_1)$  on a ship centerplane  $D$  in the plane  $y = 0$ . If we use the principle of superposition, the regular ship wave can be represented as

$$\zeta = \frac{4k_0}{V} \int_0^L \int_0^H m(x_1, z_1) \int_{-\pi/2+\delta}^{\pi/2} \exp(-k_0 z_1 \sec^2 \theta) \times \sec^3 \theta \cos(k_0 \omega \sec^2 \theta) d\theta dx_1 dz_1, \quad (10)$$

where  $L$  and  $H$  are the ship length and the ship draft respectively. This regular wave is important because this is the only wave which contributes to the wave resistance no matter what method we use to calculate the wave resistance. For example, Havelock's (4,5) formula is from the regular waves far behind the ship, while Lagally's method is from the regular waves on the singularity plane (3). If we change the order of integration in Eq. (10), we obtain

$$\begin{aligned} \zeta = & \frac{4k_0}{V} \int_{-(\pi/2)+\delta_1}^{(\pi/2)+\delta_2} d\theta \int_0^{x+y \tan \theta} U(k_0 H, x_1, \theta) \cos(k_0 \omega \sec^2 \theta) dx_1 \\ & + \frac{4k_0}{V} \int_{-(\pi/2)+\delta_2}^{\pi/2} d\theta \int_0^L U(k_0 H, x_1, \theta) \cos(k_0 \omega \sec^2 \theta) dx_1, \end{aligned} \quad (11)$$

where

$$U(k_0 H, x_1, \theta) = \int_0^H m(x_1, z_1) \exp(-k_0 z_1 \sec^2 \theta) \sec^3 \theta dz_1, \quad (12)$$

$$\delta_1 = \tan^{-1}(y/x),$$

and

$$\delta_2 = \tan^{-1}(y/\sqrt{x-L}).$$

When we consider the wave height on the  $x$  axis ( $y = 0$ ),

$$\zeta = \frac{4k_0}{V} \int_{-\pi/2}^{\pi/2} d\theta \int_0^x U(k_0 H, x_1, \theta) \cos[k_0(x-x_1) \sec \theta] dx_1 \quad (13)$$

for  $0 \leq x \leq L$  and

$$\zeta = \frac{4k_0}{V} \int_{-\pi/2}^{\pi/2} d\theta \int_0^L U(k_0 H, x_1, \theta) \cos[k_0(x-x_1) \sec \theta] dx_1 \quad (14)$$

for  $x > L$ .

If we consider, for convenience, a polynomial source distribution

$$m(x_1) = \sum_{n=0}^N a_n x_1^n \quad (15)$$

in  $0 < x < L$  and  $0 < z_1 < H$ , then Eq. (13) becomes

$$\zeta_1 = \frac{4k_0}{V} \int_{-\pi/2}^{\pi/2} A(0, \theta) \sin(k_0 x \sec \theta) + B(0, \theta) \cos(k_0 x \sec \theta) - B(x, \theta) d\theta, \quad (16)$$

where

$$A(L, \theta) = \left(1 - e^{-k_0 H \sec^2 \theta}\right) \sum_{n=0}^N \frac{(-1)^n m^{(2n)}(L)}{k_0^{2n-1} \sec^{2n} \theta}$$

and

$$B(L, \theta) = \left(1 - e^{-k_0 H \sec^2 \theta}\right) \sum_{n=0}^N \frac{(-1)^{n+1} m^{(2n+1)}(L)}{k_0^{2n} \sec^{2n+1} \theta}.$$

Equation (16) is an analytic function of  $x$ . Therefore, Eq. (16) or (13) can be analytically continued to  $x \geq L$ , and we may call these the physical bow waves on the  $x$  axis for the source distribution given by Eq. (15); then the physical stern waves will be, from Eq. (14),

$$\begin{aligned} \zeta_2 &= \frac{4k_0}{V} \int_{\pi/2}^{\pi/2} d\theta \int_x^L U(k_0 H, x, \theta) \cos[k_0(x - x_1) \sec \theta] dx_1 \\ &= -\frac{4k_0}{V} \int_{-\pi/2}^{\pi/2} A(L, \theta) \sin(k_0 \overline{x - L} \sec \theta) \\ &\quad + B(L, \theta) \cos(k_0 \overline{x - L} \sec \theta) - B(x, \theta) d\theta \end{aligned} \quad (17)$$

for  $x \geq L$ .

This is essentially a fictitious consideration of the analytically continued source distribution in  $x \geq L$  for the bow waves and a sink distribution with the same magnitude of strength from  $x = L$  to  $x \geq L$  for the stern waves, so that the total source strength in  $x \geq L$  is zero. In this sense we may extend our definition of the physical bow and stern waves to any point on the free surface  $(x, y, 0)$ .

In Eqs. (16) and (17) the integrals of  $B(x, \theta)$  are canceled out in a total wave (bow waves plus stern waves) for  $x \geq L$ . Therefore

$$\zeta_1 = \frac{4k_0}{V} \int_{-\pi/2}^{\pi/2} A(0, \theta) \sin(k_0 x \sec \theta) + B(0, \theta) \cos(k_0 x \sec \theta) - d\theta, \quad (16')$$

and

$$\begin{aligned} \zeta_2 &= \frac{4k_0}{V} \int_{\pi/2}^{\pi/2} d\theta \int_x^L U(k_0 H, x, \theta) \cos[k_0(x - x_1) \sec \theta] dx_1 \\ &= -\frac{4k_0}{V} \int_{-\pi/2}^{\pi/2} A(L, \theta) \sin(k_0 \overline{x - L} \sec \theta) + B(L, \theta) \cos(k_0 \overline{x - L} \sec \theta) - d\theta \quad (17') \end{aligned}$$

have been called (1) regular bow waves and regular stern waves respectively at  $x > L$ . (In this report, we use simply bow waves and stern waves in this sense.) Another reason for this can be attributed to the concept of elementary waves (4). Namely, Eqs. (16') and (17') can be considered to be equivalent to  $\theta$ -wise superpositions of elementary waves starting from the bow and the stern respectively. Thus, the wave resistance of an analytic ship is considered to consist of three components: the bow wave resistance, the stern wave resistance, and the interaction of the two. This concept is very helpful to treat a bulbous bow (6) for the range of low Froude numbers.

If we consider the source distribution

$$m(x_1, z_1) = a_0$$

in  $0 < x_1 < a$  and  $0 < z_1 < H$ , which is only the first term of Eq. (15) and represents a wedge ship bow approximately, then Eqs. (16) and (16') or (17) and (17') coincide with each other, Eq. (17) representing the shoulder wave rather than the stern wave. Namely,  $B(x, \theta)$  in Eqs. (16) and (17) is identically zero, and the actual regular waves in  $0 < x < a$  is represented by Eq. (16'). We notice that  $a_0$  is a dominating term (1,7) in the wave height (Eqs. (16) and (17)) for low Froude numbers ( $< 0.3$ ).

It is well known that the bow bulb flattens the regular bow waves. In fact, if the ship source distribution is represented in an even power series of  $x$  as in  $\cos \pi x$ , it was shown (2) that a doublet distribution exists along the straight infinite stemlines of the bow and the stern, which cancels all the regular ship waves.

In general, the influence of the stern shape on the wave is very complicated due to the effects of the boundary layer and the wakes in the actual fluid. These effects seem to tend to decrease the influence of the discontinuity of the slope at the stern. Therefore, in the low Froude number range, the bulb even at the bow only is supposed to decrease the wave resistance considerably.

It must be noted, however, that in  $0 < x < L$  the regular wave which contributes to the wave resistance is in general expressed not by Eq. (16') but by Eq. (16). Therefore, just canceling Eq. (16') does not necessarily mean canceling the actual regular waves on the ship hull. In addition, although Eq. (16') involves

physical quantities related to the bow only, and Eq. (17'), the stern only, to deal separately with Eq. (16') from Eq. (17') is quite dangerous for the range of higher Froude numbers. The following example may demonstrate this.

Let us consider a source distribution

$$\frac{m(x_1, z_1)}{V} = a \cos(\pi x_1) \quad (18)$$

in  $0 < x_1 < 1$  and  $0 < z_1 < H$ , which produces approximately a ship of sine waterlines. Then from Eq. (14) we have the regular wave height at  $x > 1$ , and  $y = 0$ ,

$$\zeta = 4ak_0 \int_{-\pi/2}^{\pi/2} \frac{1 - \exp(-k_0 H \sec^2 \theta)}{(k_0 \sec \theta)^2 - \pi^2} \sec \theta [\sec \theta \sin(k_0 x \sec \theta) + \sec \theta \sin(k_0 \overline{x} - 1 \sec \theta)] d\theta,$$

where the first term in the brackets corresponds to Eq. (16') and the second term corresponds to Eq. (17'). However each term becomes singular when  $k_0$  approaches to  $\pi$  (or the wavelength is 2 and Froude number  $F = 0.564$ ), although the total regular wave height is always finite. If we examine the regular wave height on  $0 < x < 1$  and  $y = 0$ , from Eq. (13),

$$\zeta = 4a \int_{-\pi/2}^{\pi/2} \frac{1 - \exp(-k_0 H \sec^2 \theta)}{(k_0 \sec \theta)^2 - \pi^2} \sec \theta [k_0 \sec \theta \sin(k_0 x \sec \theta) - \pi \sin \pi x] d\theta,$$

which is also finite for any  $k_0$ . This expresses physical bow waves defined in Eq. (16).

It is easily seen here that dealing with only bow waves, Eq. (16'), without the stern waves, Eq. (17'), in the neighborhood of Froude number  $1/\sqrt{\pi}$  is meaningless.

#### LOCAL DISTURBANCE AND IMAGE SYSTEM FOR THE ZERO WAVE SHIP

When we observe the wave height on a ship hull, it is the superposition of the regular wave height and the local disturbance. Although the local disturbance does not contribute to the wave resistance in the linear theory, a large local disturbance may induce sizable higher order effects. Therefore, it is of interest to check the behavior of the local disturbance of the zero wave ship. In addition, by this we can understand the nature of zero wave singularities better.

For convenience, first, we will consider the simplest case of the zero-bow-wave submerged body (2). A uniform source distribution of the strength  $m/V = a_0$  on a line  $0 \leq x_1 \leq 1$ ,  $y = 0$ , and  $z_1 = f$  produces the  $x$  component of the perturbation velocity (3):



$$\phi_{Sx}(x, y, z) = Q(x, y, z)$$

$$+ \operatorname{Re} \left[ -\frac{ik_0}{\pi} \int_0^1 a_0 dx_1 \int_{-\pi}^{\pi} \sec \theta d\theta \int_0^{\infty} \frac{e^{k(i\omega - |z-f|)} k}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta \right]$$

$$= Q + \operatorname{Re} \frac{k_0 a_0}{\pi} \int_{-\pi}^{\pi} \sec^2 \theta d\theta \int_0^{\infty} \frac{e^{k(i\omega - |z-f|)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta \Big|_{x_1=0}, \quad (19)$$

where

$$Q = -\frac{a_0}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} \left[ e^{k(i\omega - |f+z|)} - e^{k(i\omega - |f-z|)} \right]_{x_1=0}^1 dk d\theta.$$

A uniform doublet distribution of the strength  $b_0$  on a vertical line  $x=0$ ,  $y=0$ ,  $f < z_1 < \infty$  produces

$$\phi_{0x}(x, y, z) = F_0(x, y, z) + \operatorname{Re} \int_f^{\infty} dz_1 \frac{k_0 b_0}{\pi} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{k^2 e^{k(i\omega - |z-z_1|)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta$$

$$= F_0 + \operatorname{Re} \frac{k_0 b_0}{\pi} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{k e^{k(i\omega - |z-f|)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta, \quad (20)$$

where

$$F_0 = \frac{b_0}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} k \cos^2 \theta \left[ e^{k(i\omega_0 - |f+z|)} - e^{k(i\omega_0 - |f-z|)} \right] dk d\theta$$

and

$$\omega_0 = x \cos \theta + y \sin \theta.$$

Putting

$$b_0 = a_0/k_0 \quad (21)$$

and considering another similar doublet line as given by Eq. (20) but with the negative strength on the line  $x=1$ ,  $y=0$ ,  $f < z_1 < \infty$ , whose  $x$  component of velocity is  $\phi_{1x}$ , we combine all three cases. Then we obtain

$$\phi_{Sx} + \phi_{0x} + \phi_{1x} = Q + F_0 + F_1 - \operatorname{Re} \frac{a_0}{\pi} \int_{-\pi}^{\pi} \int_0^{\infty} e^{k(i\omega - |z-f|)} dk d\theta \Big|_{x_1=0}^1. \quad (22)$$

This indicates that on the free surface the  $x$  component of the perturbation velocity is exactly the same as that due to the double model of the ship in the infinite medium. The flow as a whole is the same as that due to the ship and bulb plus the positive mirror image of the ship and the negative mirror image of bulb with respect to the mean free surface (Fig. 1).

In other words, for this special case of the uniform source distribution, if we combine such a doublet distribution and the opposite sign of doublet images and the same sign of source images, the free surface condition is exactly satisfied without any wave images. In the general case of ship shapes, it is not that simple to express the local disturbance.

For a body symmetric with respect to its midsection, the regular bow wave is the same as the stern wave and the local disturbance is also symmetric with respect to the midsection.

When the analytic ship source distribution, antisymmetric with respect to midsection, is represented by an even power series

$$m(x_1) = \sum_{n=0}^N a_{2n} x_1^{2n} \quad (23)$$

in  $0 < x_1 < 1$ ,  $z_1 = f$ ,  $y = 0$ , there exists a doublet distribution

$$\sum b_n (z_1 - f)^n \quad (24)$$

on  $f < z_1 < \infty$ ,  $y = 0$ ,  $x = 0$  and  $y = 0$ ,  $x = 1$  by which the regular ship waves are completely canceled (2) with

$$b_n = (-1)^n \frac{(2n)!}{n!} \frac{a_{2n}}{k_0^{n+1}}. \quad (25)$$

Due to the symmetry of the body with respect to the midsection the coefficients of Eq. (23) have relations

$$\sum_{n=m}^{\infty} \frac{2n!}{(2n-2m)!} a_{2n} = -(2m)! a_{2m}, \quad m = 0, 1, 2, \dots \quad (26)$$

Using Eqs. (4), (8), and (25) we obtain formally the total wave height due to Eqs. (23), (24), and (26):

$$\zeta = \text{Re} \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \int_0^{\infty} \left[ \frac{-2(2n)! a_{2n}}{(ik \cos \theta)^{2n}} \sec^2 \theta + \frac{2b_n n!}{k^{n-1}} \right] \frac{e^{-kf} (e^{ki\omega_0} + e^{ki\omega_1})}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta \quad (27)$$

(Cont.)

$$\zeta = \operatorname{Re} 2 \sum a_{2n} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{(k - k_0 \sec^2 \theta)}{k - k_0 \sec^2 \theta - i\mu \sec \theta} \frac{(2n)!(-1)^n}{k_0^{n+1}} \times \left[ \frac{k_0^n}{k^{2n} \cos^{2n} \theta} + \sum_{j=0}^{n-1} \frac{k_0^j}{k^{n-j} (k \cos \theta)^{2j}} \right] e^{-kf} \left( e^{ki\omega_0} + e^{ki\omega_1} \right) dk d\theta, \quad (27)$$

where  $\omega_1 = (x-1) \cos \theta + y \sin \theta$ .

At the limit  $\mu \rightarrow 0$ , this expresses the disturbance  $\phi_x/k_0$  on  $z=0$  due to the source distribution

$$2 \sum_n \frac{(2n)!}{k_0^{n+1}} a_{2n} (-1)^n \sum_{j=0}^{n-1} \frac{k_0^j (z-f)^{n-j-1} x^{2j} (-1)^j}{(n-j-1)! (2j)!} \quad (28)$$

on  $0 < x < 1$ ,  $f < z < \infty$  in addition to the disturbance due to the double model of ship and the doublet distribution (24) and (25) with its negative mirror image in the infinite medium. For  $n \geq 2$ , this is not finite. The local disturbance of the ideal zero regular wave of general ships may not be finite. This is because the ideal bulb for the general body becomes large with depth and becomes infinite at  $z = -\infty$ . Of course, in practice the effect of the bulb at shallow depth is so large that we need not make the bulb so deep and big for approximate wave cancellation (2). Besides, for low Froude numbers, which is rather usual, the influence of the large  $n$  in Eqs. (23) and (24) is very small, both on the regular waves and the local disturbance. This can be shown by taking one general term of Eq. (23). The total wave height due to  $a_{2n} x_1^{2n}$ , say, is

$$\begin{aligned} \zeta_n &= -\operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} \int_0^1 a_{2n} x_1^{2n} \frac{i 2k \sec \theta e^{k(i\omega - z_1)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta dx_1 \\ &= \operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} 2a_{2n} \sum \frac{2n!}{(ik \cos \theta)^{2n}} \left[ 1 - \sum_{j=0}^{2n} \frac{(ik \cos \theta)^{2n-j}}{(2n-j)!} e^{-ik \cos \theta} \right] \\ &\quad \times \frac{i e^{k(i\omega_0 - z_1)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta. \end{aligned}$$

If we replace  $k \cos \theta$  by  $kk_0 \sec \theta$ ,

$$\begin{aligned} \zeta_n &= \operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} 2a_{2n} \sum \frac{2n!}{(ik_0 k \sec \theta)^{2n}} \left[ 1 - \sum_{j=0}^{2n} \frac{(ik_0 k \sec \theta)^{2n-j}}{(2n-j)!} e^{-ik_0 k \sec \theta} \right] \\ &\quad \times \frac{i e^{k_0 k \sec^2 \theta (i\omega_0 - z_1)}}{k - 1 - \frac{i\mu}{k_0} \cos \theta} dk d\theta, \quad (29) \end{aligned}$$

where

$$\begin{aligned}
 1 - \sum_{j=0}^{2n} \frac{(ik_0 k \sec \theta)^{2n-j}}{(2n-j)!} e^{-ik_0 k \sec \theta} \\
 = 1 - \left[ e^{ik_0 k \sec \theta} - \sum_{j=2n+1}^{\infty} \frac{(ik_0 k \sec \theta)^j}{j!} \right] e^{-ik_0 k \sec \theta} \\
 = e^{-ik_0 k \sec \theta} \sum_{j=2n+1}^{\infty} \frac{(ik_0 k \sec \theta)^j}{j!}, \quad (30)
 \end{aligned}$$

and where the absolute value of this becomes smaller when  $n$  becomes larger, since the series

$$e^{ik_0 k \sec \theta} = \sum_{j=0}^{\infty} \frac{(ik_0 k \sec \theta)^j}{j!}$$

is a converging series mainly due to the factor  $1/j!$ . Therefore,  $\zeta_n$  has no singularity except  $k = 1 + (i\mu/k_0) \cos \theta$  by which the regular wave and the local disturbance can be separated. Due to the factor  $k_0^{-2n}$ , for small Froude numbers ( $= 1/\sqrt{k_0}$ ),  $|\zeta_n|$  becomes smaller when  $n$  becomes larger. In general, for a usual smooth ship, the influence of the entrance angle or the term of  $a_0$  is dominant when the Froude number is less than 0.3. Therefore the bulb for the term  $a_0$ , in this case, will be adequate to cancel almost all the regular waves (7). Then the wave height will be mainly the local disturbance proportional to  $\phi_x$  due to the ship double model. Even cutting off the bulb at a certain distance will not affect the disturbance too much, for the same reason that truncated bulbs work effectively to reduce the regular wave height.

The surface ship whose source distribution is distributed both in the  $x$  and the  $z$  direction on the  $xz$  plane, can be treated in a similar manner as above, since only superposition in the draftwise direction (2) is involved.

#### LOCAL DISTURBANCE AND IMAGE SYSTEM OF A NO-WAVE SINGULARITY

It has been known that a combination of two higher order singularities, a doublet in the  $z$  direction and a quadrupole in the  $x$  direction produce no regular waves (2). The local disturbance due to this no-wave singularity can easily be analyzed as in the previous sections.

The wave height due to the doublet in the  $z$  direction located at  $(0, 0, -z_1)$  is

$$\zeta_{dz} = \frac{\mu_2}{2\pi V} \operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{i 2k^2 \sec \theta e^{k(i\omega - z_1)}}{k - k_0 \sec^2 \theta - i\mu_2 \sec \theta} dk d\theta, \quad (31)$$



where  $\omega = x \cos \theta + y \sin \theta$ , and  $\mu_2$  is the doublet strength. The wave height due to the quadrupole in  $x$  direction at  $(0, 0, -z_1)$  is

$$\zeta_q = \frac{1}{2\pi V} \operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{-i 2k^3 \cos \theta e^{k(i\omega - z_1)}}{k - k_0 \sec^2 \theta - i\mu \sec \theta} dk d\theta, \quad (32)$$

where  $\zeta_q$  is the quadrupole strength. By combination with  $k_0 = \mu_2$  we obtain in the limit  $\mu \rightarrow 0$

$$\zeta_q + \zeta_{dz} = -\frac{\lambda}{2\pi V} \operatorname{Re} \int_{-\pi}^{\pi} \int_0^{\infty} 2ik^2 \cos \theta e^{k(i\omega - z_1)} dk d\theta. \quad (33)$$

This indicates that the local disturbance ( $\zeta_{dz}$  at  $x = k_0$ ), is due only to the doublet at  $(0, 0, z_1)$  and the mirror image. The total flow is produced by the singularity system shown in Fig. 2, which satisfies the free surface boundary condition. Namely, the  $x$  component of the perturbation velocity on the free surface is exactly the same as that due to the doublet and its mirror image only. Since the sign of the image of the quadrupole is opposite from the real one, these do not contribute to the horizontal velocity on the free surface. It is interesting to have such a simple set of singularities and images which satisfies the free-surface boundary condition.

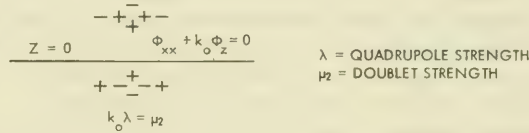


Fig. 2 - Image system of a no-wave singularity

In this manner, we can obtain the simple image system for any order no-wave point singularities, namely, for all the derivatives as well as integrals of the first-order no-wave point singularity mentioned above. For example, if we uniformly distribute the first-order no-wave singularity from  $x = 0$  to  $x = \text{infinity}$ , we obtain the zeroth-order no-wave singularity which is composed of a doublet in the  $x$  direction and a lift element at  $(0, 0, z_1)$  with the image composed of the doublet and the lift element in the opposite direction at the position of the mirror image. M. P. Tulin, in a seminar at Hydronautics, Incorporated, has brought attention to this particular no-wave singularity system. He pointed out that the existence of this system indicates the theoretical possibility of a finite three-dimensional ship without wave resistance (although, as he also pointed out, the buoyancy of the ship would be canceled by its negative dynamic lift). He has also discussed the application of these singularities to both two-dimensional and three-dimensional cavity flows beneath free surfaces.

From the composition of the no-wave point singularity and its image, it can be easily seen that any such three-dimensional point singularity with finite strength located on the free surface is nullified by its image. Thus, in the case of the zeroth-order no-wave point singularity, the flow field produced by a doublet in the  $x$  direction located on the free surface is exactly the same as that due to a vortex element at the same point. This flow field is also equivalent to that due to a pressure point on the free surface.

If we consider the strength of a no-wave singularity increasing inversely proportionally to the distance between the singularity and the image, we will obtain a higher order no-wave singularity (derivative with respect to  $z$ ) located on the free surface. In this case, with a single no-wave singularity on the free surface, the linear free surface condition is exactly satisfied without needing any image. This is the similar situation as of a vortex on the free surface in the case of infinite Froude number.

#### ANALYSIS OF FORCES DUE TO A ZERO-WAVE BOW

It is well known that the ship wave resistance can be obtained by pressure integration around the ship hull or by applying Lagally's theorem to the source distribution. We consider as an example a wedge ship for simplicity which has bow waves and shoulder waves. If we cancel the bow waves completely with a bulb, there will remain shoulder waves, yet the pressure integration on the wedge hull seems to give no resistance from the front of the wedge ship. In connection with this problem, it will be interesting to see the actual resistance force distribution of bulbous ships. For the simple case of a submerged point source and a point doublet, the analysis has been performed (6), and it was found that the negative interference due to the doublet on the source contributes to reduce the total resistance.

Lagally's theorem shows that the force at the source distribution is

$$4\pi\rho \iint \phi_x(x, z) m(x, z) dx dz \quad (34)$$

and the force at the doublet distribution is

$$4\pi\rho \iint \phi_{xx}(x, z) \mu(x, z) dx dz, \quad (35)$$

integrating over the whole source and doublet distribution respectively.

If we consider a source distribution

$$m(x_1, z_1) = a_0 \quad (36)$$

in  $0 < x_1 < a$ ,  $0 < z_1 < H$ ,  $y = 0$  and the doublet distribution

$$\mu_1 = bz_1 \quad (37a)$$

in  $0 < z_1 < H$ ,  $x_1 = y = 0$  and

$$\mu_1 = bH \quad (37b)$$

in  $H < z_1 < \infty$ ,  $x_1 = y = 0$ , this combination of singularities will not produce any regular bow waves (2). Likewise, it is easy to see that the  $x$  component of perturbation velocity minus the local disturbance (this may be called the regular part of the perturbation velocity) is zero in  $0 < x < a$ ,  $0 > z > -H$ . It is known that the local disturbance of  $\phi_x$  does not contribute to the force (3) for any source distribution. Therefore, it is evident that Eq. (34) in this case is zero. However, Eq. (35) in this case is exactly the same as the wave resistance of the doublet distribution itself, due to following reasons. When we consider a doublet line (Eqs. (37)) located at  $x = -b$ ,  $y = 0$  ( $b \rightarrow 0$ ), the regular  $\phi_x$  on the doublet line due to the source distribution (Eq. (36)) is zero because the regular wave does not propagate forward. Thus,  $\phi_{xx}(x, z)$  in Eq. (35) is due to the doublet line itself only, without any contribution from the source distribution (Eq. (36)). In fact, this wave resistance of the doublet line (Eqs. (37)) is exactly the same as the bow (or shoulder) wave resistance. Namely, when we consider the pressure distribution on the bulbous bow of the wedge ship corresponding to Eqs. (36) plus (37), the bulb has eliminated the pressure on the ship hull, making the drag due to the ship hull equal to zero, but is carrying its own drag, which is exactly the same as the shoulder wave drag.

For a symmetric ship, we consider, as an example, the source distribution of Eq. (15),

$$m(x_1) = \sum_{n=0}^N a_n x_1^n$$

in  $0 < x_1 < 1$ ,  $0 < z_1 < H$ . The  $x$  component of the regular velocity can be obtained exactly the same way as for Eq. (16):

$$\begin{aligned} \phi_x = \frac{4k_0}{V} \int_{-\pi/2}^{\pi/2} A_1(0, z, \theta) \sin(k_0 x \sec \theta) \\ + B_1(0, z, \theta) \cos(k_0 x \sec \theta) - B_1(x, z, \theta) d\theta, \end{aligned} \quad (38)$$

where

$$A_1(x, z, \theta) = \left[ e^{k_0 z \sec^2 \theta} - e^{k_0(z-H) \sec^2 \theta} \right] \sum_{n=0}^N \frac{(-1)^n m^{(2n)}(x)}{k_0^{2n-1} \sec^{2n} \theta} \quad (39)$$

and

$$B_1(x, z, \theta) = \left[ e^{k_0 z \sec^2 \theta} - e^{k_0(z-H) \sec^2 \theta} \right] \sum_{n=0}^N \frac{(-1)^{n+1} m^{(2n+1)}(x)}{k_0^{2n} \sec^{2n+1} \theta}. \quad (40)$$

Because of symmetry

$$m^{(n)}(0) = (-1)^n m^{(n)}(1). \quad (41)$$

Hence,

$$\int_0^1 m(x) m^{(2n+1)}(x) dx = 0. \quad (42)$$

In the evaluation of the wave resistance by substitution of Eqs. (38) and (42) in Eq. (34) we obtain

$$\int_0^1 B_1(x, z, \theta) m(x) dx = 0.$$

Therefore in this case the term including  $B_1(x, z, \theta)$  is like a local disturbance and does not contribute to the wave resistance. Now the situation is exactly similar to the case of the uniform source distribution of Eq. (36) mentioned before. Namely, if we use a doublet distribution to cancel the bow waves, the resistance due to the source given by Eq. (34) becomes zero, while the resistance due to the doublet given by Eq. (35) remains, and is equal to the bow wave resistance, which is equal to the stern wave resistance here.

If a ship is not symmetric, the contribution of  $B_1$  in Eq. (38) to the wave resistance is not zero, and the bow wave resistance is not equal to the stern wave resistance. If we cancel the bow waves by the doublet distribution in this case, the stern wave resistance will be exactly the same as the effect on  $B_1$  on the wave resistance plus the bow wave resistance which is exerted on the doublet distribution. To sum up, for a symmetric ship, the regular bow waves on a ship hull which are responsible for the total wave resistance can be eliminated by our ideal bow bulb, yet there remains the wave drag of the bulb itself, which is equivalent to the stern wave resistance.

This means that the regular wave heights along the ship hull is not the only origin of all the wave resistance. The stern wave energy can be propagated with the expense of the force on the bow bulb.

If we attach the ideal stern bulb to a symmetric ship in addition to the ideal bow bulb, it is known that the entire wave resistance will become zero (2). That is, the ideal stern bulb will have a negative force to cancel the force at the bow bulb. This mechanism can be seen by the same method used for the bow bulb. Namely, we consider a doublet distribution to cancel regular stern waves on the line  $x = 1, y = 0$ . Then the regular part of  $\phi_{xx}$  in Eq. (35) becomes

$$\phi_{xx} = \begin{cases} 0 & \text{on } x = 1 + 0, y = 0 \\ \phi_{xx} & \text{due to stern waves only without the} \\ & \text{stern bulb effect on } x = 1 - 0, y = 0. \end{cases}$$

The stern waves of a symmetric ship are exactly the same as the bow waves and of the opposite sign of the regular waves due to the doublet distribution. Therefore, from Eq. (35) the force at the doublet distribution at the stern is in



the opposite direction of that at the bow. Thus, we understand how the wave effect in the vicinity of the submerged bulb at the stern acts to decrease the wave resistance.

We cannot use the ideal bulb in practice, since it is infinitely long. However, this idealized situation not only introduces an easy analysis but also introduces us to a deeper insight of the mechanism. Since the lower part of the bulb deeply submerged under the free surface does not very much influence the regular waves, in general all the theory here can be applied approximately to practical bulbous ships (see Ref. 2).

## APPLICATION

Michell's ship theory has too much discrepancy with experiments for a given surface ship; hence it was suggested to build a model from a singularity distribution by tracing the body streamlines. However, since it is so complicated and time consuming to plot streamlines for the ship singularity under the free surface with the free surface condition satisfied, Inui used a double model for which the free surface is considered to be flat. Inui (1) claims that the double model is better than Michell's ship for common Froude numbers. Pien (8) also used double models for his optimum singularity distributions to build his bulbous ships which showed better performance than the corresponding ships in the Taylor series.

Here, as an application of the theory on the image system of a zero wave ship in an earlier section we may propose to use a small wave image system that is composed of a double model ship with a point doublet for a bulb and its negative mirror image as shown in Fig. 3. The approximate optimum radius of a bulb represented by a point doublet which minimizes the bow wave resistance for each Froude number ( $0.15 < F < 0.3$ ) and each entrance angle is given by Yim (7). For the case of the ideal zero wave singularity system as in Fig. 1, the free surface condition is satisfied without having any wave image. Since the general behavior of the flow field near the free surface due to the long vertical doublet distribution and its negative mirror image is similar to that due to the concentrated doublet and its negative mirror image, the free surface condition may be almost satisfied by this small wave singularity system, and the approximate local disturbance will be represented by the free surface streamlines.

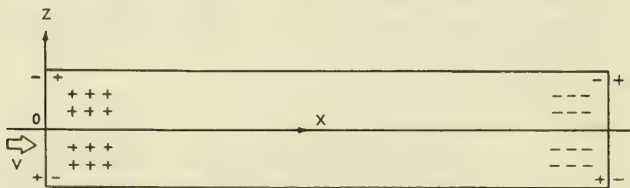


Fig. 3 - Singularity and image system of a bulbous system

# Bow and Stern Waves and Small-Wave-Ship Singularities

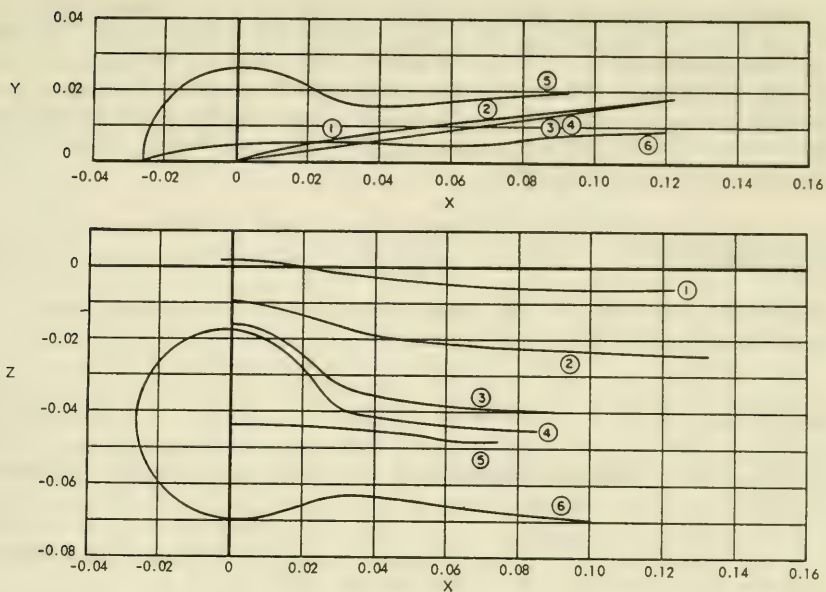


Fig. 4 - Surface streamlines of a bulbous parabolic ship  
( $F = 0.25$ ,  $B/L = 0.10$ ,  $L/H = 16.0$ )

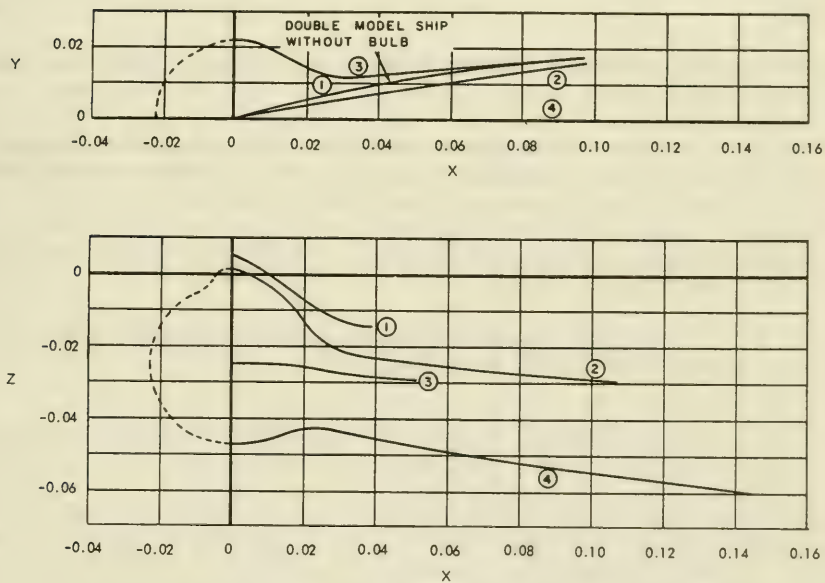


Fig. 5 - Surface streamlines of a bulbous parabolic ship  
( $F = 0.28$ ,  $B/L = 0.12$ ,  $L/H = 24.0$ )

As examples, we chose parabolic ship hulls with point doublet bulbs at the bow and the stern. Considering the image system mentioned above we plotted the body streamlines shown in Figs. 4 and 5. The interference between the ship and the bulb in reducing the ship entrance angle seems to be important, although this must be less severe than the case of the double model of the whole bulbous ship where not only the regular waves but also the local disturbances on the free surface are totally neglected.

#### ACKNOWLEDGMENT

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## DISCUSSION

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Dr. Yim's approach is definitely unorthodox. But I feel that we should deviate from usual ideas still further on his line in order to have more simple results.

Dr. Yim tries to compensate for the ship's integrated wave system by introduction of singularities at a peculiar position, namely, at vertical lines through bow and stern only. He assumes a polynomial representation of the hull form and then has to calculate derivatives at the stem, each higher derivative giving rise to singularities of a corresponding high degree, producing even infinite local wave height for the local component save for the first derivative. Certainly any continuous hull form may be approximated arbitrarily close by polynomials, but would such an approximation be adequate as well for the derivatives in the sense under consideration here?

I propose the following modification of Dr. Yim's procedure: Let us assume some spatial distribution of sources which, combined with its image above the undisturbed free surface, will represent a double body composed of a ship and its image in unbounded parallel flow, i.e., under zero Froude number. A ship form generated like this by sources may be called an Inuid, even if the sources are not necessarily on the centerplane.

Now we should add to each source element a distribution of  $\times\times$  quadrupoles on the vertical line extending from the source point  $P_1$  down to infinity, and an image system of negative intensity above. The intensity of this quadrupole distribution should be equal to the source strength times  $k_0 = g/c^2$ . This would cancel the wave element generated by the source element.

It is possible to perform closed form integration vertically. The expression for the potential  $G(P, P_1)$  at a point  $P$  due to a unit source element at  $P_1$  then is

$$G(P, P_1) = \frac{1}{r} + \frac{1}{r' - k_0^{-1}} \log \left( \frac{z_1 + z + r'}{z_1 - z + r} \right)_{\times\times}.$$

This potential meets the linearized free surface condition; only the first two terms determine the first-order wave elevation, which is entirely local.

After integration over the assumed source distribution the contribution of the third term will be singular only below some line marking the edge of the horizontal extension of the source distribution. As quadrupoles do not contribute to Kelvin-impulse, the volume of a body generated by the total system of singularities should not differ much from that of the Inuid. The wave pattern due to



the integrated singularity system will be a representation of the flow around the Inuid in the horizontal meridian plane in the limiting case of zero Froude number.

Influence of Froude number on wave pattern is only by affinity through a factor  $k_0^{-1}$ . In particular this wave pattern comes out to be independent of the choice of singularity system to represent the Inuid in unbounded flow, whereas the uncorrected wave field and wave resistance are not.\*

For a symmetrical Inuid, the corrected flow will be symmetrical as well and therefore even the third order wave resistance component  $R_2^{(3)}$  as given in my Eq. (51) in the preceding paper will be zero. This is due to a peculiar selection among all correction potentials making the second order wave resistance vanish and differing by wave free potentials.

If now, by tracing streamlines, we could for any  $k_0$  determine an associated hull form, starting with the Inuid for zero Froude number, we would have a class of zero wave ships with form dependent continuously on  $k_0^{-1}$ . But theory tells us that no such forms of finite extent exist. However, I hope that the approach sketched above will after some intuitive modification lend itself at least for determination of hull changes to achieve low wave making. The expressions for the velocity field are much simpler than for those of the uncorrected wave system due to a source.

\* \* \*

## REPLY TO DISCUSSION

B. Yim

I appreciate Dr. Eggers' interest in my paper and his cordial comment on it. The theory shows that if we are required to cancel the entire regular ship waves for a certain Froude number, we can do it, but with some features we do not want in practice such as infinite volume or infinite local disturbance. However, if we consider the dependence of the wave height on such parameters as the depth, the Froude number, and derivatives of the waterline  $y$  coordinate with respect to  $x$ , we can easily see that the deeper part of the concentrated singularity corresponding to the bulb attachment contributes mainly to the local disturbance rather than to the regular waves with which we are concerned; and the higher order derivatives on the waterline are significant for regular waves only when the Froude number is sufficiently large ( $>0.3$ ). Thus it is clear that in practice even the finite bulb works, under such a theoretical basis, with the image systems as are explained in the present paper.

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\*K. Eggers, discussion following Dr. Pien's paper in "Fifth Symposium on Naval Hydrodynamics: Ship Motions and Drag Reduction," Office of Naval Research, Department of the Navy, ACR-112, 1964.

On the modification Dr. Eggers proposed, I would like to point out that his suggested source plus the downward infinite line of  $\times\times$  quadrupoles is the uniform downward line distribution of the no-wave singularities which are composed of  $z$ -direction doublets and  $\times\times$  quadrupoles (mentioned in my present paper). As it is explained in the present paper, we can form some interesting no-wave singularity distributions. As for his suggested no-wave singularity distribution, I also wonder what kind of ship shape we will get even if we consider cutting off the quadrupole lines from the keel. Since we neglected the viscosity in our problem, the shape which not only reduces the wave resistance but also prevents the separation is most interesting to us.

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Tuesday, October 4, 1966

Afternoon Session

## **SHIP-GENERATED WAVES AND SHIP WAVE RESISTANCE**

Chairman: F. H. Todd

David Taylor Model Basin  
Washington, D.C.

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# AN APPROACH TO THE DESIGN OF LOW-RESISTANCE HULL FORMS

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## SUMMARY AND INTRODUCTION

This paper is divided into three parts. The first seeks to establish criteria for judging what is meant by a hull form of low resistance. The task of a ship is seen as that of carrying a given load at a given speed, and accordingly the question asked is "What is the minimum power required to propel a ship of given displacement (with no explicit restrictions on linear dimensions or shape) at a given speed?" It is recognized that a form which is optimum from this viewpoint may be more expensive to build and hence less attractive as an economic proposition than some more resistful form. Nevertheless it is useful to know what this purely hydrodynamic optimum is, so that one can judge what penalties may reasonably be incurred in departing from it for practical reasons.

Whilst the hull forms of present-day ships are determined by mainly empirical means, it is to be hoped that in the future mathematical theory may be employed as a design tool. This theory is largely concerned with wave resistance, so it is of interest to try to establish whether or not present theories of wave resistance, incomplete though they are, are nevertheless adequate for design purposes. Accordingly, the second part of the paper deals with an experimental investigation into the adequacy of Inui's (1) approach to the relationship between hull form and wavemaking. In this investigation boundary-layer effects, which sometimes cloud the issue, are kept small.

To see whether the theoretical approach to design can improve on the conventional empirical one, it seems useful to try to beat the target set in the first section of the paper and to design a hull which, for a given displacement and speed, requires less propulsive power than any normal ship. Unless a considerably lower power can be achieved in this artificial situation, where no restrictions are placed on hull shape, it seems unlikely that much improvement can be expected in any real design, where practical limitations will have to be allowed for. The third section of the paper, therefore, deals with an initial NPL attempt at this artificial design problem. This attempt, in a very rough and ready way, pays as much attention to minimizing friction resistance as to wavemaking. The experimental results obtained for the designed hull form, though they represent only a fairly small improvement on the best conventional practice, are considered encouraging in view of the crudity of the design methods

used. However the experimental results also raise questions as to the extent of viscous effects on wavemaking resistance. These questions will need to be satisfactorily answered (perhaps by a modification of the wavemaking theory on the lines suggested by Guilloton (2)) if the theoretical approach to design is to become much more accurate.

## PART 1 — CRITERIA FOR LOW RESISTANCE

### Waveless Bodies

Optimum ships in the sense defined above would be expected to have a low wavemaking resistance. This does not mean that the only requirement is to minimize wavemaking, which for a given displacement and speed could doubtless be done by making the ship resemble a rowing eight. This would, of course, lead to an excessive wetted area and a correspondingly high friction drag. For deeply submerged bodies, with no wavemaking, experience suggests that a streamlined airship shape, of length to diameter ratio in the region of 5 to 7, is best. For such a body of revolution, a suitable equation for the radius  $r$  as a function of the distance  $x$  from the nose is

$$r = A \left( \frac{x}{L} \right)^n \left( 1 - \frac{x}{L} \right),$$

where  $L$  is the body length. If the maximum thickness position is 40 percent of the length back from the nose,  $n = 2/3$ , and, as can be seen from Fig. 1, the body has a well streamlined shape. It is then easily shown that the displacement volume  $\nabla$  and the surface area  $S$  are given by

$$\nabla = 0.439 t^2 L^3 \quad (1)$$

and

$$S = tL^2 (2.168 + 0.946 t^2),$$

where  $t$ , the ratio of the maximum diameter to the length, is assumed to be small enough for  $ds = dx [1 + (dr/dx)^2]^{1/2}$  to be replaced by  $dx [1 + (1/2) (dr/dx)^2]$ . Hoerner (3) suggests that for turbulent boundary-layer flow the drag coefficient  $C_{DW}$ , based on wetted area for such a body, is related to the corresponding flat-plate friction coefficient  $C_F$  by

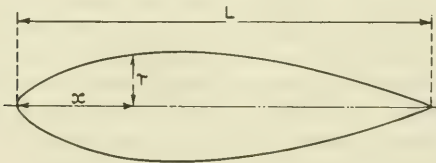


Fig. 1 - Body of the form  
 $r = A(x/L)^{2/3} [1 - (x/L)]$

# Design of Low-Resistance Hull Forms

$$C_{DW} = C_F (1 + 1.5 t^{3/2} + 7 t^3)^{**}$$

Hence, for motion at speed  $V$  through a fluid of density  $\rho$  and viscosity  $\mu$ , the drag  $D$  is given by

$$D = \frac{1}{2} \rho V^2 \frac{\nabla^{2/3} t}{0.578 t^{4/3}} (2.168 + 0.946 t^2) (1 + 1.5 t^{3/2} + 7 t^3) C_F,$$

where  $C_F$  is a function of the Reynolds number

$$R_n = \frac{\rho V L}{\mu} = \frac{1.318 \rho V \nabla^{1/3}}{t^{2/3} \mu}.$$

Most of the standard turbulent friction formulations imply that  $C_F$  varies something like  $R_n^{-1/5}$ . Hence, for minimum drag with a given speed  $V$  and volume  $V$  the factor

$$n = (2.168 + 0.946 t^2) (1 + 1.5 t^{3/2} + 7 t^3) t^{-1/5}$$

must be a minimum. It can be seen from Fig. 2 that  $n$  has a fairly flat minimum in the region of  $t = 0.15$ , and that  $n$  does not differ much from this minimum value for  $t$  in the range 0.1 to 0.25. This covers the range of beam-to-length ratios encountered in ships.

Setting  $t = 0.15$  we obtain the minimum value of  $D$ ,

$$D_{\min} = 4.0 \rho V^2 \nabla^{2/3} C_F(R_n), \quad (2)$$

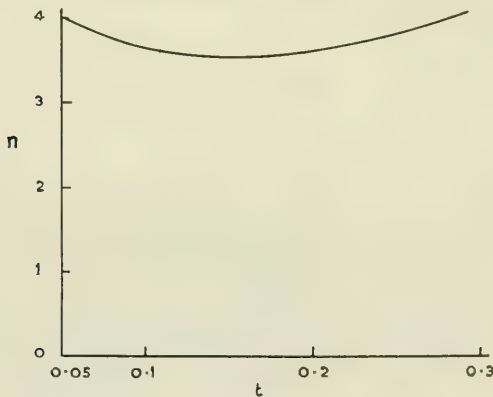


Fig. 2 - Factor  $n$  proportional to the drag of the streamlined form

\*Hoerner uses the Schoenherr line for  $C_F$ . The ITTC formula (Eq. (5) below) approximates quite closely to this for Reynolds numbers greater than  $10^7$ .



where

$$R_n = \frac{4.7}{\mu} \rho V \nabla^{1/3} \quad (3)$$

Thus, if an overall propulsive efficiency of 70 percent can be assumed, the minimum horsepower  $P_{\min}$  required to propel a body of volume  $\nabla$  at speed  $V$  is

$$P_{\min} = 1.04 \times 10^{-2} \rho V^3 \nabla^{2/3} C_F(R_n), \quad (4)$$

where  $\rho$  is in slugs/ft<sup>3</sup>,  $V$  in ft/sec, and  $\nabla$  in cu ft.

It is of interest to compare the performances of some of the old airships, whose shapes were quite similar to Fig. 1, with Eq. (4). We assume for air that  $\rho = 0.00238$  slugs/ft<sup>3</sup> and  $\mu/\rho = 1.564 \cdot 10^{-4}$  ft<sup>2</sup>/sec, and throughout this paper we employ the ITTC formulation for  $C_F$ ,

$$C_F = \frac{0.075}{(\log_{10} R_n - 2)^2} \quad (5)$$

In Table 1 the speeds  $V$  and actual powers  $P_a$  of a few airships are shown together with the estimated values of  $P_{\min}$ , calculated by assuming the displaced volume to be the gas capacity times 1.06 (an average ratio). It can be seen that for the later airships  $P_a/P_{\min}$  was lower than for the early ones, though it always remained appreciably greater than 1. This is hardly surprising, since the engine nacelles were attached to the hull by struts and bracing wires, whose drag was probably appreciable, and the fabric-covered hulls were probably not completely smooth.

Table 2 shows a similar comparison for submarines in the submerged condition. Here Eq. (4) and Eq. (3) become

$$P_{\min} = 1.061 V^3 \Delta^{2/3} C_F(R_n)$$

$$R_n = 2.11 \times 10^6 V \Delta^{1/3},$$

where  $\Delta$  is the salt-water displacement in tons (35 cu ft = 1 ton) and  $V$  is the speed in knots. Prewar submarines were not primarily designed for high speeds under water and were poorly streamlined, with consequent very high values of  $P_a/P_{\min}$ . More modern submarines seem to have performances comparable with those of airships. It has to be borne in mind that the data upon which Tables 1 and 2 are based, and the ship data to be presented below, may not be completely accurate, and that a small error in speed makes an appreciable difference to  $P_{\min}$ .

### Surface Ships

If a body of revolution of the shape shown in Fig. 1 were moved, half immersed, through water, with its axis of symmetry in the plane of the undisturbed

Table 1  
Published Data for Airships

| Airship                    | Nation-<br>ality | Date<br>Built | Length<br>(ft) | Dia.<br>(ft) | Gas<br>Capacity<br>(cu ft) | No. of<br>Engines | V<br>(mph) | P <sub>a</sub><br>(hp) | P <sub>min</sub><br>(hp) | $\frac{P_a}{P_{min}}$ |
|----------------------------|------------------|---------------|----------------|--------------|----------------------------|-------------------|------------|------------------------|--------------------------|-----------------------|
| LZ 1                       | German           | 1900          | 420            | 38           | 400,000                    | 2                 | 20         | 32                     | 7.1                      | 4.5                   |
| R 34                       | British          | 1918          | 643            | 79           | 1,960,000                  | 5                 | 60         | 1250                   | 485                      | 2.6                   |
| LZ 127<br>Graf<br>Zeppelin | German           | 1928          | 772            | 100          | 3,710,000                  | 5                 | 80         | 2650                   | 1620                     | 1.64                  |
| Akron                      | American         | 1932          | 785            | 133          | 6,500,000                  | 8                 | 80         | 4480                   | 2300                     | 1.95                  |
| LZ 129<br>Hindenburg       | German           | 1936          | 814            | 135          | 6,709,800                  | 4                 | 85         | 4400                   | 2830                     | 1.56                  |

Table 2  
Published Data for Submarines

| Submarine | Nationality | Date<br>Built | Δ<br>(tons) | V<br>(knots) | P <sub>a</sub><br>(hp) | P <sub>min</sub><br>(hp) | $\frac{P_a}{P_{min}}$ |
|-----------|-------------|---------------|-------------|--------------|------------------------|--------------------------|-----------------------|
| Porpoise  | British     | 1932          | 2060        | 8.75         | 1630                   | 213                      | 7.7                   |
| Sea Lion  | British     | 1935          | 960         | 10           | 1300                   | 194                      | 6.7                   |
| Oberon    | British     | 1959          | 2410        | 16           | 3600                   | 1320                     | 2.72                  |
| Skate     | American    | 1958          | 2861        | 19           | 6600                   | 2410                     | 2.74                  |
| Gudgeon   | American    | 1952          | 2400        | 18           | 3200                   | 1860                     | 1.72                  |

water, it would of course make waves. Let us imagine, however, that its shape could be modified so that its wavemaking were reduced virtually to zero, without altering its immersed volume or appreciably increasing its frictional drag. The resistance of such an imaginary ship would then be half that given by Eq. (2) with  $\nabla$  set equal to twice the immersed volume, since the water surface would remain practically flat and would be equivalent to a plane of symmetry for a deeply immersed body. Correspondingly, the power needed to drive this ship with 70 percent propulsive efficiency would, from Eq. (4), be given by

$$P_{\min} = 0.845 V^3 \Delta^{2/3} C_F(R_n), \quad (6)$$

where

$$R_n = 2.66 \times 10^6 V \Delta^{1/3}, \quad (7)$$

$v$  is the speed in knots, and  $\Delta$  the salt-water displacement in tons of the immersed part of the hull. This implies that, in the usual notation, the minimum value of  $C$  is  $18.9/(\log_{10} R_n - 2)^2$ , remarkably close to Hughes' estimate (4), derived by quite different procedures, for the minimum viscous contribution to  $C$ .

In reality, of course, a greater power is likely to be needed, since the modifications in shape required to reduce the wavemaking of the half-immersed body of revolution would almost certainly increase frictional drag. Nevertheless Eq. (6), in conjunction with Eq. (5) and Eq. (7), provides a reasonable datum with which to compare actual ship performances.

In Table 3 the actual powers  $P_a$  of a number of ships, whose details are given in published literature, are compared with the  $P_{\min}$  values calculated from  $\Delta$  and  $V$  using Eq. (6). Speed-to-length ratios  $V/\sqrt{L}$  are also given, where  $L$  is the LBP in feet. Some of the data of Table 3 are quite probably in error; although we have tried to ensure that the stated power corresponds to smooth-water smooth-hull operation at the stated displacement and speed, this may not be true in every case. Sometimes, for example, the displacement given may be a service one, whereas the speed may be a trials result obtained at a different displacement. Likewise, the power quoted may contain an appreciable service margin to enable the ship to reach the given speed even in rough-weather conditions and when the hull surface has deteriorated with time out of dock. Thus, some of the results for  $P_a/P_{\min}$  may be subject to quite large errors.

It can be seen that the ratios  $P_a/P_{\min}$  vary widely, even for ships of roughly the same speed-to-length ratios. For large values of  $V/\sqrt{L}$ , typical of fast warships,  $P_a/P_{\min}$  is large, in the region of 3 to 5. It is interesting that the power ratios of postwar warships are somewhat lower than those for prewar ones; this presumably reflects improvements in hull design and the use of welded rather than riveted construction. Among the vessels with  $V/\sqrt{L}$  in the region of 1, cargo liners with displacements in the region of 10,000 to 20,000 tons are much superior to passenger ferries; the latter have to be made beamier because of draught limitations and for stability reasons. Large tankers and bulk carriers show a wide range of performance, the best of them having a power ratio of as low as 1.5, whilst some have ratios of well over 2.

# Design of Low-Resistance Hull Forms

Table 3  
Published Data for Ships

| Name                 | Nationality | Type                     | Date Built | $\Delta$<br>(tons) | $v$<br>(knots) | $P_a$<br>(hp)    | $P_{min}$<br>(hp) | $\frac{P_a}{P_{min}}$ | $\frac{v}{\sqrt{L}}$ |
|----------------------|-------------|--------------------------|------------|--------------------|----------------|------------------|-------------------|-----------------------|----------------------|
| Nelson               | British     | Battle-ship              | 1927       | 33,950             | 23             | 45,000           | 15,200            | 2.96                  | 0.86                 |
| Illustrious          | British     | Aircraft carrier         | 1939       | 23,000             | 31             | 110,000          | 28,300            | 3.89                  | 1.13                 |
| London               | British     | Cruiser                  | 1929       | 9,900              | 32.25          | 80,000           | 18,600            | 4.30                  | 1.28                 |
| Newcastle            | British     | Cruiser                  | 1936       | 9,100              | 32             | 75,000           | 17,200            | 4.36                  | 1.32                 |
| Arethusa             | British     | Cruiser                  | 1934       | 5,220              | 32.25          | 64,000           | 12,500            | 5.13                  | 1.44                 |
| Lightning            | British     | Destroyer                | 1939       | 1,920              | 36.5           | 48,000           | 9,500             | 5.05                  | 1.94                 |
| Atherstone           | British     | Destroyer                | 1939       | 904                | 27.5           | 19,000           | 2,650             | 7.17                  | 1.67                 |
| Zulu                 | British     | Destroyer                | 1937       | 1,870              | 36.5           | 44,000           | 9,400             | 4.68                  | 1.94                 |
| Devonshire           | British     | Guided-missile destroyer | 1960       | 6,200              | 32.5           | 66,000           | 14,100            | 4.25                  | 1.45                 |
| Hermes               | British     | Aircraft carrier         | 1953       | 27,800             | 28             | 78,000           | 23,600            | 3.30                  | 1.10                 |
| Eagle                | British     | Aircraft carrier         | 1946       | 54,100             | 31.5           | 152,000          | 50,200            | 3.03                  | 1.17                 |
| Ashanti              | British     | Frigate                  | 1959       | 2,700              | 28             | 20,000           | 5,480             | 3.65                  | 1.50                 |
| Whitby               | British     | Frigate                  | 1956       | 2,560              | 31             | 30,430           | 7,100             | 4.28                  | 1.63                 |
| Benloyal             | British     | Cargo liner              | 1959       | 9,000              | 22.3           | 11,371           | 6,100             | 1.86                  | 0.99                 |
| Yamashiro Maru       | Japanese    | Cargo liner              | 1963       | (11,180<br>18,170) | 22.45<br>19.5  | 13,322<br>11,050 | 7,060<br>6,330    | 1.89<br>1.75          | 1.01<br>0.88         |
| Media                | British     | Cargo liner              | 1963       | 11,500             | 17             | 6,930            | 3,210             | 2.16                  | 0.85                 |
| Vienne               | French      | Cargo liner              | 1962       | 15,000             | 20             | 14,400           | 6,100             | 2.36                  | 0.92                 |
| Glenogle             | British     | Cargo liner              | 1962       | 11,270             | 22.43          | 16,850           | 7,070             | 2.38                  | 1.00                 |
| Neder Rijn           | German      | Cargo liner              | 1962       | 10,500             | 20.9           | 13,300           | 5,500             | 2.42                  | 0.94                 |
| Straat Frazer        | Dutch       | Cargo liner              | 1962       | 9,660              | 20.0           | 11,340           | 4,630             | 2.45                  | 0.93                 |
| Dlugosz              | Danish      | Cargo liner              | 1963       | 18,300             | 16.25          | 6,500            | 3,750             | 1.73                  | 0.75                 |
| Glenfalloch          | British     | Cargo liner              | 1962       | 19,040             | 20             | 16,600           | 7,090             | 2.34                  | 0.89                 |
| Mormacargo           | American    | Cargo liner              | 1964       | 12,300             | 24             | 17,500           | 9,130             | 1.92                  | 1.02                 |
| Manx Maid            | British     | Channel ferry            | 1961       | 3,033              | 21             | 9,500            | 2,560             | 3.71                  | 1.16                 |
| Lord Warden          | British     | Channel ferry            | 1963       | 3,470              | 20             | 8,000            | 2,285             | 3.50                  | 1.09                 |
| Ilmatar              | Finnish     | Channel ferry            | 1963       | 3,470              | 16.6           | 4,500            | 1,430             | 3.14                  | 0.92                 |
| Malaspina            | American    | Channel ferry            | 1962       | 3,585              | 18             | 8,000            | 1,840             | 4.35                  | 0.96                 |
| Princess of Tasmania | Australian  | Channel ferry            | 1959       | 4,550              | 17.75          | 7,500            | 2,060             | 3.64                  | 0.94                 |

(Table continues)



Table 3 (Continued)

| Name                    | Nation-<br>ality | Type                                     | Date<br>Built | $\Delta$<br>(tons) | V<br>(knots) | P <sub>a</sub><br>(hp) | P <sub>min</sub><br>(hp) | $\frac{P_a}{P_{min}}$ | $\frac{V}{\sqrt{L}}$ |
|-------------------------|------------------|------------------------------------------|---------------|--------------------|--------------|------------------------|--------------------------|-----------------------|----------------------|
| Empress of<br>Australia | Australian       | Passenger<br>liner &<br>vehicle<br>ferry | 1964          | 8,750              | 19.5         | 13,500                 | 4,050                    | 3.33                  | 0.92                 |
| Ivan Franko             | E. German        | Passenger<br>liner                       | 1964          | 19,018             | 20           | 21,000                 | 7,080                    | 2.96                  | 0.89                 |
| Oceanic                 | Italian          | Passenger<br>liner                       | 1962          | 30,660             | 25.1         | 44,500                 | 18,400                   | 2.42                  | 0.95                 |
| Santa<br>Magdalena      | American         | Cargo-<br>passenger                      | 1962          | 18,110             | 20.5         | 19,800                 | 7,400                    | 2.68                  | 0.88                 |
| Chatwood                | Spanish          | Cargo                                    | 1962          | 17,936             | 16.5         | 7,200                  | 3,890                    | 1.85                  | 0.76                 |
| Letaba                  | British          | Fruit<br>carrier                         | 1963          | 15,750             | 17.5         | 9,665                  | 4,270                    | 2.26                  | 0.80                 |
| Geestbay                | Dutch            | Banana<br>carrier                        | 1964          | 13,425             | 19.5         | 10,500                 | 5,280                    | 1.99                  | 0.93                 |
| Fort<br>Josephine       | French           | Fruit<br>carrier                         | 1964          | 8,520              | 18.5         | 6,664                  | 3,410                    | 1.95                  | 0.95                 |
| Sinclair<br>Texas       | American         | Tanker                                   | 1963          | 60,700             | 16.5         | 13,600                 | 8,400                    | 1.62                  | 0.61                 |
| British<br>Ensign       | British          | Tanker                                   | 1963          | 88,980             | 15.5         | 20,000                 | 8,920                    | 2.24                  | 0.54                 |
| Sirella                 | French           | Tanker                                   | 1963          | 99,946             | 17.7         | 26,750                 | 14,900                   | 1.80                  | 0.62                 |
| Mobil                   | Swedish          | Tanker                                   | 1963          | 118,350            | 17.25        | 28,000                 | 14,500                   | 1.93                  | 0.59                 |
| Brilliant               | British          | Tanker                                   | 1963          | 89,503             | 15.5         | 20,000                 | 8,930                    | 2.24                  | 0.56                 |
| Mariner                 | Swedish          | Tanker                                   | 1962          | 74,456             | 17.5         | 21,740                 | 11,350                   | 1.91                  | 0.64                 |
| Vestalis                | Japanese         | Tanker                                   | 1964          | 85,512             | 17.09        | 23,000                 | 11,450                   | 2.01                  | 0.62                 |
| Atlantic<br>Atares      | Irish            | Bulk<br>carrier                          | 1963          | 38,075             | 15.1         | 9,300                  | 4,900                    | 1.90                  | 0.62                 |
| Amstelhof               | British          | Bulk<br>carrier                          | 1963          | 26,400             | 14           | 7,500                  | 3,100                    | 2.42                  | 0.61                 |
| Carlton                 | British          | Bulk<br>carrier                          | 1962          | 68,500             | 16.25        | 15,000                 | 8,680                    | 1.73                  | 0.59                 |
| Essi Gina               | British          | Logistics<br>ship                        | 1963          | 5,550              | 17           | 4,125                  | 2,040                    | 2.02                  | 0.89                 |
| Sir<br>Lancelot         | British          | Coastal<br>tanker                        | 1963          | 2,217              | 11           | 1,230                  | 331                      | 3.72                  | 0.75                 |
| Hamble                  | Spanish          | Trawler                                  | 1963          | 2,525              | 12           | 1,850                  | 459                      | 4.03                  | 0.76                 |
| Toula                   | British          | Tug                                      | 1963          | 942                | 14           | 2,000                  | 388                      | 5.16                  | 1.17                 |
| Britonia                | British          | Corvette                                 | 1964          | 440                | (20<br>(18   | 3,340<br>2,000         | 670<br>495               | 4.99<br>4.04          | 1.58<br>1.42         |

We now attempt to find a formula for the power required to propel a ship of salt-water displacement  $\Delta$  tons at a speed of  $V$  knots, when this ship is designed by the best current procedures to have minimum resistance, with no other restrictions imposed. A reasonable form of expression, based on Eq. (6), is

$$P_{\text{best}} = V^3 \Delta^{2/3} [0.845 C_F(R_n) + F(N)] , \quad (8)$$

where  $N$ , approximately equal to  $V/\sqrt{L}$ , is given by

$$N = \frac{0.228 V}{\Delta^{1/6}} \quad (9)$$

from Eq. (1), where we put  $t$  equal to 0.15, as for the optimum body of revolution, and  $\nabla$  equal to twice the displaced volume in cu ft of the ship, or  $70\Delta$ . Eq. (8) was fitted to the best data of Table 3 and of a further collection of trials data, not reproduced here. These best data were:

|             |                        |                   |                   |
|-------------|------------------------|-------------------|-------------------|
| Tanker      | $\Delta = 24,230$ tons | $V = 13.78$ knots | $P_a = 4,120$ hp  |
| Cargo liner | $\Delta = 11,070$ tons | $V = 22$ knots    | $P_a = 11,290$ hp |
| Frigate     | $\Delta = 2,700$ tons  | $V = 28$ knots    | $P_a = 20,000$ hp |

and so we obtained an approximate relation

$$P_{\text{best}} = V^3 \Delta^{2/3} [0.845 C_F(R_n) + 0.5 \times 10^{-3} + 0.3 \times 10^{-3} N^4] . \quad (10)$$

An alternative, equally plausible, relation roughly fitting the same optimum data is

$$P_{\text{best}} = V^3 \Delta^{2/3} [1.20 C_F(R_n) + 0.3 \times 10^{-3} N^4] .$$

This does not differ much from Eq. (10) in the low Reynolds number range, but shows up most of the large tankers in an even poorer light\* than does Eq. (10), which we therefore assume to be more appropriate. In a sense, this may be departing from the intention that  $P_{\text{best}}$  should represent the minimum power attainable, since large tankers, which provide most of the low Froude-number data, are always designed with very bluff lines, and there is little doubt that a finer-lined ship of the same displacement and speed would require less power. To this extent, therefore, Eq. (10) may represent some compromise with practical requirements. This compromise could perhaps have been avoided by using model results in deriving Eq. (10). However, we have consistently used full-scale data to avoid uncertainties in ship-model correlations and to include effects of any hull-surface imperfections which may be normally present on the full scale. There is a scarcity of full-scale data for large fine-lined ships at low Froude numbers, but, since most tanker powers are considerably in excess

\*That is, of course, when looked at purely from the point of view of resistance. As pointed out earlier, practical considerations (in this case the desirability of minimizing length, so reducing construction costs) may often require large departures from the minimum-resistance "optimum."

of  $P_{best}$  as given by Eq. (10), the latter may not be too much of an overestimate at the low Froude-number end. At the other extreme, it cannot be expected to apply for speed-to-length ratios much larger than for the frigate results used in its derivation; hence we restrict  $N$ , given by Eq. (9), to values less than 2.

Figure 3 shows curves of  $P_{best}$ , given by Eq. (10), as a function of displacement  $\Delta$  for various speeds  $V$ . It also shows the corresponding curves of  $P_{min}$ , given by Eq. (6). Even with very sophisticated design, ship powers are never likely to come down as low as  $P_{min}$ , but the difference  $P_{best} - P_{min}$  nevertheless gives some measure of the improvement over the best current designs which might conceivably be achievable. Most of the ship powers given in Table 3 lie above the  $P_{best}$  curves, in some cases a great deal above, as is only to be expected, since the curves of Fig. 3 are (subject to the possible reservation

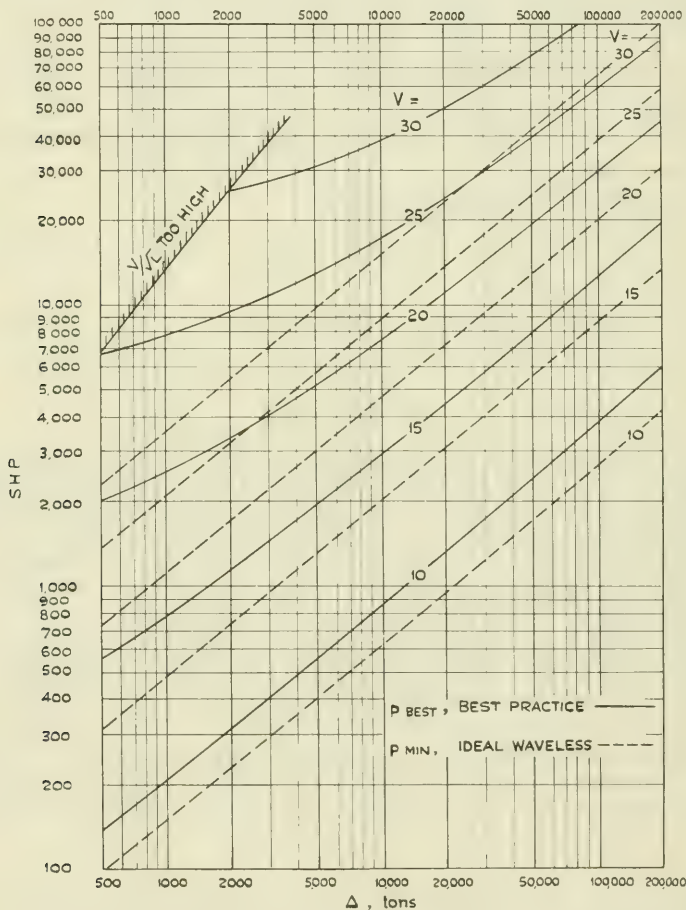


Fig. 3 - Curves of horsepower versus displacement for various speeds  $V$  (knots)

noted above concerning the tanker end of the data) intended to represent minimum powers achieved by current optimum designs when subject to the minimum of extraneous practical requirements.

## Conclusions from Part 1

It has been shown that the propulsive shaft horsepowers  $P_{best}$ , of those present-day ships which for a given displacement  $\Delta$  and speed  $V$  have the least resistance, approximately fit the formula

$$P_{best} = V^3 \Delta^{2/3} [0.845 C_F(R_n) + 0.5 \times 10^{-3} + 0.3 \times 10^{-3} N^4], \quad (10)$$

where

$$C_F(R_n) = 0.075 (\log_{10} R_n - 2)^{-2}, \quad (5)$$

$$R_n = 2.66 \times 10^6 V \Delta^{1/3}, \quad (7)$$

and

$$N = 0.228 V \Delta^{-1/6}. \quad (9)$$

It is unlikely that, however much design can be improved, the power could ever be brought down much below

$$P_{min} = 0.845 V^3 \Delta^{2/3} C_F(R_n), \quad (6)$$

to achieve which, wavemaking would have to be virtually eliminated without increasing the frictional drag per ton above that of a fully submerged optimum airship form. The difference between  $P_{best}$  and  $P_{min}$ , therefore, represents the margin there is for improvement on present-day designs.

Three points should be borne in mind.

1. Equation (10) represents an envelope for different ships, each "optimised" (by empirical processes) for its particular operational conditions, and so it does not represent what happens to a given hull form as its speed is varied. Thus the  $N^4$  factor cannot be identified simply with wave resistance, though undoubtedly it forms an important part of the wave resistance.

2. A ship whose propulsive power greatly exceeds  $P_{best}$  is not necessarily a badly designed ship, since Eq. (10) represents only a hydrodynamic optimum, and a more resistful ship may well be cheaper to build and hence a more economic design than one conforming to Eq. (10). The latter form may also be unable to meet practical requirements of stability, draught limitation, etc.

3. Nevertheless, Fig. 2 suggests that even for very beamy, and hence short and cheap, ships, the viscous component of resistance need not necessarily greatly exceed its optimum value. If, therefore, we were sufficiently adept at reducing wave resistance for large beam-to-length ratios, we ought to be able to design a ship which was not only cheap to build, but which also compared favourably with Eq. (10).



## PART 2 — EXPERIMENTS ON INUIDS

One possible technique for handling the wavemaking side of design is that adopted by Inui (1) and Pien (5). A distribution of wavemaking sources, giving suitably low resistance, is chosen and the corresponding hull surface calculated by, in effect, assuming the water surface to remain flat. Thus the source distribution is treated as one of ordinary potential sources reflected in the undisturbed water plane. This procedure cannot be rigorously justified, even for very low Froude numbers, but it has been plausibly argued by Pien that for forms of low wavemaking resistance, in which we are mainly interested, the theory should be adequate, since the water surface then does remain fairly flat.

Inui has made experimental comparisons between models (which have been termed "Inuids") derived from source distributions in this way and other models derived from the same source distribution by the Michell thin-ship approximation. The latter is equivalent to relating the waterline slope to the source density on the central plane. In general the Inuids showed better agreement with experiment, but discrepancies remained and were attributed by Inui largely to viscous effects.

A third possible way in which the source distribution may be related to hull shape is that proposed by Guilloton (2). Although this method looks very promising it has not been investigated here. Rather, it was attempted to establish whether in experiments with Inuids the discrepancies between theory and experiment could reasonably be ascribed to viscous effects or whether instead the discrepancies resulted mainly from the Draconian treatment of the free surface condition.

With normal ship models viscous interference with wavemaking arises mainly at the stern. If, then, a head form is used, followed by a long parallel afterbody from which it is separated by a small gap, the viscous effects on the wavemaking of the head should be small. Such a head form in deeply submerged potential flow would have zero drag if the pressure in the gap were equal to that in the undisturbed stream. In a free surface, therefore, the net pressure force on the head, acting over all its surfaces, plus the force represented by the excess of mean pressure in the gap over free-stream pressure times the wetted area of the rear face of the head abutting the gap should equal the wave resistance. A net *total* force measurement on the head will, of course, include a friction component, but this ought to be reliably calculable, since the boundary layer should remain thin and well-behaved.

Experiments on models of this kind were independently suggested by Breslin (6). At the NPL, two such head-form models were designed as shown in Figs. 4 and 5. Each model was generated, by the Inui method, from source distributions defined over a rectangular area of length  $L$  and depth  $D = L/4\pi$  on the central plane. The source distribution was uniform for the first model, which did not have a particularly low theoretical wave resistance. For the second, quasi bulbous-bow, model the source distribution varied both in the longitudinal or  $x$  direction and the depth or  $z$  direction, as indicated in Fig. 5. However, the source density was everywhere positive to ensure that the waterlines were everywhere inclined at a positive angle to the oncoming flow, in the interests of

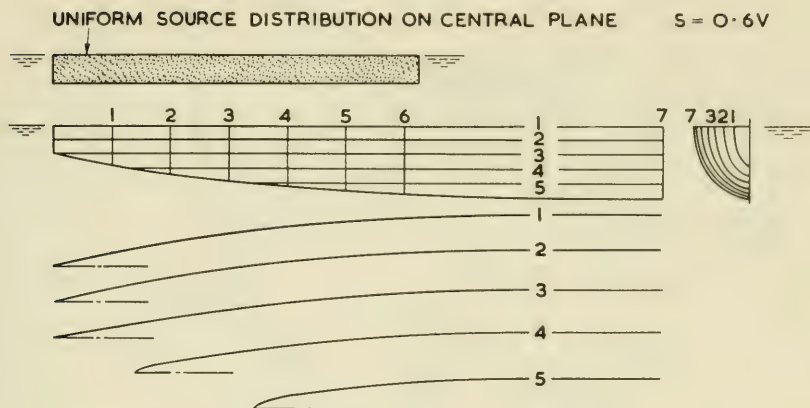


Fig. 4 - Uniform source distribution head form

a well-behaved boundary layer. Also, at each depth within the source-distribution rectangle the integral of the source density was the same as for the uniform source-density model, thus ensuring that the afterbody was of roughly the same shape for the two models. Within these imposed limitations the form chosen for the source density in Fig. 5 was arrived at by a cut-and-try procedure to give a resistance which, if not necessarily the lowest possible, was at least substantially below that for the uniform density model, as can be seen from Fig. 6. On Pien's arrangement, therefore, experimental results for the quasi-bulbous model should agree closer with theory than those for the uniform source-density model.

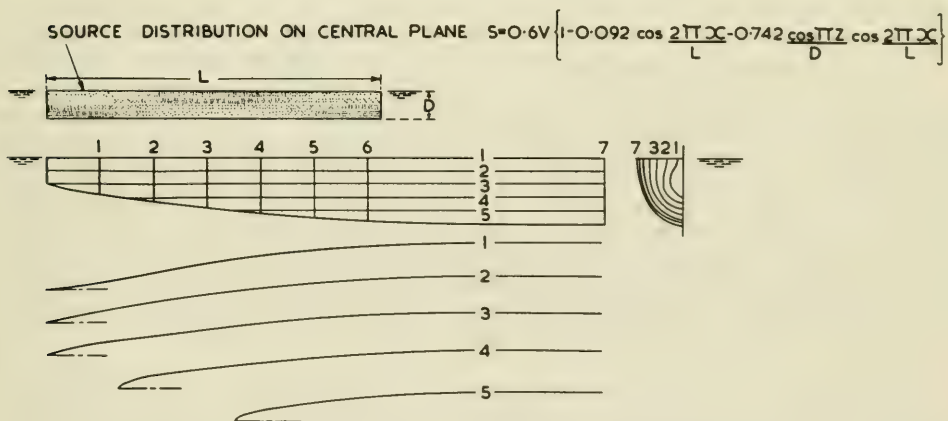


Fig. 5 - Quasi-bulbous head form

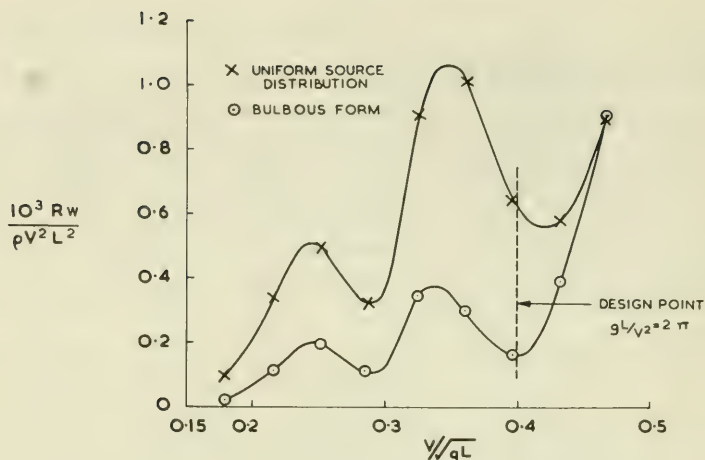


Fig. 6 - Calculated wave resistance curves for the two head forms

The models as made had  $L = 6$  ft, and the junction with the afterbody was at 10 ft from the bow. Figures 7 and 8 show the experimental results obtained for the net force on the head of the uniform density model and the pressure force on its rear bulkhead as functions of speed. The pressure involved in Fig. 8 was measured at the centre of the bulkhead. It is possible that some variation of pressure occurred on the bulkhead close to the hull surface, and this would introduce errors in Fig. 8. However, it is expected that these would be small since the gap,  $1/4$  in., was small compared with the local boundary layer thickness of about 2 in. It can be seen from Figs. 7 and 8 that each of the force components oscillates more wildly than their sum, in Fig. 9, representing wave drag plus friction drag. Also shown in Fig. 9 are the theoretical friction drag  $D_F$ , based on wetted area and the ITTC line, Eq. (5), and the sum of  $D_F$  and the theoretical wave drag  $D_W$ , derived from Fig. 6. It will be seen that the lower speed hump is bigger experimentally than theoretically, the reverse of what usually happens on a complete ship model. However, the addition of the oscillatory components, Figs. 7 and 8, may lead to inaccuracies, especially at the lower speeds where the forces are smaller, so perhaps too much attention should not be paid to this discrepancy. More significant probably is the fact that the experimental humps and hollows are displaced to the right of the theoretical ones, in exactly the same way as for complete models. Thus this displacement seems unlikely to be the result mainly of viscous effects, but rather to arise from the inadequacies of the approximations of the potential-flow problem.

However, as was mentioned above, the agreement between theory and experiment was not necessarily expected to be as good for the resistful uniform-source form as for the quasi-bulbous form. The results corresponding to Fig. 9 are shown for the latter form in Fig. 10.

# Design of Low-Resistance Hull Forms

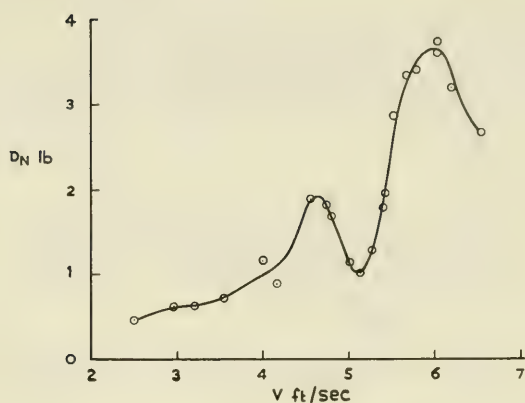


Fig. 7 - Net drag  $D_N$  of the head of the uniform source density model

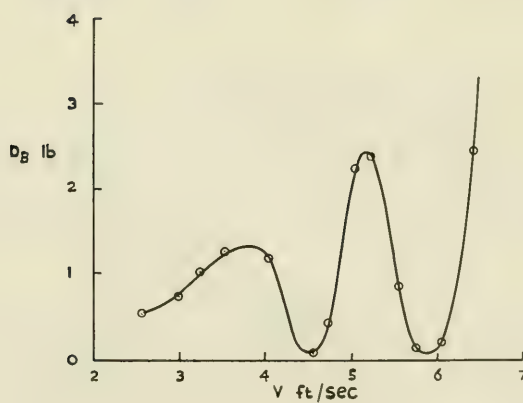


Fig. 8 - Bulkhead pressure force  $D_B$  of the uniform source density model



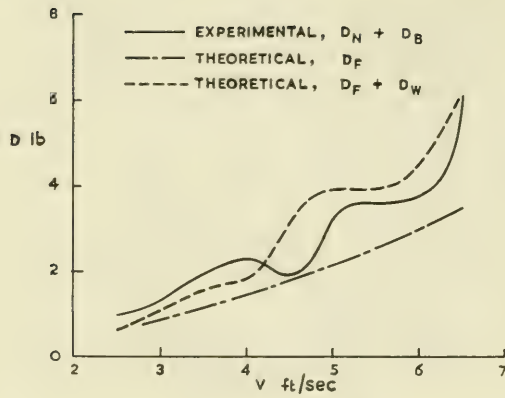


Fig. 9 - Theoretical and experimental drags of the uniform source density model

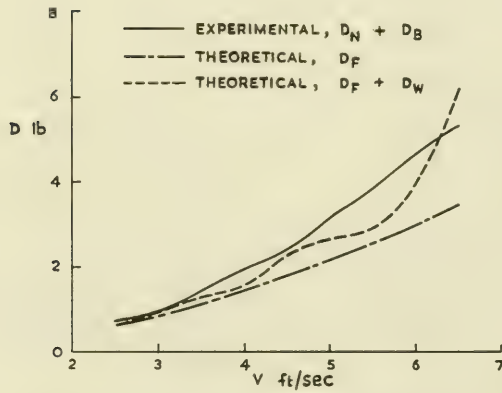


Fig. 10 - Theoretical and experimental drags of the quasi-bulbous model

It can be seen that the agreement with experiment is certainly no better for the quasi-bulbous form, except for speeds below 3.5 ft/sec where the water surface does indeed remain nearly flat as assumed in the theory. The expected reduction of resistance as compared with the uniform-source form is not for the most part realized. The experimental resistance curve is much less undular in Fig. 10 than in Fig. 9, and in this qualitative respect the experimental results resemble the theoretical ones. However, if the experimental results can be believed (and as mentioned above they are subject to some uncertainties) it would appear that wavemaking theory applied in the Inui manner may be a very unsafe guide to hull design.

Computations are in hand to determine the wave resistance according to the Michell thin-ship theory for the two head forms tested. It will be interesting to see if this theory agrees better or worse with the experimental results than the Inui theory, and it is hoped to present results at the meeting.

### PART 3 — AN ATTEMPT TO DESIGN A LOW-RESISTANCE FORM

If a more theoretical approach to ship design is to lead to better results than current practice can achieve, its use ought to make possible the design of a ship with a substantially lower power than is predicted by Eq. (10). In this section a very crude attempt is made at such a design. This follows in the wake of the remarkable achievements of Inui (1) and Pien (5) but lays more stress on the frictional component of resistance.

In the discussion of Part 1 we talked of regarding a ship as half an airship, so modified as to reduce its wavemaking without losing the viscous-drag advantages of the airship form. Airships, unlike ships, have their maximum-thickness sections well forward of amidships. Such a form would, for a ship, usually lead to high wavemaking, but if we were clever enough at reducing the latter, might not this shape then have advantages due to reduced frictional drag? These somewhat naive questionings prompted the design shown in Fig. 11.

The starting point of this design was the observation that in Fig. 5 the afterbody of the quasi-bulbous form is roughly semielliptical in section. An airship has circular cross sections, as this gives the minimum wetted area. For the same reason a low-resistance ship ought to have semicircular cross sections. (We ignore here matters such as lack of stability and high costs of construction which sections of such a shape might give rise to, since we are for the moment only concerned with the artificial design task of achieving a purely hydrodynamic optimum.) The head form of Fig. 5, stretched laterally so that it ended as a semicircle rather than an ellipse, might therefore, it was thought, make a forebody of sufficiently low wave resistance to be joined onto an afterbody derived from an airship form. (These ideas were conceived prior to the actual testing of the quasi-bulbous model, and would have seemed less attractive in the light of the unfavourable experimental results for it.) The airship form chosen was as in Fig. 1, and the afterbody was basically the part of this downstream of the maximum section. However, to avoid the draught becoming zero at the stern, the after sections were smoothly changed from semicircles to semiellipses of

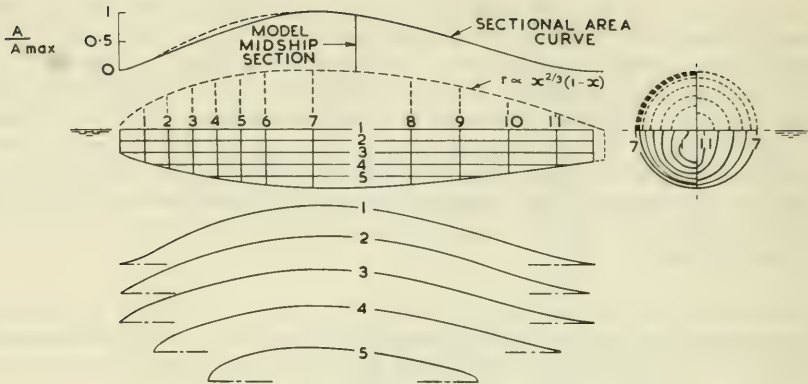


Fig. 11 - Widened bulbous head form matched with the afterbody. The longitudinal dimensions are compressed by a factor of  $1/2$ .

the same area to produce a stern draught of half the maximum. The forebody length of 8 ft corresponded to 1.28 times the source length  $L$  in Fig. 5, and the beam was 2.5 ft. The initially designed total length was 20 ft, but the stern was found to be so finely cusped that it was truncated by 6 inches to give a model 19.5 ft long.

As will be seen from Fig. 11 the sectional area curve for the resulting model is nearly the same as for the dotted airship form, and is highly asymmetrical. Wave theory suggests that this must be a bad thing, but the hope was to recoup any losses on this score from a reduction in frictional resistance. Although it was intended that the bulbous end of the model should form the bow, an over-rigid dogmatism on this point was felt to be undesirable, and hence the model was fitted with transition-fixing studs at both ends.

The wisdom of this decision is illustrated in Fig. 12, which shows that the resistance was, for the most part, much less when run with the bulbous end aft. At low speeds it is true the airship-way round shows some advantages, but these are very small. Perhaps even here wavemaking is still significant, or perhaps the viscous drag penalties due to running a sufficiently fine airship form backwards rather than forwards are smaller than I had supposed. (Such penalties might well be greater for beamier forms.) However, once wavemaking becomes appreciable the bulb-aft condition shows a clear advantage, with a resistance curve well below that for the other condition, though parallel to it for speeds above 5 ft/sec.

The residuary resistance is thus much less for the model with the fuller end aft, in apparent contradiction to theory, which states that a given source distribution should have the same wave drag run in either direction. The directional effect found in the experiments reinforces Wigley's findings (7) on asymmetrical forms. In the present experiments the wave-pattern resistance was determined from wave-pattern measurements which were fed into computer



# Design of Low-Resistance Hull Forms

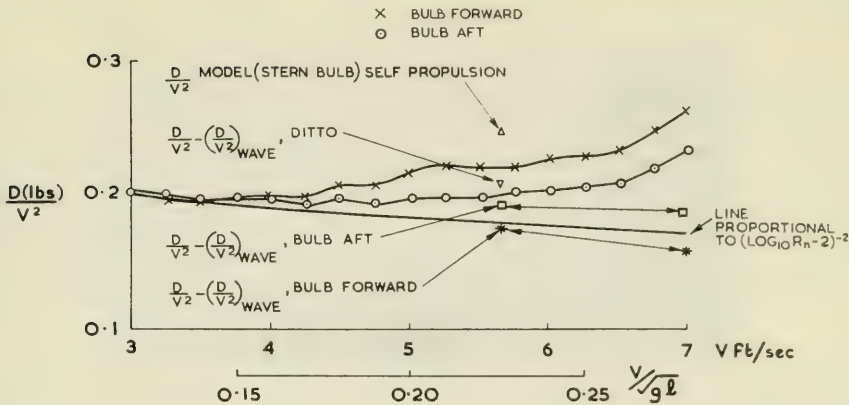


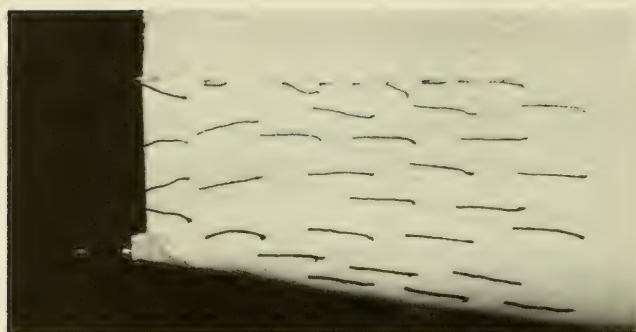
Fig. 12 - Experimental results for the 19.5-ft-long would-be low-drag hull form

programmes (8) based on the Eggers method (9). In Fig. 12 points are shown corresponding to the total resistance minus the wave-pattern resistance. For the bulb-forward condition these points lie close to a line which may be regarded as representing the frictional resistance. Thus in this condition the residuary and wave-pattern resistances agree. For the bulb-aft condition, however, the points of total minus wave-pattern resistance lie well above the frictional line. Thus it appears that the wave-pattern resistance is much smaller than even the reduced residuary resistance for this condition. Flow photographs taken in the Ship Division circulating water channel explain this discrepancy. They show, as in Figs. 13a and 13b, that at 5.66 ft/sec there is more evidence of separation from the bulb at the stern than at 3 ft/sec. This is doubtless due to the stern wave being higher at 5.66 ft/sec, causing an adverse pressure gradient over the bulb. Thus part of the residuary resistance at the higher speeds with the bulb aft can be attributed to an increase in form drag above that acting on the vessel at lower speeds.

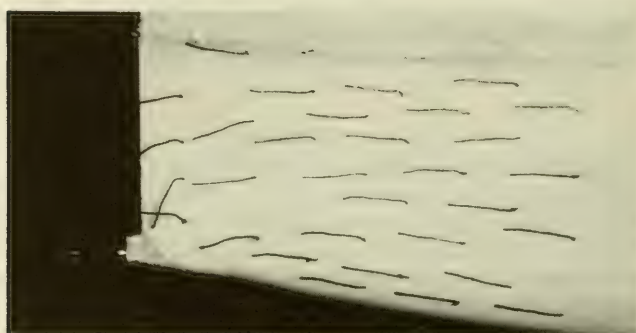
In Figs. 13a and 13b a fairing can be seen over a propeller shaft which was fitted in the bulb end after it was found that the bulb-aft condition was best. When the model was run at 5.66 ft/sec, and propelled, as appropriate for a full scale ship 693 ft long, at 20 knots, the quasi-propulsive coefficient was found to be 0.74. When run at the model self-propulsion point the model resistance was, as can be seen from Fig. 12, appreciably higher than when towed. However the wave-pattern resistance was also higher then,  $(D/V^2)_{\text{wave}}$  being 0.038, much greater than the unpropelled value of 0.009, and nearly as great as the value 0.045 for the towed model run bulb forward at the same speed of 5.66 ft/sec. The running of the propeller suppressed the separation evident in Fig. 13b as can be seen from Fig. 13c.

All this raises some interesting questions. Does the lower wave-pattern resistance with the bulb aft result mainly from the boundary layer thickness at the stern being greater than when the fine end is run aft? Is this true also





(a)



(b)



(c)

Fig. 13 - Flow over the stern bulb: (a) 3 ft/sec, (b) 5.66 ft/sec, and (c) 5.66 ft/sec but with the propeller running

of the residuary resistance? If so, it means that boundary layer thickening can have a net beneficial effect, a very surprising result.

Of course, the increase in wave-pattern resistance when the propeller is running might be due, not to a mere suppression of separation, but rather to an induction by the screw, over the stern bulb, of low pressures such as would occur in potential flow. It appears from Figs. 13b and 13c that there is little effect on wave height at the stern, but a very small change of local wave height could be associated with a relatively large change in pressure drag. For this latter reason it cannot be accepted as conclusive proof in favour of Guilloton's theory (2) that he demonstrates good agreement between theory and experiment regarding the wave profiles along the side of the model.

Guilloton's theory does, however, suggest an alternative explanation of the discrepancy between the residuary resistances of the model run in the different directions. As mentioned above, a given source distribution should have the same wave drag run in either direction. According to either Michell's theory or to Inui's theory the same hull run in opposite directions corresponds to the same source distribution, but this is not so for Guilloton's theory. (I cite Guilloton's method rather than other potentially more exact methods of taking account of the distortion of the free surface because these latter methods have not so far been developed to the point of being practicable for application, whereas Guilloton's method is practicable.) If, then, Inui's theory is valid, the discrepancies in Fig. 12 must be attributed to viscosity, but they need not be if Guilloton's theory is more correct. On the other hand even Guilloton's theory predicts that a hull of minimum wavemaking should be symmetrical about the centre line. The bulb-aft model has such a low wave resistance that it is hard to imagine that its effective hydrodynamic shape (including boundary layer displacement effects) can be far from an optimum, unless this is such that some kinds of large departure from the ideal shape can be tolerated without much increase of resistance.

If then we fall back on viscous effects as representing probably at least part of the explanation, it seems important to develop a theoretical treatment of them. Of course Havelock (10), Wigley (11), Emerson (12), and Inui (1) have attempted to do this, but their methods lack a strong theoretical justification. More satisfactory in principle is the method of Wu (13), but its development to take proper account of the three-dimensional, turbulent nature of the boundary layer would be difficult. However, the following arguments suggest, in support of Havelock's intuition (10), that probably only the local boundary-layer thickening near the stern is of any significance. Hogben's experimental measurements (14) show that the mean displacement thickness on a mathematical-form model with parabolic waterlines varies roughly linearly with distance from the bow till quite near the stern, where there is a rapid increase. Let us therefore consider a central-plane source distribution whose density  $S$  is defined by

$$S = AV \left( 1 - \frac{2x}{L} \right) + AV\epsilon, \quad 0 < x < L, \quad 0 < z < D,$$

where  $\epsilon = 0$  for  $x < X$ . If  $X = 0$  such a distribution defines, according to the Michell approximations, a body of the shape shown in Fig. 14a, with parabolic

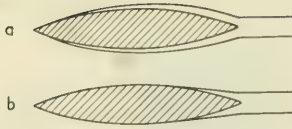


Fig. 14 - Mathematical models of viscous effects

waterlines and a linearly growing boundary layer with a constant-thickness wake. If however  $X$  is close to  $L$ , as in Fig. 14b, the boundary-layer thickness is assumed to be negligible until close to the stern, where a wedge-shaped region of separation occurs followed by a constant-thickness wake. It may be shown that the theoretical wave resistance corresponding to this source distribution is given by

$$C_W = \frac{R_W}{\frac{1}{2} \rho V^2 (AL)^2} = \frac{4}{\pi K_0^2 L^2} \int_0^{\pi/2} [1 - \exp(-K_0 D \sec^2 \theta)]^2 \cos^3 \theta G(\theta) d\theta,$$

where

$$K_0 = g/V^2,$$

$$G(\theta) = a - \frac{2 \cos \theta}{K_0 L} b - \epsilon a [1 - \cos (K_0 X \sec \theta)] \\ + \epsilon b \sin (K_0 X \sec \theta) + \epsilon^2 \{1 - \cos [K_0 (L - X) \sec \theta]\},$$

$$a = 1 + \cos (K_0 L \sec \theta) - \frac{2 \cos^2 \theta}{K_0 L} \sin (K_0 L \sec \theta),$$

and

$$b = \sin (K_0 L \sec \theta) - \frac{2 \cos^2 \theta}{K_0 L} [1 - \cos (K_0 L \sec \theta)].$$

Thus if  $X = 0$ , as in Fig. 14a, the  $\epsilon$  terms in  $G(\theta)$  vanish, and the wave resistance is simply a linear combination of the inviscid resistance with  $\epsilon = 0$  and that due to a uniform source distribution, as in Fig. 6. There is no interaction between the two components. To obtain the right kind of displacement thickness at the stern we need  $\epsilon$  in the region of  $0.003/A$ , and since this enters only as the square it is likely to make only a negligible contribution to the wave resistance. On the other hand if  $X$  is close to  $L$ , say  $0.9L$ ,  $\epsilon$  must be much bigger, say  $0.03/A$ , if an appreciable growth of displacement thickness within the separated region is to take place. Then, since  $\epsilon$  now enters in its first power, we see that there is likely to be a much greater effect on wave resistance. Moreover this latter effect is approximately independent of the small distance  $t (= L - X)$  from the stern at which separation takes place, provided  $\epsilon t$  is kept constant. Since  $\epsilon t$  defines the displacement thickness at the stern, or perhaps rather that part of it which is due to the rapid growth of displacement thickness near the stern, it could be determined experimentally fairly easily. Thus a simple method of correcting theoretical wavemaking calculations for viscous effects may be to locate concentrated sources of the proper strength at the stern. Alternatively

doublers might be used, since although the displacement thickness of the far wake will be nonzero, it will be considerably less there than it is at the stern.

Figure 15 provides concrete evidence in favour of the general argument advanced above that the rapid boundary layer thickening near the stern matters far more than the gradual thickening over the complete hull. However the effects are smaller than one would perhaps expect for the action of viscosity, so it may well be equally important to model the rapid contraction of the wake just downstream of the stern. This, of course, would be to return very nearly to Havelock's original proposals (10).

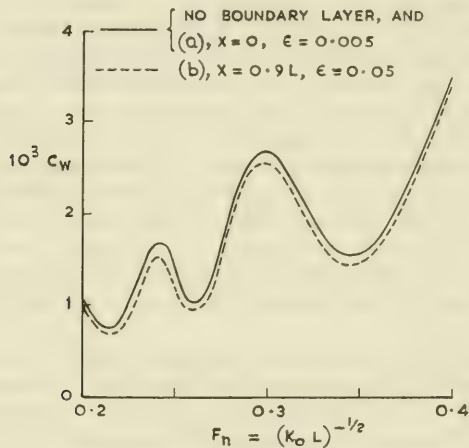


Fig. 15 - Computed effects of (a) and (b) in Fig. 14 on wave resistance

Whatever the complete explanation may be for the discrepancy between the two resistance curves in Fig. 12, the outcome of the present exercise has been that, by happy accident rather than skillful design, a hull form of lower resistance than Eq. (10) has been achieved. Thus, scaled to a 693-ft-long ship, of 34,300 tons displacement, at 20 knots, the results for the model in the bulb-aft condition at 5.66 ft/sec lead to an estimated horsepower of 13,800. Here a conservative quasi-propulsive coefficient of 70 percent has been taken rather than the measured value of 74 percent. Run bulb forwards, the horsepower would be 16,430. Equation (10) gives  $P_{best} = 15,520$  for 34,300 tons displacement and 20 knots. Thus it appears that with the vessel run the opposite way round to what was originally envisaged the target has been beaten by a small margin. The change of direction does not necessarily mean that the ideas behind the original design were entirely unsound, although the low-speed results do suggest that, at any rate for a ship of this slenderness, the viscous-drag advantages of having the maximum thickness forward are very small, so that to utilize them one would have to be far more clever than I have been in reducing wave resistance.



The other way in which viscous effects have been taken account of, the reduction of wetted area by making the sections as nearly semicircular as possible, has been shown to be justified for the present limited purposes of striving towards a purely hydrodynamic optimum. In view of the crude way in which wave resistance has been dealt with in the present design, the results encourage the belief that more sophisticated techniques could lead to substantial improvements over current practice.

## FINAL REMARKS

Part 3 of the paper has left unresolved the doubts raised by Part 2 as to the adequacy of the Inui approach to the relationship between hull form and wave-making. It is possible, but far from certain, that the large differences in residuary resistance of the asymmetrical model in the different directions of motion are due to viscous effects. If they are not, then theories, such as the Inui one, for which the relationship between hull shape and source distribution is independent of the direction of motion must be wrong. If they are, then potential flow theory, even if accurate in itself, must be an unreliable guide to experimental residuary resistance, especially for forms with full afterbodies. In either event, improvements in our present theories are called for. These might take the form of an improvement in the theory for potential flow, and possibly Guiloton's theory (2) already represents such an improvement. Alternatively it may be more important to take account of viscous effects.

Despite these uncertainties, there are grounds for hope that considerable reductions of resistance of ship forms are possible. It seems worthwhile to make further attempts to beat the target set in Part 1, without any practical application in mind, as a means of forging the theoretical tools which will ultimately, it is hoped, be useful for practical design, but which, as we have seen, are at present only rather blunt instruments. Moreover, if really radical improvements on current designs could be demonstrated, this might lead to some practical restrictions, normally insisted upon, being relaxed. Even if, in most practical cases, the form of least possible resistance may not be the shape that should be chosen for a ship, knowledge concerning this purely hydrodynamic optimum must be of assistance in making a correct technical assessment of ship design.

## ACKNOWLEDGMENT

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# AN ATTEMPTED APPLICATION OF WAVE ANALYSIS TECHNIQUES TO ACHIEVE BOW-WAVE REDUCTION

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## ABSTRACT

Two longitudinal-cut methods of analyzing the wave system of a ship model, originally proposed by the author in 1963, have been numerically and experimentally verified. It has been found that the free wave spectrum of a given model can be derived with fair accuracy from measurements of wave height or slope along a single track parallel to the direction of motion, provided suitable truncation corrections are applied to account for the asymptotic behaviour of the wave profile behind the model. A concrete example shows that such empirical information on the wave making characteristics of a main-hull form can be used as a basis for the theoretical design of an additive bow bulb which would achieve optimum wave cancellation and thus indirectly reduce ship resistance and improve performance.

## PART I

### INTRODUCTION.

This paper presents the first results of a research project initiated about a year ago with a view to devising a systematic method for the design and appropriate dimensioning of bulbous bows. It is generally recognized that the beneficial action of the bow bulb in reducing ship resistance and thereby improving performance results primarily from an attenuation of the bow wave system which is usually accompanied by a reduction of wave making resistance. Even in such cases, with fuller ships for example, where there is good reason to believe that the bulb tends to reduce the viscous rather than wave resistance by smoothening the flow around the forebody, the wave reduction can be used as a good practical criterion for bulb efficiency, since it is an empirical fact that optimum designs of least total resistance generally display the least surface disturbance in the vicinity of the bow. Therefore, it seems to be a sound working principle to try to match the bow wave system of the main hull against that of an additive bulb.



In theory, this matching can be accomplished by purely analytical means. Using the well-known concept of Havelock sources and representing the main-hull/bulb combination by distributed and concentrated singularities respectively, one can derive systems of extremely low, or even zero, theoretical wave resistance. Perhaps the most striking of such results have been obtained by Yim, whose paper (1) to the last ONR Symposium on Naval Hydrodynamics at Bergen, Norway, in 1964 may be consulted also for references to previous literature on the subject. However, the underlying linearized theory is known to be quite inadequate for practical purposes. It generally fails to predict with sufficient accuracy the actual wave-making characteristics of even simple ship forms.

Therefore, for the present work a semiempirical approach as advocated by Inui, for example in his 1962 paper (2), was adopted. In this method theory is only used to estimate the bulb wave system, while the wave-making characteristics of the main hull are determined essentially from model experiments. As rightly emphasized by Inui the concept of a free wave spectrum is extremely useful in matching the bulb wave pattern to that of the main hull, for this enables one to adjust the two systems by considering the individual amplitudes and phases of component plane waves rather than the complicated geometry of the integrated two-dimensional wave pattern. In actual application, however, Inui and his collaborators seem to have used the general configuration of the measured wave crests and troughs to define a fictitious point of origin ahead of the bow and have then taken radial wave profiles through this point for further evaluation, cf. Takahei (3). This method requires laborious stereophotogrammetric measurements of the entire wave system and, in effect, fails to take full advantage of the simple and elegant concept of the free wave spectrum.

For the present investigation it was decided to make a consistent use of the elementary wave concept by first analyzing the measured wave system of the main hull to obtain the spectrum and then adjusting the bulb to it. The basic step, therefore, was to find a convenient and practical method of obtaining the empirical spectrum from wave measurements capable of being taken, if possible, during a routine resistance test. Extensive testing previously conducted by the author (4-6) on a mathematical model, designated as the Inuid S-201, had established the validity of a transverse-cut method of deriving the free wave spectrum from Fourier transforms of two or more wave profiles measured along straight lines normal to the direction of motion of the model. However, this method appeared to be somewhat inconvenient for the present purpose, since transverse cuts can be measured only by moving wave probes across the towing carriage during the model run, unless the even more difficult stereophotogrammetric technique is used. Moreover, there is no unique way of breaking up the waves measured behind the model into contributions from the bow and the stern which may be desirable from the point of view of bulb design.

Hence it was decided to try out the two longitudinal-cut methods of wave analysis originally suggested by the author in 1963, see Ref. 4a. These require only the knowledge of wave height or transverse wave slope along a single straight track parallel to the direction of motion of the model. This information can be obtained rather easily by locating a stationary wave probe at some suitable point in the towing tank and taking time-dependent records while the model passes by, without any interference with the routine testing. Incidentally, this

method is also best suited to application on a full-size ship for investigating scale effects.

Although the longitudinal-cut methods had been proposed with a certain reservation regarding their feasibility, see Ref. 4b, the numerical and experimental experience gained so far and reported in this paper seems to confirm their validity and practicability.

## DEFINITION OF AN AMPLITUDE SPECTRUM

The surface disturbance produced by a moving ship cannot in general be described in terms of a one-dimensional wave spectrum. However, if one neglects certain local effects and considers the asymptotic behavior of the far field, it turns out that only one wavelength, the so-called free wave, predominates in each direction of propagation. The remaining terms decay so rapidly that the wave-making resistance in an ideal fluid is determined entirely by these free waves. For most practical purposes, therefore, the wave-making characteristics of a ship form may be considered as established if the free wave spectrum, that is, the pair of functions relating the amplitudes and phases of the component plane waves to their wave numbers, is known. Since each component wave has a different direction of propagation and wavenumber, the spectrum can be defined in various ways depending upon whether the total free wave system is expressed as an integral in the direction of propagation or in one of the possible wavenumbers. It will be found convenient for the present work to first define an unambiguous amplitude spectrum.

Consider a right-handed Cartesian coordinate system  $Oxyz$  moving steadily with the ship at speed  $V$  in the direction of  $Ox$ . Let  $Oxy$  be the undisturbed free surface,  $Ozx$  the center plane, and  $Oyz$  the midship section of the ship. Take  $Oz$  vertically upward, i.e., against the acceleration of gravity  $g$ . The steady surface deformation produced by the moving ship can then be expressed by the equation

$$z = \zeta(\underline{x}, \underline{y}), \quad (1)$$

where  $\zeta$  is an arbitrary function of  $(\underline{x}, \underline{y})$ .

From now on it will be advantageous to use only nondimensional quantities. This can be easily achieved by multiplying or dividing all lengths and wavenumbers by the quantity  $k_0 = g/V^2$ , which is a fundamental unit of the problem. The only relevant dimensional quantity other than a length or a wavenumber needed here is the wave resistance  $R_w$ , which will be rendered nondimensional by the relation

$$R_w = \underline{R}_w k_0^2 / \rho V^2, \quad (2)$$

where  $\rho$  denotes the density of water. (The term "wave resistance" will be used in this paper to designate the dynamic reaction of any given free wave system as predicted by linearized potential theory.) When dimensional quantities are used in the following pages for special reasons they will be identified by underlining.

The nondimensional free surface deformation due to the moving ship is now given by

$$z = \zeta(x, y). \quad (3)$$

Following Havelock (7) this function may be represented for large distances ( $x^2 + y^2$ ) by a general superposition of elementary plane waves

$$\zeta(x, y) = \int [f(\theta) \sin(sx + uy) + g(\theta) \cos(sx + uy)] d\theta, \quad (4)$$

$$s = \sec \theta, \quad u = \sec \theta \tan \theta,$$

where  $\theta$  is the direction of propagation,  $s$  and  $u$  are the kinematically linked longitudinal and transverse (circular) wavenumbers respectively,  $f(\theta)$  and  $g(\theta)$  are two arbitrary functions, and the limits of integration may depend on the location of the point  $(x, y)$ . In the relevant literature  $f(\theta)$  and  $g(\theta)$  are sometimes called the amplitude functions of the sine and cosine wave systems respectively. However, this terminology can be misleading, since  $f(\theta)$  and  $g(\theta)$  do not actually have the physical significance of wave amplitudes. Thus Takahei (3a) and Inui (2a) introduce a "wave slope" as the ratio of amplitude to wavelength in direction  $\theta$ :

$$f(\theta)/\lambda(\theta) = f(\theta)/2\pi \cos^2 \theta \quad (5)$$

and then question the validity of linearized wave theory because this ratio for a simple cosine ship (Inuid C-201) approaches infinity as  $\theta$  tends to  $\pi/2$ .

In the author's opinion such fallacious conclusions can be avoided by representing  $\zeta(x, y)$  as an integral in the transverse wavenumber  $u$ :

$$\zeta(x, y) = \int [F(u) \sin(sx + uy) + G(u) \cos(sx + uy)] du, \quad (6)$$

$$u = s \sqrt{s^2 - 1}, \quad s^2 = (\sqrt{1 + (2u)^2} + 1)/2.$$

The redefined amplitudes  $F(u)$  and  $G(u)$  contain an additional factor

$$d\theta/du = \cos^3 \theta / (1 + \sin^2 \theta)$$

as compared to  $f(\theta)$  and  $g(\theta)$ ; therefore the ratios

$$F(u)/2\pi \cos^2 \theta \quad \text{and} \quad G(u)/2\pi \cos^2 \theta$$

always remain finite, except when the linearized theory is genuinely overstrained (for example by locating a singularity on the free surface).

Even otherwise, the functions  $F(u)$  and  $G(u)$  come quite close to what may be considered as physical amplitudes except for a constant factor of proportionality. For instance, the average rate of flow of total energy across a transverse vertical plane for large negative  $x$  is given by



$$\pi \int_{-\infty}^{\infty} [F^2(u) + G^2(u)] du$$

and the wave resistance is given by

$$R_w = \frac{\pi}{2} \int_{-\infty}^{\infty} [F^2(u) + G^2(u)] (2 - \cos^2 \theta) du, \quad (7)$$

as was first shown by Havelock (7).

In previous work (4-6) the author has used, following Eggers (8), the convenient "deep tank representation," which is obtained immediately from Eq. (6) by replacing the integral with an infinite series. Assuming a transverse symmetric disturbance in a deep tank with straight vertical walls  $y = \pm b/2$  one obtains the following general expression for a linearized free wave system far abaft:

$$\zeta(x, y) = \sum_{\nu=0}^{\infty} \epsilon_{\nu} [\alpha_{\nu} \cos(s_{\nu}x) + \beta_{\nu} \sin(s_{\nu}x)] \cos(u_{\nu}y), \quad (8)$$

$$u_{\nu} = 2\pi\nu/b, \quad \epsilon_{\nu} = 1/2 \text{ for } \nu = 0 \text{ and } 1 \text{ elsewhere,}$$

or alternatively

$$\zeta(x, y) = \sum_{\nu=0}^{\infty} \epsilon_{\nu} [A_{\nu} \sin s_{\nu}(x - \xi_{\nu})] \cos(u_{\nu}y), \quad (9)$$

$$A_{\nu} = \sqrt{\alpha_{\nu}^2 + \beta_{\nu}^2}, \quad \tan(s_{\nu}\xi_{\nu}) = -\alpha_{\nu}/\beta_{\nu};$$

see, for instance, Ref. 6a. The coefficients  $A_{\nu}$  and  $\xi_{\nu}$  in Eq. (9) are, in fact, actual amplitudes and phases of elementary tank waves, each of which can be considered as a separate physical entity. For any given disturbance producing the free wave in a tank of width  $b$  the coefficients  $\alpha_{\nu}$  and  $\beta_{\nu}$  are simply related to the continuous functions  $F(u)$  and  $G(u)$  which constitute the free wave spectrum by the proportionality

$$\alpha_{\nu} = (4\pi/b) G(u), \quad \beta_{\nu} = (4\pi/b) F(u). \quad (10)$$

Assuming any sufficiently large hypothetical tank width  $b$ , Eqs. (9) and (10) can also be considered as a numerical approximation to the exact integral in Eq. (6) for laterally unrestricted fluid. The corresponding approximation for wave resistance is then

$$R_w = (b/8) \sum_{\nu=0}^{\infty} \epsilon_{\nu} (\alpha_{\nu}^2 + \beta_{\nu}^2) (2 - 1/s_{\nu}^2), \quad (11)$$



which is again "exact" for a tank of width  $b$ . The quantity  $A$  (in an actual tank) or  $Ab$  (if tank width  $b$  is only a numerical artifice) considered as a continuous function of the wavenumber  $u$  or  $s$  seems to be a quite reasonable definition of the "amplitude spectrum" and will, therefore, be used for its graphical representation in this paper, as has already been the practice in previous publications of the author.

## PRINCIPLE OF LONGITUDINAL-CUT ANALYSIS

For the sake of completeness the theory of longitudinal-cut analysis, already derived and discussed in Ref. 4, will be briefly recapitulated here. Consider the following Fourier transforms

$$C(s, y) + iS(s, y) = \int_{-\infty}^{\infty} \zeta(x, y) \exp(isx) dx, \quad 1 \leq s < \infty, \quad (12)$$

of any longitudinal cut  $y = \text{const.}$  through a transverse symmetric wave system  $z = \zeta(x, y)$  generated by a ship or model in laterally unrestricted deep water. Now it was shown in Ref. 4a, by exploiting the asymptotic behavior of  $\zeta(x, y)$  for large  $y$ , that the sine and cosine amplitude functions characteristic of the wave-generating agency, as defined in the foregoing, can be directly obtained from the above transforms by the simple relationship

$$b\alpha(u) = 4\pi G(u) = \frac{4\sqrt{s^2 - 1}}{2s^2 - 1} [C \sin(uy) - S \cos(uy)] \quad (13a)$$

$$b\beta(u) = 4\pi F(u) = \frac{4\sqrt{s^2 - 1}}{2s^2 - 1} [C \sin(uy) - S \cos(uy)], \quad (13b)$$

while the wave resistance can also be computed directly, without reference to the amplitude or phase spectrum, by the formula

$$R_w = \frac{1}{\pi} \int_0^{\infty} \frac{s^2 - 1}{s^2(2s^2 - 1)} (C^2 + S^2) du. \quad (14)$$

It may be noted in passing that if the wave pattern is transversally unsymmetric, one longitudinal cut on either side of the disturbance is necessary and sufficient for the determination of the entire spectrum in the range  $-\infty < u < \infty$  and hence for the computation of the original wave resistance integral in Eq. (7).

It was further pointed out in Ref. 4 that similar relations can be derived for the analogous Fourier transforms of wave slopes  $\zeta_x$  or  $\zeta_y$  measured along longitudinal cuts. Especially, a decided numerical advantage was claimed in favor of the transforms  $C_y$  and  $S_y$  of the transverse wave slope  $\zeta_y$  over those of wave height  $\zeta$  because of a more reasonable weighting function in the following relations corresponding to Eqs. (13) and (14):

$$b\alpha(u) = 4\pi G(u) = \frac{-4}{s(2s^2 - 1)} [C_y \sin(uy) + S_y \cos(uy)] , \quad (15a)$$

$$b\beta(u) = 4\pi F(u) = \frac{4}{s(2s^2 - 1)} [C_y \cos(uy) - S_y \sin(uy)] , \quad (15b)$$

and

$$R_w = \frac{1}{\pi} \int_0^\infty \frac{1}{s^4(2s^2 - 1)} (C_y^2 + S_y^2) du . \quad (16)$$

It is obvious from an inspection of the foregoing equations that in the most important and interesting region of the spectrum at  $u = 0$ , that is at  $s = 1$ , a numerical zero must be multiplied with a numerical infinity on the right-hand side of Eqs. (13) and (14) in order to obtain a nonzero value of the transverse wave amplitude  $A_0$ , while no such difficulty occurs with the Eqs. (15) or (16).

Finally, it should be mentioned that Newman (9a) independently derived an equivalent of Eq. (14) and presented it simultaneously with the author (4) to the International Seminar on Theoretical Wave Resistance at Ann Arbor, Michigan, in August 1963, while Pien and Moore (10) and Shor (11) proposed somewhat different methods of longitudinal-cut analysis on the same occasion.

## TRUNCATION CORRECTIONS

The numerical difficulty just mentioned finds another expression in the fact that under certain circumstances the Fourier transforms  $C$ ,  $S$  and  $C_y$ ,  $S_y$ , etc., do not converge at all. In fact the theoretical free wave spectrum of any ship form is such that in general the function  $\zeta(x, y)$  is not absolutely integrable at the afterend  $-x \rightarrow \infty$ , and  $C^2 + S^2$  becomes infinite at the point  $s = 1$ .

Similarly, for a surface disturbance (as distinguished from a submerged disturbance of arbitrarily small but nonzero depth of submergence) also the function  $\zeta_y(x, y)$  is not absolutely integrable and produces an infinity of  $C_y^2 + S_y^2$  at the other end of the spectrum, namely, as  $s \rightarrow \infty$ . However, this problem can be circumvented by restricting the computation to a finite range of wave number  $s$  without prejudice to the practical aim of wave analysis, since this end of the spectrum contributes but little to wave resistance. In one typical case, for instance, it was estimated that truncating the spectrum at  $s = 3$  would cause an error of less than 0.5 percent in wave resistance.

Fortunately, the aforesaid convergence and the associated truncation problem (arising from the practical restriction of the  $x$  integration to a finite range) in the analysis of height records can be approximately solved by a simple trick. Since the amplitudes are in any case obtained by multiplying the Fourier transforms  $C$  and  $S$  with the factor  $4\sqrt{s^2 - 1}/(2s^2 - 1)$ , which becomes zero at  $s = 1$ , one might as well start right away with the following weighted Fourier transforms:

$$C^*(s, y) + iS^*(s, y) = \int_{-\infty}^{\infty} \sqrt{s^2 - 1} \, \zeta(x, y) \exp(isx) \, dx. \quad (17)$$

Now, if the asymptotic behavior of wave height

$$-x \rightarrow \infty: \quad \zeta(x, y) \approx (c_1 \cos x + c_2 \sin x) / \sqrt{c_3 - x} \quad (18)$$

is postulated in analogy to the result obtained by the usual stationary phase analysis of any theoretical wave system, then the function  $\zeta(x, y)$  can indeed be extrapolated to  $-x \rightarrow \infty$ , once the constants  $c_1$ ,  $c_2$ , and  $c_3$  have been determined by, say, a least-squares fit to the known function  $\zeta$  in some finite region  $x_1 \leq x \leq x_2$ , where  $-x_1, -x_2 \gg 1$ .

The contribution of the integration  $-\infty < x \leq x_e$ , where  $x_e$  is some large negative value of  $x$ , can then be derived by closed evaluation of the following integrals:

$$\Delta C^* + i\Delta S^* = \int_{-\infty}^{x_e} \sqrt{s^2 - 1} \frac{c_1 \cos x + c_2 \sin x}{\sqrt{c_3 - x}} \exp(isx) \, dx. \quad (19)$$

After some rewriting and simplification one obtains the following results:

$$\Delta C^* = \sqrt{\frac{\pi}{2}} [d_1 C_F(z^+) + d_2 S_F(z^+) + d_3 C_F(z^-) + d_4 S_F(z^-)] \quad (20a)$$

$$\Delta S^* = \sqrt{\frac{\pi}{2}} [-d_1 S_F(z^+) + d_2 C_F(z^+) - d_3 S_F(z^-) + d_4 C_F(z^-)] \quad (20b)$$

with the abbreviations

$$d_1 = \sqrt{s-1} [c_1 \cos c_3(s+1) + c_2 \sin c_3(s+1)], \quad (21a)$$

$$d_2 = \sqrt{s-1} [c_1 \sin c_3(s+1) - c_2 \cos c_3(s+1)], \quad (21b)$$

$$d_3 = \sqrt{s+1} [c_1 \cos c_3(s-1) - c_2 \sin c_3(s-1)], \quad (21c)$$

$$d_4 = \sqrt{s+1} [c_1 \sin c_3(s-1) + c_2 \cos c_3(s-1)]. \quad (21d)$$

Herein  $C_F$  and  $S_F$  denote the easily computable Fresnel integrals

$$C_F(z) + iS_F(z) = \int_z^{\infty} \exp\left(i\frac{\pi}{2} t^2\right) dt, \quad (22)$$

e.g., see Ref. 12, and the argument  $z$  takes the values

$$z^{\pm} = \sqrt{2(c_3 - x_e)(s \pm 1)/\pi} . \quad (23)$$

The further evaluation of the weighted Fourier transforms  $C^*$  and  $S^*$ , thus corrected for any truncation errors at the after end  $-x \rightarrow \infty$ , proceeds as before but with modified weighting factors. Thus

$$b\alpha(u) = 4\pi G(u) = \frac{4}{2s^2 - 1} [C^* \cos(uy) - S^* \sin(uy)] , \quad (24a)$$

$$b\beta(u) = 4\pi F(u) = \frac{4}{2s^2 - 1} [C^* \sin(uy) + S^* \cos(uy)] , \quad (24b)$$

and

$$R_w = \frac{1}{\pi} \int_0^{\omega} [(C^*)^2 + (S^*)^2] \frac{du}{s^2(2s^2 - 1)} . \quad (25)$$

It is apparent that the numerical ambiguity at  $s = 1$  has been eliminated. It may be noted in passing that the function  $\zeta(x, y)$  decays so rapidly for  $x \rightarrow \infty$  that no truncation corrections are in practice needed at this end.

For the sake of record it may be recalled that Newman (9) also derived a certain truncation correction for wave resistance in a different way but starting with an assumed asymptotic behavior of wave height similar to Eq. (18).

## NUMERICAL CHECKS ON THEORETICAL WAVE SYSTEMS

Before trying to apply the longitudinal-cut methods in practice it appeared advisable to check them first on a few theoretical wave patterns of predetermined spectra. As a simple example, typical of a bulb wave system, the linearized wave pattern of a submerged sphere of unit radius  $r = 1$ , located at  $x = y = 0$ ,  $z = -f$  was investigated. This is represented to a good order of approximation by the functions

$$\zeta(x, y) = 2 \int_{-\infty}^{-x/y} s^2 \exp(-fs^2) \sin(sx + uy) dt , \quad (26a)$$

$$\zeta_y(x, y) = 2 \int_{-\infty}^{-x/y} us^2 \exp(-fs^2) \cos(sx + uy) dt , \quad (26b)$$

$$s = \sqrt{t^2 + 1} , \quad u = t \sqrt{t^2 + 1} , \quad y > 0 .$$

which are adapted from Havelock (13) after neglecting certain local terms involving double integrals (which had to be dropped in view of limited computer facilities at the disposal of the author).



The above integrals were numerically evaluated at 241 points, in the range  $x = -20\pi(\pi/10)4\pi$ , for typical parameter values  $f = 1$ ,  $y = \pi$  by the Romberg method specifying an absolute accuracy of  $\pm 0.0005$ . The results are listed in Tables 1 and 2. It may be noted that at the afterend of the cut the slope oscillations have decayed down to about 1 percent of their peak values (which is a realistic simulation of experimental accuracy), while the corresponding figure for height is much higher (about 25 percent) as may also be expected.

Table 1  
List of Computed  $\zeta_y (\times 10^4)$  Values for a Submerged Sphere  
(Parameters:  $r = 1$ ,  $f = 1$ ,  $y = \pi$ ; Variable,  $x = 4\pi$  stepped  $-\pi/10$  until  $-20\pi$ )

|       |       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | 0     | 0     | 0     | -1    | -1    | -2    |
| -4    | -6    | -9    | -14   | -22   | -32   | -47   | -64   | -90   | -120  |
| -156  | -197  | -242  | -287  | -332  | -370  | -397  | -408  | -399  | -368  |
| -313  | -235  | -138  | -25   | 95    | 217   | 335   | 443   | 545   | 635   |
| 715   | 792   | 870   | 949   | 1037  | 1119  | 1187  | 1220  | 1190  | 1064  |
| 799   | 377   | -220  | -969  | -1823 | -2676 | -3392 | -3805 | -3750 | -3098 |
| -1804 | 58    | 2264  | 4457  | 6178  | 6966  | 6453  | 4517  | 1347  | -2523 |
| -6278 | -8996 | -9891 | -8558 | -5156 | -438  | 4409  | 8085  | 9586  | 8545  |
| 5379  | 1165  | -2724 | -5121 | -5486 | -4067 | -1745 | 380   | 1479  | 1325  |
| 314   | -811  | -1375 | -1100 | -196  | 830   | 1488  | 1563  | 1175  | 639   |
| 233   | 55    | 15    | -42   | -203  | -442  | -650  | -742  | -701  | -560  |
| -406  | -213  | -76   | 93    | 258   | 401   | 496   | 536   | 516   | 436   |
| 344   | 216   | 69    | -79   | -215  | -339  | -399  | -432  | -407  | -371  |
| -289  | -196  | 51    | 65    | 194   | 277   | 331   | 345   | 354   | 314   |
| 260   | 153   | 50    | -54   | -149  | -224  | -282  | -264  | -304  | -269  |
| -209  | -132  | -44   | 45    | 129   | 198   | 245   | 263   | 263   | 234   |
| 183   | 116   | 39    | -40   | -113  | -173  | -215  | -235  | -232  | -206  |
| -162  | -102  | -34   | 35    | 100   | 1088  | 191   | 208   | 206   | 184   |
| 145   | 92    | 31    | -31   | -89   | -136  | -171  | -187  | -189  | -165  |
| -131  | -82   | -27   | 29    | 80    | 124   | 152   | 169   | 173   | 151   |
| 117   | 71    | 22    | -26   | -73   | -112  | -139  | -153  | -153  | -136  |
| -107  | -63   | -23   | 23    | 65    | 102   | 128   | 147   | 111   | 123   |
| 92    | 62    | 22    | -21   | -58   | -94   | -118  | -129  | -128  | -114  |
| -91   | -58   | -19   | 20    | 56    | 87    | 108   | 117   | 119   | 106   |
| 81    |       |       |       |       |       |       |       |       |       |

The slope data of Table 1 were analyzed to yield the amplitude spectrum according to Eq. (15), truncating once at  $-x = 10\pi$  and then at  $-x = 20\pi$  without any correction for the residual terms. The resulting amplitudes  $bA(u)$  are plotted vs transverse wavenumber  $u$  as curves 2 and 3 in Fig. 1 for the sake of comparison with curve 1, the exact theoretical curve, which is of course known a priori:

$$b\alpha(u) = 0 \quad (27a)$$

and

$$b\beta(u) = bA(u) = 8\pi s^3 \exp(-fs^2)/(2s^2 - 1). \quad (27b)$$

Table 2

List of Computed  $\zeta$  ( $\times 10^4$ ) Values for a Submerged Sphere  
(Parameters:  $r = 1$ ,  $f = 1$ ,  $y = \pi$ ; Variable,  $x = 4\pi$  stepped  
 $-\pi/10$  until  $-20\pi$ )

|       |       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| 0     | -1    | -2    | -3    | -6    | -9    | -14   | -22   | -34   | -50   |
| -73   | -103  | -147  | -198  | -266  | -348  | -448  | -565  | -699  | -848  |
| -1010 | -1180 | -1355 | -1529 | -1700 | -1861 | -2013 | -2152 | -2278 | -2393 |
| -2501 | -2602 | -2700 | -2794 | -2885 | -2963 | -3016 | -3028 | -2974 | -2824 |
| -2549 | -2118 | -1516 | -736  | 197   | 1230  | 2274  | 3205  | 3881  | 4147  |
| 3874  | 2981  | 1479  | -527  | -2808 | -5036 | -6821 | -7781 | -7618 | -6204 |
| -3631 | -243  | 3422  | 6706  | 8983  | 9789  | 8962  | 6670  | 3386  | -227  |
| -3482 | -5842 | -7031 | -7073 | -6219 | -4837 | -3256 | -1727 | -300  | 1027  |
| 2279  | 3420  | 4339  | 4878  | 4904  | 4343  | 3340  | 1951  | 417   | -1085 |
| -2407 | -3446 | -4126 | -4425 | -4255 | -3690 | -2763 | -1573 | -247  | 1076  |
| 2267  | 3209  | 3819  | 4042  | 3864  | 3308  | 2435  | 1334  | 119   | -1090 |
| -2173 | -3025 | -3566 | -3748 | -3560 | -3017 | -2199 | -1167 | -32   | 1076  |
| 2090  | 2870  | 3359  | 3513  | 3335  | 2804  | 2017  | 1039  | -29   | -1081 |
| -2013 | -2739 | -3188 | -3320 | -3125 | -2635 | -1868 | -939  | 72    | 1065  |
| 1944  | 2625  | 3043  | 3157  | 2960  | 2473  | 1749  | 860   | -104  | -1061 |
| -1882 | -2525 | -2915 | -3015 | -2817 | -2347 | -1648 | -796  | 128   | 1030  |
| 1826  | 2436  | 2803  | 2890  | 2692  | 2236  | 1564  | 742   | -146  | -1010 |
| -1774 | -2356 | -2702 | -2781 | -2587 | -2141 | -1489 | -696  | 161   | 994   |
| 1726  | 2282  | 2612  | 2683  | 2491  | 2057  | 1422  | 656   | -172  | -951  |
| -1677 | -2216 | -2531 | -2595 | -2406 | -1981 | -1366 | -621  | 180   | 959   |
| 1639  | 2155  | 2456  | 2511  | 2328  | 1914  | 1315  | 591   | -187  | -940  |
| -1616 | -2090 | -2389 | -2443 | -2269 | -1853 | -1269 | -564  | 193   | 926   |
| 1564  | 2047  | 2326  | 2376  | 2194  | 1797  | 1227  | 540   | -197  | -911  |
| -1531 |       |       |       |       |       |       |       |       |       |

Similarly, the height data of Table 2 were analyzed according to Eqs. (13) and (24) truncating in either case at  $-x = 20\pi$ . The results are indicated by the two sets of discrete points in Fig. 1, the open dots being without truncation correction, Eq. (13), and the solid dots being with truncation correction, Eq. (24).

The wave resistance was computed from each of the five amplitude functions shown in Fig. 1 by the approximate formula (compare with Eq. (11))

$$R_w = \frac{b}{8} \sum_{\nu=0}^{60} \epsilon_{\nu} A_{\nu}^2 (2 - 1/s_{\nu}^2), \quad (28)$$

assuming a sufficiently large hypothetical tank width  $b = 30^-$ . The following results were obtained. The exact theoretical value from curve 1 was 1.528. The slope analysis, curves 2 and 3, yielded 1.400 and 1.493 respectively, while the height analysis resulted in 1.511 and 1.407 with and with truncation correction.

As a next step the wave pattern generated by a pair of submerged spheres was examined to investigate interference effects. For this purpose a second

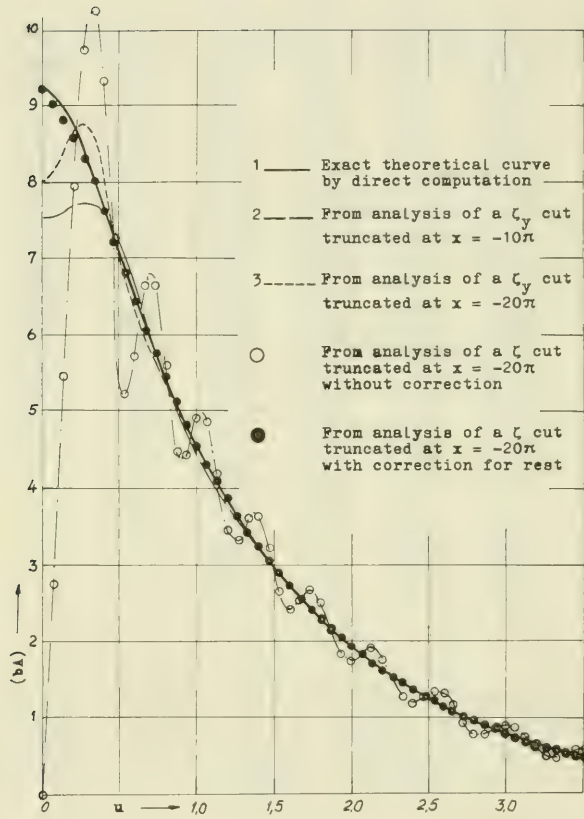


Fig. 1 - Amplitude spectrum of an isolated submerged sphere of unit radius

sphere of unit radius  $r = 1$  and submergence  $f = 1$  was located at  $x = -3\pi$ ,  $y = 0$  and the total wave system obtained by superposing the data listed in Table 1 or 2 on itself with a phase shift of 30 steps. This corresponds to a Froude number  $F_n = \sqrt{1/L}$  of about 0.326, since the nondimensional separation  $L$  of the two spheres is here  $3\pi$  taken center to center.

The result of this analysis is depicted in Fig. 2 where, in order to avoid overcrowding, only two sets of points are indicated in addition to the continuous theoretical curve, namely, the solid dots from slope analysis and the open dots from height analysis (with truncation correction), each cut running through 240 steps from  $x = 4\pi$  to  $-20\pi$ . The corresponding figures for wave resistance are 2.018 (theory), 1.994 (slope analysis), and 1.892 (height analysis).

Finally, in order to test the method on a typical ship-wave pattern generated by a continuous distribution of sources and sinks the theoretical wave system of the Inuid S-201 was selected. This model has been described frequently in previous work (see Ref. 6b for instance); therefore details will be omitted.

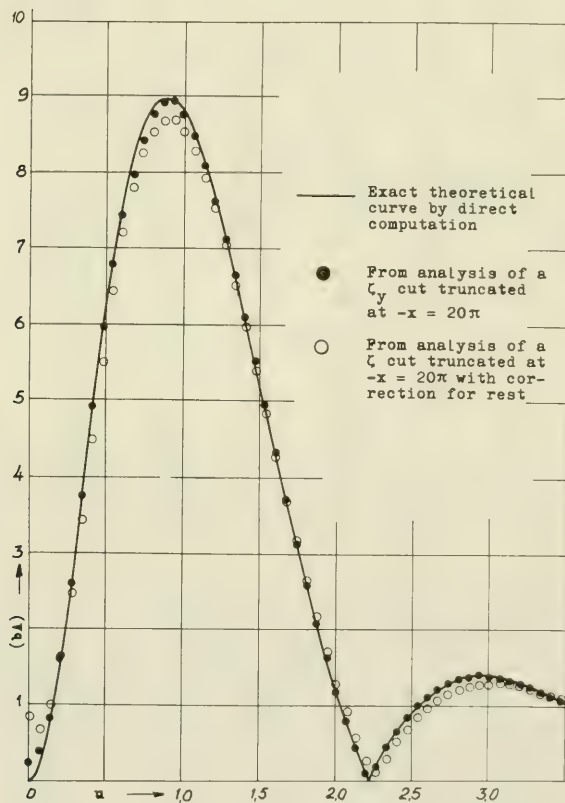


Fig. 2 - Amplitude spectrum of a pair of submerged spheres  $F_n = 0.326$



The test cut was a height record for a nondimensional model length  $L = 20$  (corresponds to  $F_n = 0.2236$ ) at transverse location  $y = 8.01$  in a tank of width  $b = 90$ , taken in 250 equal steps from  $x = -100$  to  $+25$ . The theoretical heights were computed by Dr. Klaus Eggers of the Institut für Schiffbau der Universität Hamburg by the usual linearized theory including all local effects and with a specified absolute error limit of  $\pm 0.0005$ . The data as used by the author are reproduced in Table 3.

Table 3  
List of Computed  $\zeta$  ( $\times 10^4$ ) Values for Inuid S-201  
(Parameters:  $\gamma_0 = 10$ ,  $b = 90$ ,  $y = 8.01$ ; Variable,  
 $x = 25$  stepped  $-0.5$  until  $-100$ )

|       |       |      |      |      |      |      |      |       |       |
|-------|-------|------|------|------|------|------|------|-------|-------|
| 0     | 0     | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     |
| 0     | 0     | 0    | 0    | 12   | 13   | 13   | 14   | 15    | 16    |
| 17    | 18    | 19   | 30   | 31   | 32   | 33   | 33   | 33    | 32    |
| 43    | 25    | 13   | 15   | 11   | 2    | -14  | -23  | -38   | -57   |
| -67   | -85   | -96  | -99  | -106 | -98  | -87  | -75  | -52   | -38   |
| -28   | -20   | -35  | -61  | -100 | -154 | -195 | -222 | -218  | -162  |
| -70   | 60    | 196  | 288  | 311  | 223  | 22   | -243 | -516  | -653  |
| -579  | -300  | 201  | 744  | 1131 | 1179 | 769  | 3    | -871  | -1496 |
| -1549 | -962  | 59   | 1099 | 1662 | 1506 | 741  | -249 | -925  | -1043 |
| -711  | -256  | -24  | -42  | -55  | 160  | 595  | 906  | 726   | 89    |
| -603  | -865  | -565 | -89  | 148  | 88   | 12   | 183  | 400   | 277   |
| -177  | -524  | -326 | 271  | 648  | 528  | 162  | -143 | -424  | -873  |
| -1173 | -729  | 379  | 1361 | 1556 | 1066 | 368  | -445 | -1264 | -1499 |
| -867  | 19    | 363  | 242  | 319  | 611  | 720  | 713  | 590   | -23   |
| -963  | -1344 | -827 | -224 | -38  | 160  | 442  | 609  | 882   | 992   |
| 359   | -517  | -739 | -564 | -649 | -711 | -438 | -38  | 577   | 1027  |
| 700   | 392   | 519  | 171  | -490 | -801 | -858 | -593 | -299  | -246  |
| 230   | 862   | 808  | 553  | 493  | 249  | -229 | -692 | -783  | -583  |
| -564  | -368  | 266  | 603  | 581  | 718  | 705  | 261  | -177  | -370  |
| -599  | -832  | -640 | -244 | -66  | 282  | 733  | 692  | 562   | 530   |
| 75    | -359  | -457 | -706 | -753 | -394 | -179 | 119  | 560   | 646   |
| 639   | 592   | 216  | -119 | -363 | -661 | -686 | -536 | -354  | 23    |
| 355   | 545   | 693  | 606  | 363  | 95   | -265 | -529 | -634  | -640  |
| -441  | -141  | 158  | 465  | 626  | 625  | 506  | 224  | -95   | -375  |
| -595  | -637  | -529 | -310 | 15   | 312  | 531  | 639  | 562   | 363   |
| 80    |       |      |      |      |      |      |      |       |       |

The amplitude spectra obtained from an analysis of this theoretical wave profile for two different lengths of run (terminating at  $-x = 50.5$  and  $100.0$ ) each time with truncation correction are shown in Fig. 3 (open and solid dots respectively), where the continuous curve represents the exact theoretical value taken from Fig. 8 in Ref. 4. In the case of the longer run the amplitudes obtained without truncation correction are also displayed as the third set of points (crosses) in Fig. 3. The corresponding values of wave resistance are as follows: theory, 0.1165; long run, 0.1109 with and 0.0980 without truncation correction; and short run, 0.0964 with and 0.0800 without truncation correction.

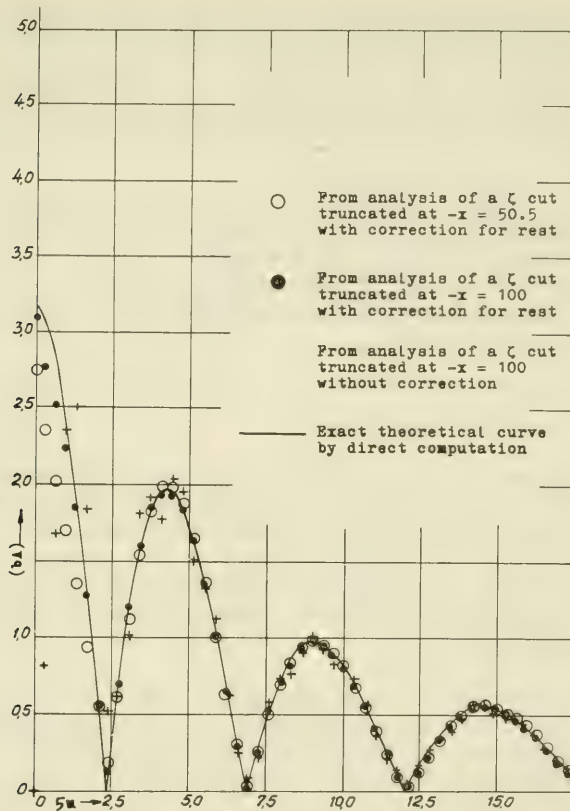


Fig. 3 - Amplitude spectrum of  
Inuid S-201 for  $\gamma_0 = 10$

The principal results of the three foregoing numerical tests on theoretical wave systems can be summarized as follows. Both, the wave height and the transverse slope, methods of longitudinal-cut analysis are essentially correct and feasible. The accuracy of analysis is practically limited by two factors: the finite transverse location  $y$  of the cut and the finite length of run in the negative  $x$  direction. The truncation errors are most prominent at the lower end of the spectrum (about  $u = 0$ , that is,  $s = 1$ ) and are relatively less serious in slope analysis than in height analysis (see Figs. 1 and 2). The latter, conducted without truncation correction for any finite length of run however large, will always yield zero amplitudes at the point  $u = 0$  and will generally lead to large errors of amplitude and phase in a significant portion of the spectrum (see Figs. 1 and 3), which may, however, sometimes cancel out in the integration and yield a surprisingly reasonable value of wave resistance (see Fig. 1). The truncation correction for height analysis as applied by the author seems to be most effective in the case of an isolated point disturbance and yields strikingly accurate results (Fig. 1), while the improvement achieved in the case of a pair of point

disturbances (Fig. 2) and of a distributed singularity (Fig. 3) is not quite as good. In any case the truncation correction itself becomes more and more effective with increasing length of run (see Fig. 3). The residual errors are relatively larger at hump speeds (Fig. 3) than at hollow speeds (Fig. 2) as may be expected in view of the different relative importance of the lower end of the spectrum.

On the whole, it appears that the analysis of a longitudinal cut located at a realistic transverse distance (say one fundamental wavelength,  $y = 2\pi$ ) and continued to a realistic extent behind the disturbance (say  $-x = 40\pi$ ) in realistic steps (say  $\Delta x = \pi/10$ ) would yield, after application of suitable truncation corrections, the free wave spectrum with fair accuracy and the wave resistance with an error of less than 5 percent.

## EXPERIMENTAL VERIFICATION

A further demonstration of the feasibility of the longitudinal-cut method of wave analysis was obtained from a special test run on the mathematical model Inuid S-201. This same model had been previously subjected to a series of extensive tests, including wave analysis, based on the transverse-cut method; hence sufficient reliable data for crucial comparisons were available. References 4, 5, and 6 may be consulted for a description of the model and the details of past experiments and analysis.

For the present purpose a new resistance-wire-type wave probe which simultaneously records wave height and partial slope in any one desired orientation was designed and constructed by Mr. Hans Luft of the HSVA (Hamburgische Schiffbau-Versuchsanstalt, Hamburg, Germany) with the expert advice of Professor L. W. Ward of the Webb Institute of Naval Architecture, Long Island, N. Y.

The test runs on Inuid S-201 were conducted in the HSVA on May 9, 1966, in collaboration with Dr. Klaus Eggers of the Institut für Schiffbau der Universität Hamburg and with the financial support of the Deutsche Forschungsgemeinschaft. The wave probe was located at a fixed point in the tank at a transverse distance of 1655 mm from the model center plane, and time records of wave height  $\zeta$  and transverse slope  $\zeta_y$  were taken as the model passed by at six different speeds between 1.4 and 2.0 m/sec. The comparative geometry of the longitudinal cuts thus obtained and the transverse cuts measured in August 1962 with the same model in the same tank but using a sonic wave probe is shown in Fig. 4. It may be noted that actual tank dimensions allowed recording of very much longer runs, but these were deliberately restricted to keep them well outside of the wave system reflected from the tank walls, which is an extraneous effect not accounted for in the present methods of longitudinal-cut analysis.

Owing to an unexpected ambiguity in the calibration of the slope signal it has not yet been possible to evaluate the  $\zeta_y$  records, but four height records have already been analyzed. The relevant parameters of these runs are listed in Table 4 and the height data as used for analysis are recorded in Tables 5 to 8. It should be noted that although the data are reproduced to four digits, the accuracy of measurement is not claimed to be any better than about 2 percent of



the peak values. Table 4 also shows relevant information on comparable transverse cuts of the older test series and the final result (wave resistance) obtained in each case.

Table 4  
Comparison of Parameters and Results of Transverse and Longitudinal-Cut Analysis Conducted on a 4-m Model of the Mathematical Form Inuid S-201

| Parameter                              | Case 1  | Case 2  | Case 3  | Case 4  |
|----------------------------------------|---------|---------|---------|---------|
| $v$ (m/sec)                            | 1.40    | 1.60    | 1.80    | 2.00    |
| $k_0$ (1/m)                            | 5.00    | 3.85    | 3.04    | 2.45    |
| $F_n$                                  | 0.224   | 0.255   | 0.287   | 0.319   |
| $R_w$ (Theoretical)                    | 0.1185  | 0.0824  | 0.1120  | 0.0674  |
| <u>Transverse Cuts</u>                 |         |         |         |         |
| $\gamma_0$                             | 9.993   | 7.710   | 6.087   | 4.923   |
| $-x_1$                                 | 29.39   | 22.06   | 16.69   | 13.00   |
| $-x_2$                                 | 30.39   | 22.99   | 17.60   | 13.87   |
| $-x_3$                                 | 31.39   | 23.91   | 18.51   | 14.73   |
| $-x_4$                                 | 32.39   | 24.84   | 19.43   | 15.59   |
| $-x_5$                                 | 33.39   | 25.76   | 20.34   | 16.45   |
| $R_w$ (Least squares fit)              | 0.0282  | 0.0385  | 0.0282  | 0.0396  |
| <u>Longitudinal Cut</u>                |         |         |         |         |
| $\gamma_0$                             | 10.01   | 7.690   | 6.067   | 4.903   |
| $y$                                    | 8.28    | 6.36    | 5.02    | 4.06    |
| $x_{\text{fore-end}}$                  | 10.54   | 8.82    | 9.10    | 7.55    |
| $-x_{\text{aft-end}}$                  | 86.46   | 69.47   | 64.70   | 73.68   |
| $-y/x_{\text{aft-end}}$                | 0.0958  | 0.0916  | 0.0776  | 0.0551  |
| $\zeta$ List                           | Table 5 | Table 6 | Table 7 | Table 8 |
| $R_w$ (Uncorrected)                    | 0.0287  | 0.0383  | 0.0287  | 0.0370  |
| Ratio to $R_w$ (trans.)                | 1.02    | 0.99    | 1.02    | 0.93    |
| $R_w$ (Corrected for truncation error) | 0.0296  | 0.0386  | 0.0312  | 0.0376  |
| Ratio to $R_w$ (trans.)                | 1.05    | 1.00    | 1.11    | 0.95    |
| Plot of amplitude vs wavenumber        | Fig. 5  | Fig. 6  | Fig. 7  | Fig. 8  |



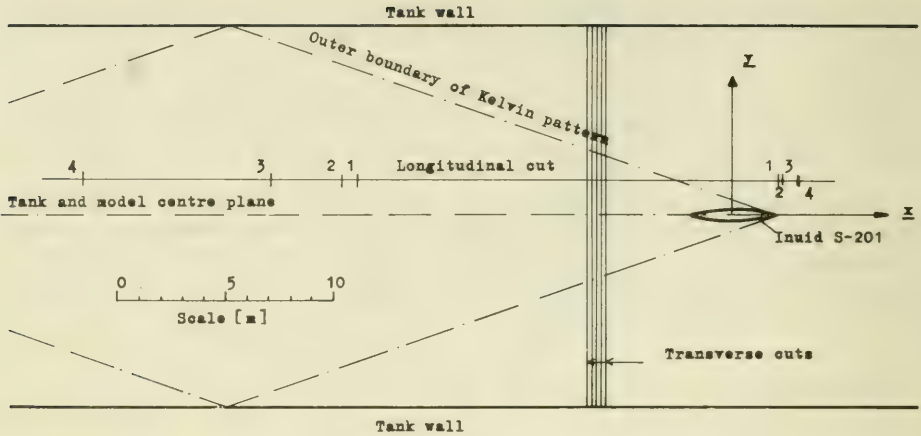


Fig. 4 - Comparative geometry of transverse and longitudinal cuts actually analyzed. The numbers 1, 2, 3, 4, on the longitudinal cut, mark the range of data listed in Tables 5, 6, 7, and 8.

Table 5  
List of Measured  $\zeta$  ( $\times 10^4$ ) Values for Inuid S-201  
(Parameters:  $\gamma_0 = 10.01$ ,  $b = 90.09$ ,  $y = 8.28$ ;  
Variable,  $x = 10.54$  stepped  $-0.3502$  until  $-86.46$ )

|      |      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|------|
| 35   | 32   | 32   | 26   | 26   | 26   | 23   | 13   | -3   | -16  |
| -32  | -51  | -61  | -58  | -64  | -68  | -77  | -84  | -80  | -93  |
| -93  | -93  | -93  | -90  | -80  | -77  | -64  | -64  | -51  | -32  |
| -16  | -16  | -39  | -48  | -71  | -96  | -138 | -164 | -196 | -225 |
| -209 | -186 | -113 | -61  | 32   | 116  | 257  | 347  | 383  | 325  |
| 151  | -74  | -363 | -608 | -775 | -849 | -743 | -492 | -39  | 482  |
| 1238 | 1640 | 1280 | 617  | -80  | -547 | -813 | -878 | -842 | -707 |
| -453 | -209 | 77   | 315  | 569  | 730  | 714  | 579  | 408  | 225  |
| -16  | -209 | -322 | -395 | -428 | -434 | -389 | -325 | -203 | -42  |
| 183  | 450  | 579  | 656  | 540  | 354  | 158  | -16  | -225 | -354 |
| -428 | -460 | -444 | -350 | -222 | -71  | 135  | 286  | 434  | 498  |
| 453  | 376  | 190  | 58   | -145 | -267 | -350 | -370 | -334 | -286 |
| -161 | 3    | 154  | 241  | 276  | 228  | 161  | 87   | 26   | -29  |
| -48  | -32  | 0    | 26   | 32   | 35   | -23  | -106 | -177 | -225 |
| -209 | -132 | -32  | 96   | 203  | 241  | 235  | 186  | 129  | 55   |
| 23   | -6   | -29  | -90  | -145 | -193 | -238 | -209 | -129 | -39  |
| 71   | 145  | 190  | 199  | 190  | 203  | 174  | 151  | 71   | -16  |
| -125 | -193 | -225 | -209 | -170 | -116 | -77  | -32  | 16   | 129  |
| 177  | 238  | 238  | 209  | 141  | 61   | 3    | -77  | -129 | -170 |
| -203 | -199 | -177 | -113 | -55  | 16   | 71   | 141  | 186  | 244  |
| 238  | 170  | 122  | 61   | 23   | -35  | -100 | -158 | -199 | -186 |
| -161 | -103 | -42  | 10   | 80   | 138  | 177  | 174  | 158  | 125  |
| 90   | 29   | -51  | -106 | -158 | -183 | -180 | -170 | -135 | -80  |
| 0    | 55   | 122  | 167  | 206  | 219  | 199  | 161  | 113  | 42   |
| -32  | -87  | -138 | -158 | -170 | -167 | -138 | -87  | -26  | 55   |
| 113  | 141  | 167  | -148 | 151  | 113  | 58   | -6   | -68  | -125 |
| -167 | -190 | -170 | -145 | -109 | -48  | 16   | 80   | 125  | 183  |
| 193  | 193  | 170  | 119  | 61   | -3   | -61  | -109 |      |      |

Table 6  
List of Measured  $\zeta$  ( $\times 10^4$ ) Values for Inuid S-201  
(Parameters:  $\gamma_0 = 7.690$ ,  $b = 69.21$ ,  $y = 6.36$ ; Variable,  $x = 8.82$  stepped  $-0.3070$  until  $-69.47$ )

|      |      |      |      |      |       |      |       |       |      |
|------|------|------|------|------|-------|------|-------|-------|------|
| 7    | 0    | 7    | 11   | 18   | 14    | 25   | 25    | 25    | 28   |
| 28   | 32   | 39   | 46   | 49   | 35    | 39   | 35    | 42    | 42   |
| 39   | 53   | 53   | 56   | 53   | 46    | 39   | 39    | 28    | 25   |
| 11   | -4   | -11  | -21  | -39  | -46   | -64  | -81   | -106  | -113 |
| -120 | -141 | -152 | -155 | -152 | -152  | -120 | -99   | -74   | -35  |
| 14   | 53   | 88   | 109  | 124  | 113   | 95   | 35    | -53   | -169 |
| -286 | -395 | -491 | -533 | -529 | -438  | -296 | -81   | 201   | 565  |
| 949  | 1331 | 1461 | 939  | 169  | -529  | -896 | -1087 | -1112 | -967 |
| -720 | -332 | 35   | 395  | 664  | 868   | 935  | 879   | 692   | 459  |
| 226  | -25  | -212 | -360 | -455 | -498  | -498 | -462  | -395  | -304 |
| -191 | -14  | 162  | 356  | 544  | 692   | 741  | 702   | 558   | 318  |
| 78   | -176 | -371 | -529 | -632 | -1380 | -653 | -540  | -385  | -159 |
| 131  | 448  | 724  | 907  | 960  | 836   | 572  | 208   | -116  | -406 |
| -604 | -709 | -702 | -604 | -427 | -180  | 60   | 293   | 434   | 448  |
| 378  | 272  | 187  | 127  | 131  | 155   | 162  | 134   | 42    | -95  |
| -226 | -307 | -356 | -293 | -215 | -81   | 32   | 106   | 148   | 138  |
| 127  | 148  | 176  | 233  | 247  | 194   | 92   | -39   | -109  | -124 |
| -113 | -99  | -116 | -145 | -152 | -120  | -67  | 18    | 71    | 106  |
| 113  | 124  | 138  | 162  | 155  | 134   | 81   | 32    | -7    | -28  |
| -49  | -81  | -99  | -116 | -116 | -92   | -71  | -32   | 21    | 60   |
| 102  | 109  | 109  | 102  | 106  | 120   | 120  | 64    | 28    | -14  |
| -49  | -78  | -81  | -81  | -81  | -74   | -67  | -46   | -11   | 25   |
| 42   | 56   | 71   | 74   | 88   | 99    | 92   | 67    | 49    | 14   |
| -25  | -46  | -78  | -78  | -95  | -116  | -64  | -42   | -28   | 4    |
| 28   | 56   | 81   | 95   | 95   | 99    | 92   | 67    | 56    | 39   |
| 7    | -7   | -35  | -56  | -78  | -85   |      |       |       |      |

Table 7  
List of Measured  $\zeta$  ( $\times 10^4$ ) Values for Inuid S-201  
(Parameters:  $\gamma_0 = 6.067$ ,  $b = 54.60$ ,  $y = 5.02$ ; Variable:  $x = 9.10$  stepped  $-0.2733$  until  $-64.70$ )

|      |      |      |       |      |      |      |      |      |      |
|------|------|------|-------|------|------|------|------|------|------|
| 8    | 8    | 12   | 12    | 16   | 16   | 19   | 19   | 23   | 19   |
| 19   | 23   | 23   | 27    | 23   | 19   | 16   | 12   | 12   | 16   |
| 0    | -12  | -47  | -51   | -74  | -117 | -140 | -168 | -195 | -222 |
| -249 | -269 | -281 | -273  | -253 | -210 | -175 | -113 | -62  | 51   |
| 121  | 234  | 308  | 382   | 398  | 382  | 316  | 148  | -58  | -327 |
| -569 | -783 | -947 | -1002 | -959 | -780 | -503 | -47  | 546  | 1236 |
| 1696 | 1489 | 1091 | 647   | 234  | -113 | -378 | -561 | -682 | -748 |
| -725 | -651 | -526 | -331  | -148 | 8    | 152  | 234  | 304  | 366  |
| 417  | 456  | 483  | 460   | 421  | 324  | 199  | 66   | -78  | -203 |
| -304 | -378 | -429 | -437  | -417 | -378 | -327 | -277 | -187 | -90  |
| 43   | 183  | 320  | 448   | 534  | 577  | 585  | 538  | 417  | 246  |
| 43   | -164 | -343 | -476  | -554 | -592 | -589 | -518 | -398 | -277 |
| -136 | -8   | 140  | 304   | 417  | 499  | 557  | 573  | 538  | 456  |
| 324  | 148  | -47  | -207  | -327 | -417 | -437 | -437 | -413 | -409 |
| -394 | -335 | -230 | -74   | 109  | 304  | 409  | 421  | 417  | 401  |
| 390  | 382  | 339  | 214   | 51   | -125 | -253 | -347 | -382 | -394 |
| -401 | -401 | -374 | -320  | -207 | -86  | 47   | 175  | 261  | 331  |
| 398  | 452  | 452  | 394   | 292  | 183  | 86   | -12  | -117 | -230 |
| -324 | -378 | -398 | -382  | -335 | -296 | -230 | -152 | -47  | 82   |
| 175  | 253  | 296  | 324   | 359  | 355  | 324  | 238  | 133  | 35   |
| -47  | -113 | -195 | -277  | -335 | -363 | -335 | -300 | -261 | -187 |
| -101 | -4   | 117  | 172   | 242  | 304  | 343  | 347  | 316  | 273  |
| 214  | 133  | 27   | -58   | -140 | -226 | -285 | -316 | -308 | -285 |
| -273 | -203 | -148 | -82   | 16   | 101  | 187  | 265  | 308  | 312  |
| 324  | 292  | 242  | 187   | 113  | 19   | -58  | -160 | -218 | -269 |
| -288 | -300 | -288 | -273  | -203 | -117 | -43  | 55   | 125  | 214  |
| 253  | 277  | 296  | 292   | 261  | 207  | 148  | 74   | 4    | -86  |
| -136 |      |      |       |      |      |      |      |      |      |

Table 8  
List of Measured  $\zeta (\times 10^4)$  Values for Inuid S-201  
(Parameters:  $\gamma_0 = 4.903$ ,  $b = 44.13$ ,  $y = 4.06$ ; Variable,  $x = 7.55$  stepped  $-0.2462$  until  $-73.68$ )

|      |      |      |       |       |       |       |       |      |      |
|------|------|------|-------|-------|-------|-------|-------|------|------|
| 16   | 16   | 9    | 9     | 9     | 16    | 22    | 25    | 32   | 28   |
| 35   | 35   | 35   | 38    | 44    | 47    | 44    | 38    | 28   | 22   |
| 9    | 3    | -13  | -41   | -72   | -104  | -139  | -176  | -211 | -249 |
| -287 | -318 | -350 | -369  | -375  | -359  | -321  | -261  | -198 | -98  |
| 13   | 151  | 296  | 473   | 614   | 740   | 797   | 759   | 580  | 224  |
| -214 | -703 | -996 | -1178 | -1219 | -1156 | -961  | -602  | -189 | 350  |
| 766  | 1046 | 1178 | 1169  | 1046  | 819   | 545   | 246   | 3    | -211 |
| -359 | -463 | -532 | -570  | -599  | -608  | -599  | -564  | -507 | -406 |
| -277 | -72  | 142  | 416   | 674   | 926   | 1084  | 1122  | 1018 | 788  |
| 466  | 104  | -249 | -554  | -778  | -951  | -1021 | -1005 | -885 | -680 |
| -397 | -66  | 261  | 473   | 529   | 495   | 447   | 410   | 397  | 400  |
| 410  | 384  | 309  | 180   | 25    | -101  | -189  | -258  | -302 | -334 |
| -369 | -384 | -394 | -375  | -318  | -230  | -126  | -6    | 104  | 198  |
| 280  | 347  | 378  | 362   | 324   | 284   | 252   | 224   | 180  | 107  |
| 0    | -120 | -221 | -280  | -312  | -318  | -312  | -274  | -230 | -186 |
| -164 | -135 | -82  | 3     | 107   | 202   | 265   | 306   | 312  | 290  |
| 249  | 211  | 167  | 132   | 85    | 16    | -60   | -139  | -192 | -230 |
| -246 | -246 | -239 | -230  | -211  | -186  | -145  | -82   | -3   | 85   |
| 154  | 192  | 221  | 243   | 252   | 249   | 217   | 180   | 123  | 63   |
| 3    | -44  | -88  | -129  | -183  | -224  | -246  | -252  | -236 | -198 |
| -161 | -117 | -60  | 3     | 66    | 113   | 148   | 186   | 205  | 221  |
| 221  | 205  | 189  | 151   | 98    | 35    | -32   | -104  | -154 | -186 |
| -180 | -173 | -176 | -183  | -173  | -161  | -145  | -95   | -38  | 13   |
| 57   | 85   | 110  | 151   | 173   | 189   | 180   | 148   | 110  | 76   |
| 44   | 19   | -9   | -50   | -95   | -142  | -170  | -176  | -180 | -176 |
| -154 | -129 | -101 | -66   | -32   | 6     | 60    | 113   | -154 | 189  |
| 202  | 198  | 186  | 148   | 117   | 82    | 38    | -3    | -60  | -107 |
| -151 | -180 | -189 | -176  | -158  | -139  | -123  | -104  | -66  | -28  |
| 25   | 66   | 95   | 107   | 123   | 135   | 154   | 158   | 145  | 117  |
| 88   | 57   | 9    | -28   | -63   | -104  | -129  | -145  | -151 | -154 |
| -139 | -123 | -95  | -63   | -25   | 13    | 32    | 60    | 79   | 101  |
| 123  | 139  | 142  | 123   | 95    | 69    | 57    | 22    | 0    | -32  |
| -63  | -91  | -110 | -126  | -123  | -135  | -142  | -135  | -117 | -91  |
| -60  |      |      |       |       |       |       |       |      |      |

A more detailed comparison of the results of transverse- and longitudinal-cut analyses is possible with the aid of Figs. 5 to 8, which exhibit the amplitude spectrum as a function of wave number in each case. Dimensional representation was preferred here to facilitate direct comparison with previously published material (see for example Ref. 4c). Each diagram shows in addition to the two empirical spectra the corresponding theoretical spectrum and two sets of semi-empirically modified spectra. These latter have been discussed at length in previous work and will not be commented upon any further.



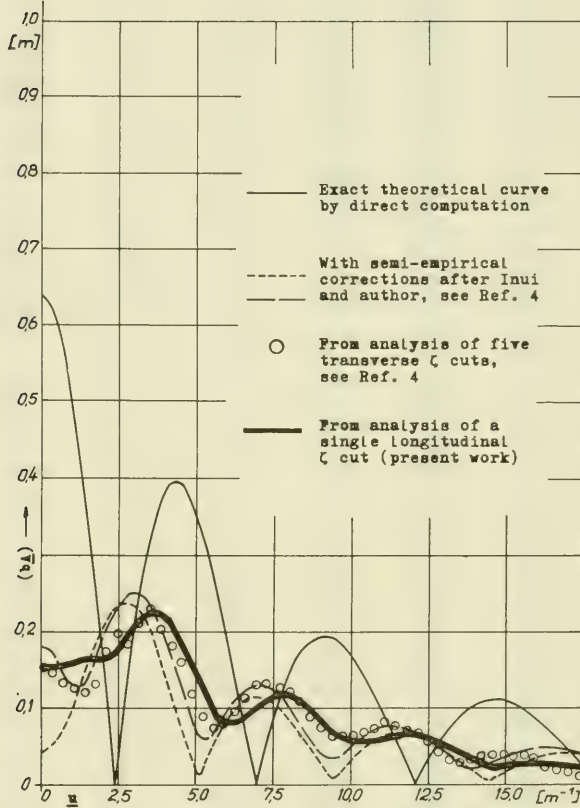


Fig. 5 - Amplitude spectrum of Inuid S-201 for  $\gamma_0 = 10.0$  or  $F_n = 0.224$

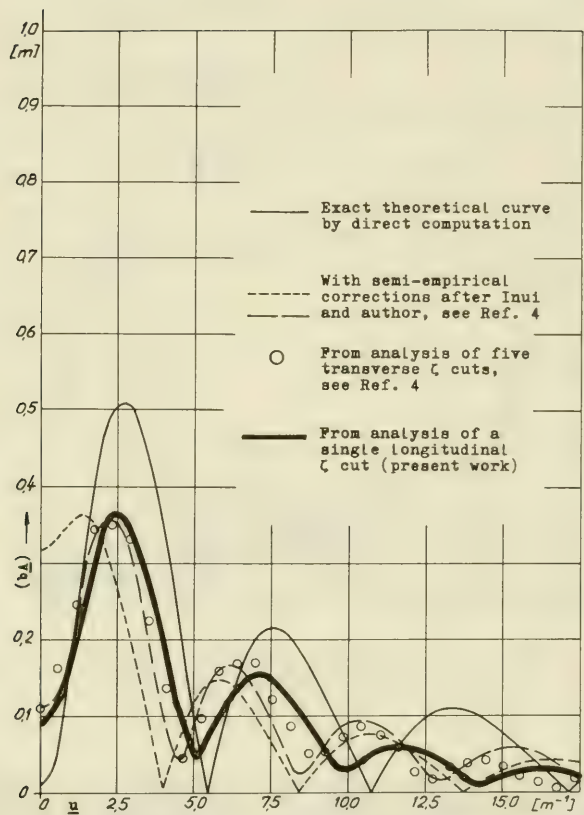


Fig. 6 - Amplitude spectrum of Inuid S-201 for  $\gamma_0 = 7.7$  or  $F_n = 0.255$

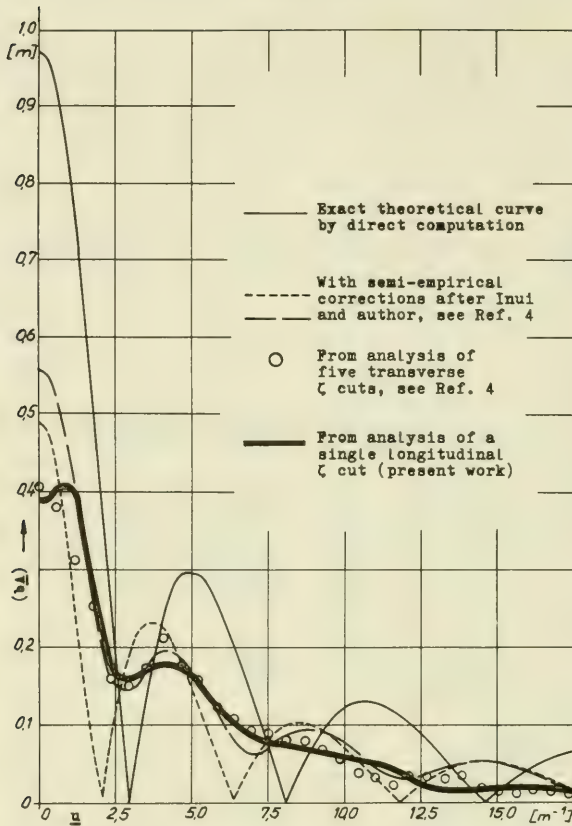


Fig. 7 - Amplitude spectrum of Inuid S-201 for  $\gamma_0 = 6.1$  or  $F_n = 0.287$

The overall result of this experimental verification can be tentatively summarized as follows. The transverse- and longitudinal-cut methods yield fairly consistent results and appear to be essentially equivalent, thus allowing the great practical advantages of the longitudinal-cut methods to be fully exploited in the future. If wave resistance alone is used as a criterion, then the differences manifested in Table 4 can be almost dismissed as due to experimental error. However, it should be noted that the discrepancy is least in Case 2 (Fig. 6), where the amplitude  $A_0$  is relatively small, and highest in Case 3 (Fig. 7), where the amplitude  $A_0$  is relatively large. In view of the numerical experience gained on theoretical wave systems (Figs. 2 and 3) this could be evidence of a systematic error arising from inadequate truncation correction and should therefore be further investigated.

Another possibility of improving the accuracy of the longitudinal-cut analysis would be to record two or more cuts simultaneously and use the redundant information to obtain more reliable least-squares fits. It may be recalled that

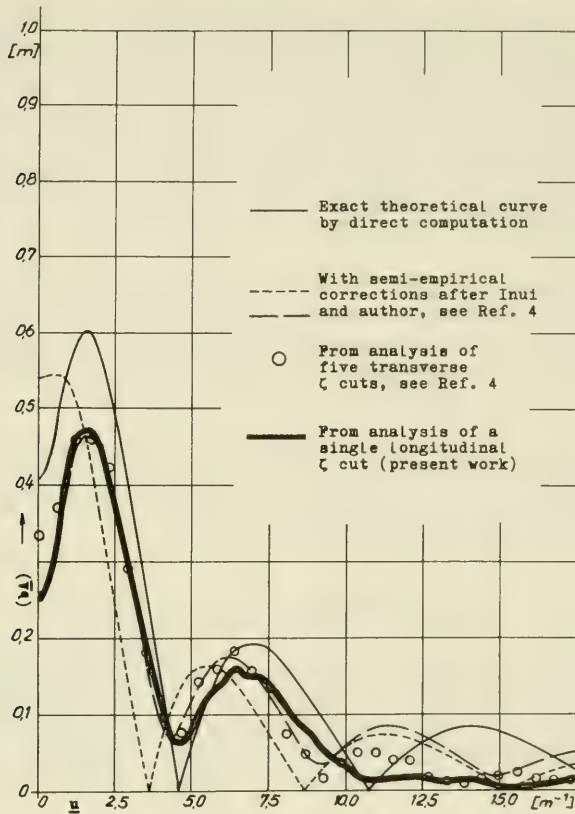


Fig. 8 - Amplitude spectrum of Inuid S-201 for  $\gamma_0 = 4.9$  or  $F_n = 0.319$

the same experience has been made previously with the transverse-cut method where as many as five cuts had to be taken (instead of a theoretical minimum requirement of only two cuts) to reduce the mean error to less than 5 percent (see Ref. 5a).

By the way, this mutual confirmation of the results of transverse- and longitudinal-cut analyses seems to further justify the working definition of wave resistance in a real fluid as the dynamic reaction of the associated free wave system. This quantity does seem to be a significant invariant property of the model not unduly sensitive to an arbitrary choice of parameters selected for analysis.

## CONCLUSIONS

It seems that a convenient and practical method of wave analysis has now been established. This involves the measurement of a time record of wave



height at a suitable fixed point in the towing tank while the model passes by at a steady speed, during a routine resistance test for example. A simple Fourier transform analysis of this record based on linearized potential theory then yields the free wave spectrum and the wave resistance of the model. The method has been numerically checked on different theoretical wave systems of predetermined spectra and experimentally verified on the mathematical model Inuid S-201. All essential numerical and experimental data leading to this conclusion have been reproduced in the paper, and others working in the field are invited to check the author's or their own methods on it.

One of the possible practical applications of this new experimental tool in ship-model testing would be to use the empirical free-wave spectrum of a main hull determined in this way as a basis for the design of an optimum additive bulb adapted to minimize free surface deformation, reduce resistance, and improve performance. (Such an application is described in Part II.)

Another possible application would be to analyze geosim model and full-size ship wave patterns by this method with a view to investigate scale effects and improve ship model correlation. It is hoped that those with the appropriate facilities at their disposal will take up the problem.

## PART II — SUPPLEMENT TO PART I

The numerical and experimental verification of a longitudinal-cut method of analyzing the measured wave system of a ship model with a view to determine its free wave spectrum was reported in Part I. The purpose of the following addendum is to round out this report by including a concrete example of possible practical application of this method to the problem of optimum bow design.

The basic empirical information needed for this work consisted of longitudinal wave profiles, measured during routine resistance tests at the HSVA, of two alternative designs (one with and the other without a bulbous bow) of a cargo ship model. Both the main hull and the bow bulb were of purely empirical design. The relevant parameters of the models tested are reproduced in Table 9. It may be noted that two operating conditions were specified; hence four sets of resistance and wave measurements were taken. The data analyzed and reported in the following were obtained at model speeds corresponding to the designed service speeds of the full-size ship in the fully loaded and ballast conditions. The transverse location and longitudinal extent of the profiles recorded and evaluated was very similar to that shown in Fig. 4.

An analysis of the wave height records by the method described in Part I yielded directly the free wave spectrum of the model with and without a bulb. From this the differential bulb spectrum was derived by taking the vector difference between the wave amplitudes with a bulb and those without a bulb at the same speed assuming the principle of linear superposition of the wave patterns created by the main hull and the bow bulb. The three sets of amplitudes obtained in this way are shown as functions of the transverse wavenumber  $u$  in Fig. 9 for the fully loaded condition and in Fig. 10 for the ballast condition. It will be noted that the amplitude spectrum of the model (both with and without a bulb)

Table 9  
Particulars of a Cargo Ship Model Tested with and  
without a Bulb at Two Different Loading Conditions

| Parameter                                                     | Model 1877<br>(without bulb) | Model 1877-1<br>(with bulb) |
|---------------------------------------------------------------|------------------------------|-----------------------------|
| Length between perpendiculars                                 | 6.306 m                      | 6.306 m                     |
| Breadth amidships                                             | 0.922 m                      | 0.922 m                     |
| Ratio of maximum bulb length<br>protruding beyond FP to LBP   | —                            | 0.0256                      |
| Ratio of maximum bulb thickness<br>to model breadth amidships | —                            | 0.145                       |
| <u>Load Condition</u>                                         |                              |                             |
| Draft                                                         | 0.420 m                      | 0.420 m                     |
| Displaced volume                                              | 1.678 m <sup>3</sup>         | 1.689 m <sup>3</sup>        |
| Block coefficient                                             | 0.687                        | 0.692                       |
| Wetted surface area                                           | 8.506 m <sup>2</sup>         | 8.716 m <sup>2</sup>        |
| Maximum length under water                                    | 6.500 m                      | 6.660 m                     |
| Ratio of total bulb volume<br>to main hull volume             | —                            | 0.006                       |
| Test speed analyzed and<br>reported in this paper             | 1.937 m/sec                  | 1.936 m/sec                 |
| <u>Ballast Condition</u>                                      |                              |                             |
| Draft at AP                                                   | 0.283 m                      | 0.283 m                     |
| Draft at FP                                                   | 0.167 m                      | 0.167 m                     |
| Mean draft                                                    | 0.225 m                      | 0.225 m                     |
| Displaced volume                                              | 0.822 m <sup>3</sup>         | 0.831 m <sup>3</sup>        |
| Block coefficient                                             | 0.628                        | 0.635                       |
| Wetted surface area                                           | 5.883 m <sup>2</sup>         | 6.056 m <sup>2</sup>        |
| Maximum length under water                                    | 5.997 m                      | 6.306 m                     |
| Ratio of total bulb volume<br>to main hull volume             | —                            | 0.011                       |
| Test speed analyzed and<br>reported in this paper             | 2.065 m/sec                  | 2.057 m/sec                 |

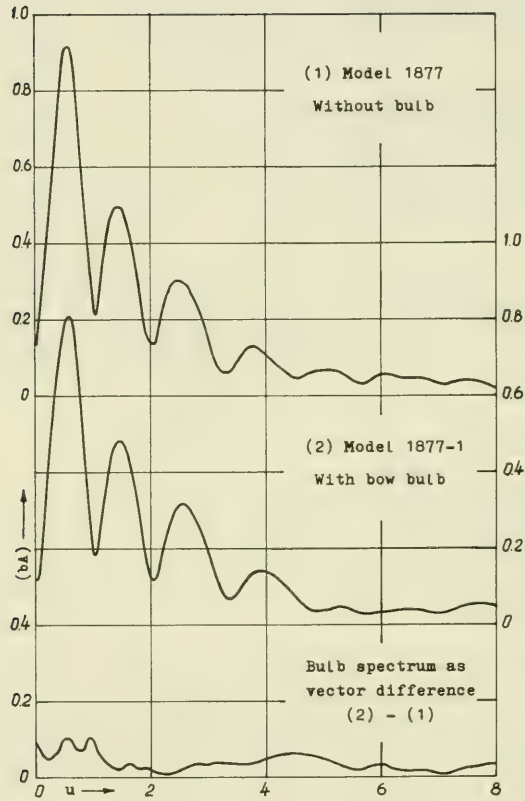


Fig. 9 - Amplitude spectrum of a cargo ship model at load draft for  $\gamma_0 = 8.25$  or  $F_n = 0.246$

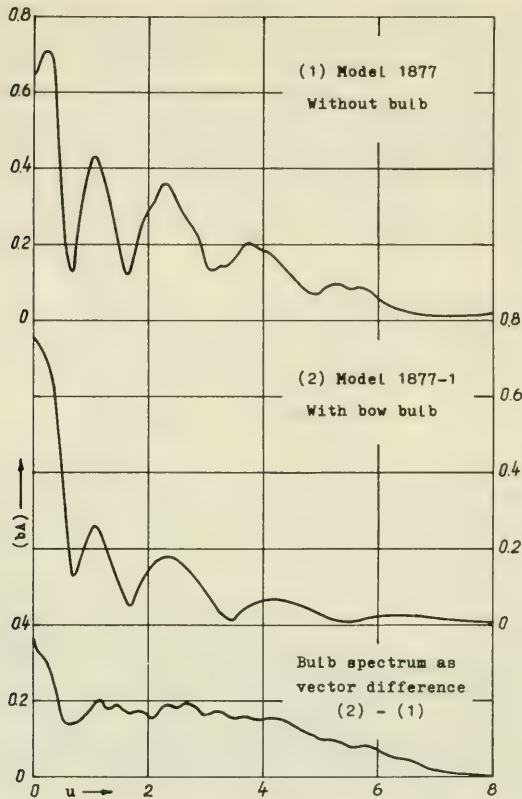


Fig. 10 - Amplitude spectrum of a cargo ship model at ballast draft for  $\gamma_0 = 7.27$  or  $F_n = 0.262$

exhibits regular oscillations arising from interference between bow and stern waves typical of any singularity distribution (see Fig. 3) in the linearized theory, while the differential bulb spectrum is a fairly monotonic curve characteristic of a concentrated point disturbance (see Fig. 1). The irregular high frequency fluctuations visible in the bulb amplitude function could be attributed to experimental error, while the somewhat more regular low frequency oscillations could result from nonlinear interaction between the main hull and bulb spectra.

Before passing on to the bulb optimization problem it may be useful to refer to the total resistance curves reproduced in Fig. 11. It will be seen that the effect of the bulb is by no means spectacular but that there is a definite reduction of total resistance over the entire speed range tested, amounting to about 12 percent at the ballast draft and 2 percent at the load draft at the respective design speeds. (The corresponding figures for the full-size ship would be presumably somewhat more favorable.) However, in view of the modest size of the bulb (less than 1 percent of the main hull volume, see Table 9) this seems to be



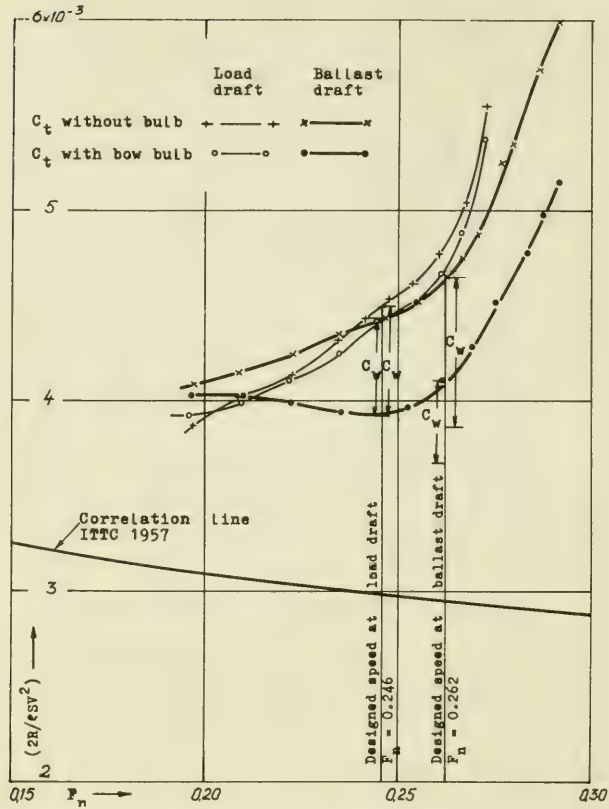
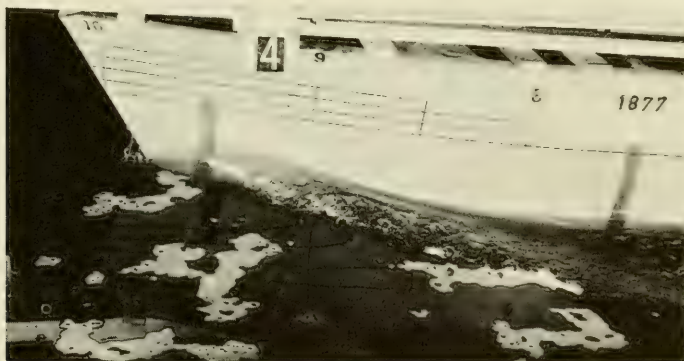


Fig. 11 - Measured total resistance coefficients of a cargo ship model with and without a bulb, at two drafts

a good result. In the present context it is more important to observe the following features in Fig. 11. The wave resistance coefficient  $C_w$  as computed from the spectra displayed in Figs. 9 and 10 is also plotted in Fig. 11 as a deduction from the measured total resistance coefficient  $C_t$ . Now, if the standard correlation line recommended by the ITTC 1957 is used as an estimate of frictional resistance  $C_f$ , it is obvious that the residuary resistance  $C_t - C_f$  is not accounted for by wave-making alone. Thus there is indirect evidence of considerable pressure drag of viscous origin. Considering differences, it is worth noting that the resistance reduction due to the bulb at the load draft can be fully explained by the observed attenuation of the wave system, whereas at the ballast draft only about two thirds of the total resistance reduction can be attributed to the attenuation of the visible surface deformation. This indicates that in the latter condition either the bulb is simultaneously acting to reduce viscous resistance or that the present method tends to underestimate the "actual" wave resistance. That this second possibility cannot be ruled out is suggested by the

appearance of the bow waves on the models with and without a bulb as compared in Fig. 12. Although the photographs are only a poor reproduction of the flow conditions actually observed during the tests, and there are certain extraneous features such as irrelevant foam patches on the free surface (foreground of Fig. 12(a)), it should be evident that the bow wave of the bulbless model has a broken crest and produces violently turbulent motion, while the bulbous model exhibits a fairly smooth flow. It is quite conceivable that in face of such highly nonlinear (and in fact nonpotential) flow phenomena the present linearized method of wave analysis would yield inaccurate results and generally tend to underestimate the "actual" wave-making resistance.

In spite of such possible shortcomings of the present method of analysis it was attempted to derive some practical results relevant to the optimization problem. To this end the theoretical effect of changing bulb size and location



(a) BOW WAVE OF BULBLESS MODEL 1877 AT 2.065 m/s



(b) BOW WAVE OF BULBOUS MODEL 1877-1 AT 2.057 m/s

Fig. 12 - Comparison of bow waves with and without bulb at the operating speed in the ballast condition: (a) bow wave of bulbless model 1877 at 2.065 m/sec and (b) bow wave of bulbous model 1877-1 at 2.057 m/sec

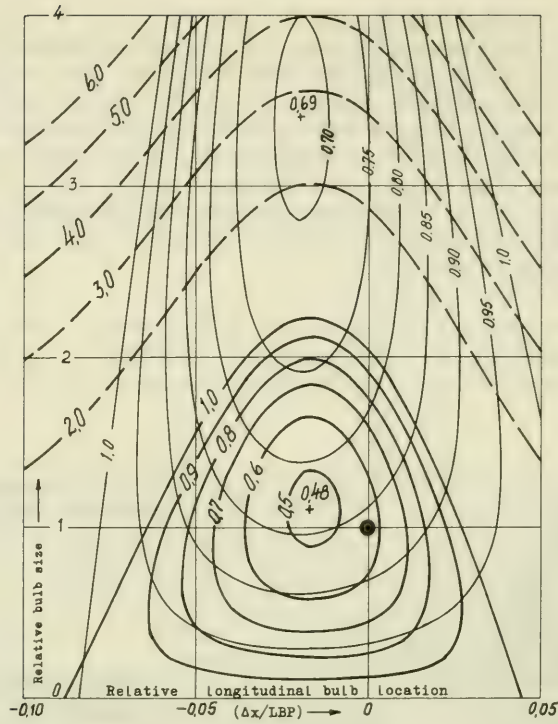


Fig. 13 - Contour lines of the ratio of wave resistance with bulb to that without bulb for ballast condition (thick lines) and load condition (thin lines)

was investigated by computing the wave resistance or corresponding linear combinations of the empirically derived free wave spectra of the main hull and the bow bulb. The results of this calculation are plotted in Fig. 13 as contour lines of the ratio of wave resistance with a bulb to that without a bulb. The set of thick lines holds for the ballast condition while the thin lines apply to the load condition.

It will be seen that for each condition there is a definite range of bulb size and location (bounded by the contour line marked 1.0) inside which the bulb produces beneficial action. There also exists in each case a definite optimal choice. The best that can be achieved with this model by merely changing bulb size and longitudinal location (i.e., without altering the basic bulb form) is a reduction of wave resistance down to about 48 percent in the ballast condition or 69 percent in the load condition. Evidently the two optima cannot be realized simultaneously and it is the designer's problem to choose a compromise somewhere along a straight line connecting the two optimal points in Fig. 13. The picture also confirms the empirical fact already known to designers that a bulb optimized for the load draft is generally oversized and detrimental at the ballast



draft. It may be noted in passing that the empirical bulb design actually tested — being represented by the coordinates (0,1) in the diagram — is a near hit to the theoretical optimum for the ballast condition, and this was to the author's knowledge also the aim of the designer in the present case. However, there is still some room for minor overall improvement, and this diagram serves to illustrate the type of practical recommendations that can be made to the designer with the aid of the wave analysis techniques in this paper.

## ACKNOWLEDGMENTS

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The wave profile measurements on the mathematical model Inuid S-201 reported in Part I of the paper were conducted in collaboration with Professor Lawrence W. Ward of the Webb Institute of Naval Architecture and Dr. Klaus Eggers of the Institut für Schiffbau der Universität Hamburg as part of a related research program aimed at comparing different methods of wave analysis. A complete report on this work is currently in preparation.

The wave records of the cargo ship model with and without a bulb, presented in Part II, were obtained during routine testing for a German shipyard, Orenstein-Koppel und Lübecker Maschinenbau Aktiengesellschaft, to whom the author is greatly indebted for kind permission to conduct these measurements and to publish most of the information.

The author is also grateful to Mr. Harald Keil of the Institut für Schiffbau for his assistance in automatic data acquisition and to Mr. Walter Alef of the HSVA for his help in data reduction and computer programming.

## NOMENCLATURE

(Unless otherwise specified, all lengths have been rendered nondimensional by multiplication with the fundamental wavenumber  $k_0$  and all wavenumbers have been rendered nondimensional by division with  $k_0$ .)

$\underline{A}$  Dimensional amplitude of an elementary wave in a tank of width

$A = k_0 \underline{A}$  Nondimensional wave amplitude

$b$  Tank width

$C, S$  Fourier transforms of  $\zeta(x)$ , see Eq. (12)

$C_y, S_y$  Analogous Fourier transforms of  $\zeta_y(x)$

$C^*, S^*$  Weighted Fourier transforms, see Eq. (17)

$C_F, S_F$  Fresnel integrals, see Eq. (22)



|                                   |                                                                      |
|-----------------------------------|----------------------------------------------------------------------|
| $c_1, c_2, c_3$                   | Constants for asymptotic representation of wave height, see Eq. (18) |
| $F_n$                             | Froude number $V/\sqrt{gL}$                                          |
| $F(u), G(u)$                      | Amplitude functions, see Eq. (6)                                     |
| $f(\theta), g(\theta)$            | Arbitrary functions for definition of spectrum, see Eq. (4)          |
| $f$                               | Depth of submergence of a dipole representing a spherical bulb       |
| $g$                               | Acceleration of gravity ( $m/sec^2$ )                                |
| $k_0 = g/V^2$                     | Fundamental wave number ( $1/m$ )                                    |
| $\underline{L}$                   | Usual dimensional model length                                       |
| $L = k_0 \underline{L}$           | Nondimensional model length                                          |
| $O$                               | Origin of coordinate system moving with the ship                     |
| $\underline{Oxyz}$                | Usual dimensional coordinate system                                  |
| $Oxyz$                            | Nondimensional coordinate system                                     |
| $Oxy$                             | Undisturbed free surface                                             |
| $Oyz$                             | Midship section                                                      |
| $Ozx$                             | Longitudinal center plane                                            |
| $R_w$                             | Nondimensional wave resistance $\underline{R}_w k_0^2 / \rho V^2$    |
| $\underline{R}_w$                 | Usual dimensional wave resistance                                    |
| $s = \sec \theta$                 | Longitudinal wavenumber                                              |
| $\underline{s} = k_0 \sec \theta$ | Dimensional wavenumber                                               |
| $t = \tan \theta$                 | Variable of integration                                              |
| $u = \sec \theta \tan \theta$     | Transverse wavenumber                                                |
| $\underline{u} = k_0 u$           | Dimensional wavenumber                                               |
| $u_v = 2\pi v/b$                  | Wavenumber of a discrete elementary wave in a tank of width $b$      |
| $x = k_0 \underline{x}$           | Coordinate in the direction of motion of the ship                    |

## Wave Analysis Techniques to Achieve Bow-Wave Reduction

$y = k_0 y$     Horizontal coordinate pointing to portside

$z = k_0 z$     Vertical coordinate positive upward

$\alpha_\nu$     Amplitude of the cosine component of an elementary tank wave, see Eq. (8)

$\beta_\nu$     Amplitude of the sine component of an elementary tank wave, see Eq. (8)

$\gamma_0 = gL/2V^2$     Physical Froude number =  $L/2$

$\epsilon_\nu$     Numerical factor =  $1/2$  for  $\nu = 0$  and  $1$  elsewhere

$\zeta(x, y)$     Wave height, free surface deformation

$\zeta_y(x, y)$     Transverse wave slope (partial derivative of  $\zeta$  with respect to  $y$ )

$\theta$     Direction of propagation of a component plane wave

$\lambda(\theta)$     Wavelength of a component plane wave in its direction of propagation  $\theta$

$\nu$     Tank wavenumber

$\xi_\nu$     Phase of elementary tank wave in a prescribed coordinate system, see Eq. (9)

$\rho$     Density of water

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\* \* \*

## DISCUSSION

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It is again a pleasure for me today to comment on a paper by a colleague with whom I have had the opportunity this past academic year to carry on interesting and valuable discussions and experimental investigations; this now, with Dr. Eggers, completes a triumvirate at the Institut für Schiffbau and at the Hamburgische Schiffbau-Versuchsanstalt which, as the author mentioned, attempted to bring both theoretical and experimental methods to bear during this period on the question of which methods of wave analysis are both valid and practical for determination of the wave resistance and other related questions, such as that of the bulbous bow which the author has focused on in the paper. Some of the results of this joint effort, in particular the results of a longitudinal cut analysis of the wave elevation due to the model Inuid S-201, are given in Table 4 in Part I. I would like to take this occasion to present other results we obtained on the same model by the "X-Y" method, in order to draw a comparison, the first that has been possible using the same model under the same conditions, between these two methods as well as with the author's previously obtained results using the Eggers transverse-cut relationships. These results are summarized in Fig. D1, in which the wave resistance coefficient  $C_w$  is presented as a function of the Froude number  $F_r$  for the three test series. It is seen that essential agreement is obtained between the three methods, except in two regions, one in a hollow and the other at the highest Froude numbers

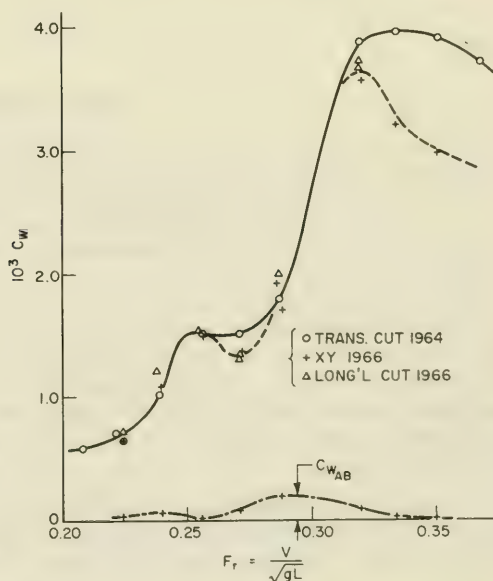


Fig. D1 - Results of various wave surveys on Inuid S-201:  $L = 4.0$  m and  $C_w = R_w / (\rho/2) S V^2$



considered, where the transverse cut analysis result is higher. Thus the comparison is encouraging and lends further support to the author's comment, in the last paragraph of the section "Experimental Verification," that this measured quantity, which N. Hogben has aptly termed "wave pattern resistance," seems to be definable in a practical sense. Further details on these results, including if possible results on a longitudinal cut method using the wave slope, should be forthcoming in a joint paper as mentioned by the author in his acknowledgments.

Let me make a comment on the specific question of the paper, that is, the experimental design of a bulb for a given ship. This is that the obvious suitability of the spectrum result in the present method of wave analysis for investigating size and location of the bulb does not preclude the consideration of other methods under the same assumptions, to solve the same problem in the tank. These are (a) linearity of the bulb wave system to the bulb strength and (b) fixed relative location of the bulb wave system with the bulb itself. In particular, one could construct the bulb X and Y signals (not the product) from the test data or from theory and apply this record through a computer analysis to the basic ship X and Y records to find optimum locations and strengths, without substantially more effort than required in analyzing and integrating the spectrum for the same purpose.

I am glad to see, that the author has included a plug for full-scale tests of wave resistance using one of these methods; a task which would require great effort but which would also, in my opinion, afford correspondingly great promise to learn something useful as well. Let me conclude by again thanking the author for preparing such an interesting paper; indeed one of a growing number of papers in this field with results and techniques which are of practical use to the ship designer.

\* \* \*

## DISCUSSION

K. W. H. Eggers

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Dr. Sharma showed us some computed results regarding the efficiency of wave analysis based on wave height along a longitudinal cut. In his Fig. 3 we may observe to what degree the spectrum of the free wave system may be determined from a profile of finite length. He also gives values for the ratio of wave resistance calculated from information of a finite cut to exact theoretical resistance.

I want to complete his presentation by a general survey of the dependence of this ratio on the sidewise distance  $y$  of a cut from the ship's path as well as from location  $x$  of the truncation point. For the same model and for a speed

# Wave Analysis Techniques to Achieve Bow-Wave Reduction

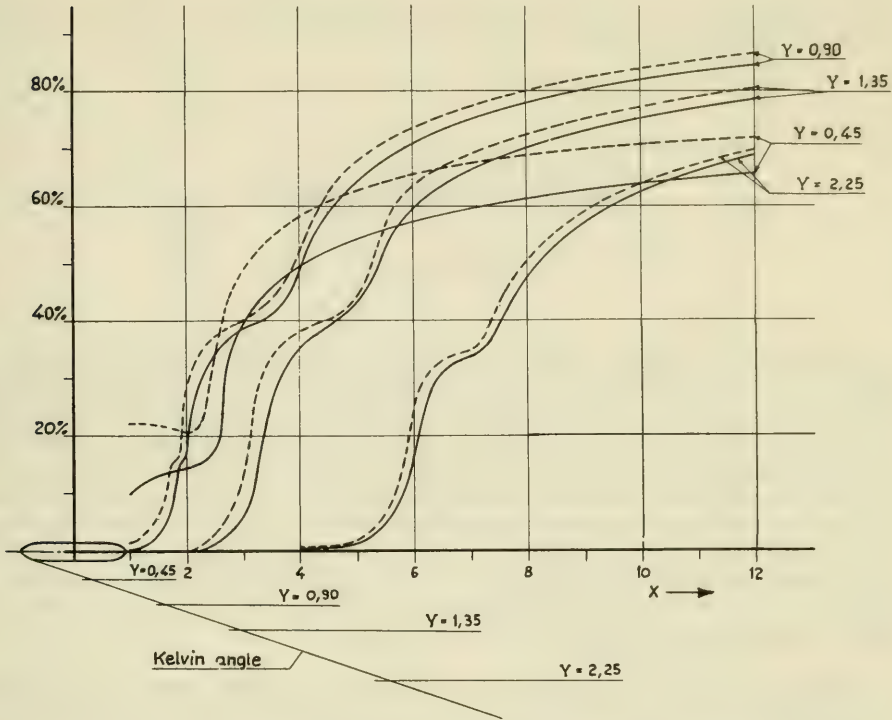


Fig. D2 - Portion of the resistance obtainable from Ward's cylinder force (broken lines) and longitudinal cut Fourier-transformation (solid lines) for different distances  $Y$  from the centerline as a function of truncation point  $X$ .  $\gamma_0 = 10$  and no truncation corrections have been applied.

parameter  $\gamma_0 = 10$ , I calculated additional data in comparison of this method with the approach of Ward, which is based on force measurements on a vertical cylinder at fixed position in the same sidewise distance. For four different values of  $Y$ , Fig. D2 shows curves of the portion of resistance gained from a cut truncated at  $X$  behind the model; no corrections for truncation are applied yet. The values for Ward's method correspond to those given in a different representation in Fig. 2 of my contribution to this symposium. All units of length are reduced by a factor  $1/10$  against the scale used by Dr. Sharma; i.e., the ship's half-length is taken as unity.

For both methods, we see that a distance  $y = 0.45$  will still bring considerable influence of local waves on the initial part of the curves; therefore even for a long cut only a limited degree of accuracy can be expected. A distance of  $0.9$  appears to be optimal to obtain a high percentage of resistance from a short cut, i.e., if the truncation point has to be close to the ship to avoid influence of a tank wall reflection.

The strong correlation between the curves for larger  $y$  is very surprising in view of the different origin of the two methods.

\* \* \*

## REPLY TO DISCUSSION

S. D. Sharma

Professor Ward has presented an instructive diagram (Fig. D1) showing a comparison of the experimental wave resistance of the Inuid S-201 determined in three different ways. This is the first time such a verification of the mutual consistency of these methods has been attempted. It may appear from the figure that the overall agreement is not quite as good as would be desirable, but it should be remembered that wave resistance is only a fraction of the total resistance, and the discrepancies regarded as percentages of the latter are small. This is encouraging in two respects. First, it shows that an approximately invariant wave resistance can be defined in a real fluid on the basis of these methods. Second, it seems permissible to use, alternatively, any of these three methods depending upon the facilities available.

Further, it is true that the "X-Y" method can also be used to predict the wave-making resistance of certain linear combinations of the bulb and the main hull under the same assumptions as used by the author. However, the spectral approach is more powerful. For example, given a certain wave profile (longitudinal or transverse) of a particular hull form, it is an easy matter to predict by linear superposition the corresponding profile (i.e., longitudinal or transverse respectively but not vice versa) of a pair of such hulls moving in tandem or side by side (catamaran) respectively. However, if one takes the trouble of deriving the spectrum by either method, one can predict any profile (longitudinal, transverse, or inclined) of any parallel configuration of any number of similar hulls! Of course, in principle, one could also derive the spectrum from the X and Y signals, but it would be a poor way of doing so.

The additional results of a computer study on the Inuid S-201 supplied by Dr. Eggers in Fig. D2 are welcome. It is evident that if  $y$  is not too small, both longitudinal cut methods would asymptotically yield the right value of wave resistance as the truncation point  $-x_e \rightarrow \infty$ . Figure D3 has been prepared by taking two cross curves from Dr. Eggers' data and supplementing it with the corresponding values corrected for truncation error as explained in the paper. It is obvious that for any given finite length of the cut there is an optimum transverse location (e.g., ca.  $y = 8$  in the present case). This results from the fact that the truncation error increases with increasing  $y$ , while the error due to local effects decreases.

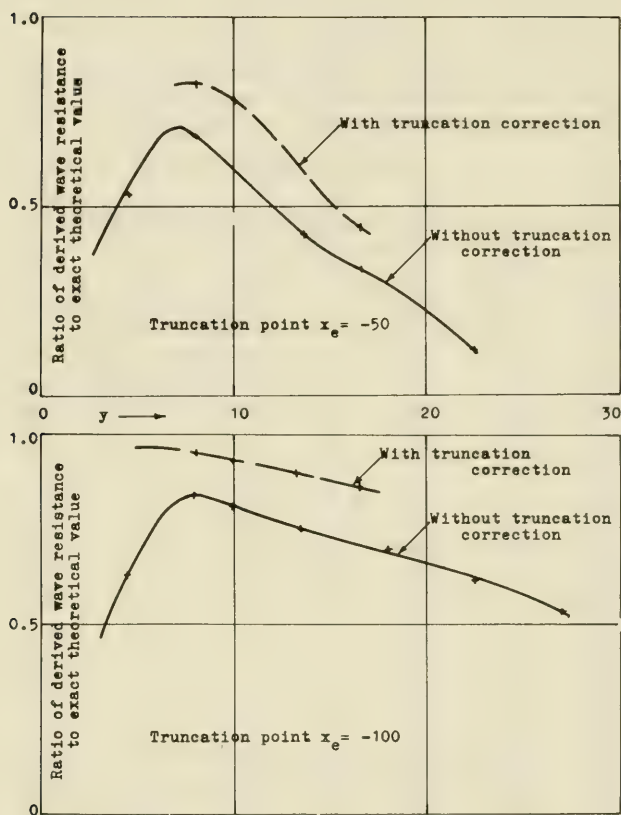


Fig. D3 - Computed effect of the transverse location of a  $\zeta$  cut on the wave resistance derived from it



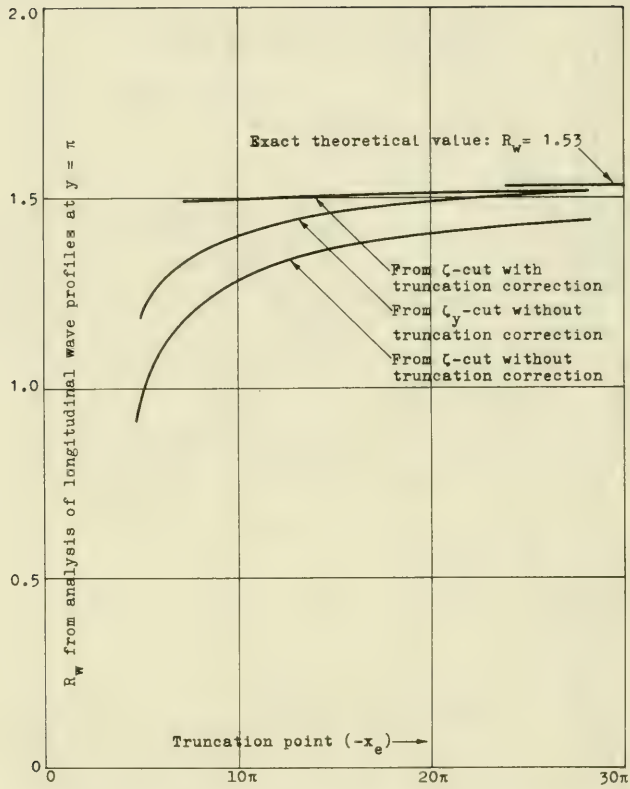


Fig. D4 - Relative magnitude of truncation errors in the wave resistance derived from analysis of theoretical wave profiles of a point dipole ( $f = 1$ ,  $r = 1$ )

A further result showing the relative magnitudes of the truncation error in the  $\zeta$ -cut and the  $\zeta_y$ -cut methods is presented in Fig. D4. Incidentally, a couple of numerical errors in Table 1 were discovered in the course of this work, and opportunity is now taken to provide a revised version in Table D1. The conclusions of the paper are not significantly affected by this correction.

Table D1

Revised and Enlarged Version of Table 1. List of Computed  $\zeta_y$  ( $\times 10^4$ ) Values for a Submerged Sphere (Parameters:  $r = 1$ ,  $f = 1$ ,  $y = \pi$ ; Variable,  $x = 4\pi$  stepped  $-\pi/10$  until  $-26\pi$ )

|       |       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | 0     | 0     | 0     | -1    | -1    | -2    |
| -4    | -6    | -9    | -14   | -22   | -32   | -47   | -66   | -90   | -120  |
| -156  | -197  | -242  | -288  | -333  | -370  | -397  | -408  | -399  | -368  |
| -313  | -235  | -138  | -25   | 95    | 217   | 336   | 446   | 545   | 634   |
| 715   | 792   | 870   | 951   | 1036  | 1119  | 1187  | 1220  | 1190  | 1062  |
| 800   | 377   | -219  | -971  | -1823 | -2676 | -3392 | -3805 | -3749 | -3098 |
| -1804 | 57    | 2264  | 4456  | 6177  | 6962  | 6453  | 4517  | 1347  | -2523 |
| -6277 | -8996 | -9891 | -8558 | -5156 | -438  | 4409  | 8085  | 9587  | 8546  |
| 5378  | 1165  | -2724 | -5122 | -5487 | -4067 | -1745 | 380   | 1480  | 1326  |
| 314   | -812  | -1375 | -1100 | -196  | 830   | 1488  | 1562  | 1176  | 639   |
| 234   | 55    | 15    | -42   | -204  | -441  | -650  | -743  | -701  | -569  |
| -405  | -240  | -76   | 92    | 259   | 402   | 497   | 534   | 515   | 450   |
| 347   | 217   | 70    | -80   | -215  | -325  | -398  | -431  | -422  | -372  |
| -289  | -180  | -59   | 65    | 177   | 269   | 332   | 361   | 354   | 313   |
| 244   | 153   | 51    | -54   | -150  | -228  | -283  | -308  | -303  | -269  |
| -210  | -132  | -44   | 46    | 129   | 197   | 245   | 267   | 263   | 234   |
| 183   | 116   | 39    | -39   | -113  | -173  | -215  | -235  | -232  | -206  |
| -162  | -102  | -34   | 35    | 99    | 153   | 190   | 209   | 206   | 184   |
| 145   | 93    | 31    | -31   | -89   | -137  | -171  | -187  | -185  | -165  |
| -130  | -82   | -27   | 28    | 80    | 123   | 154   | 169   | 166   | 149   |
| 117   | 75    | 25    | -25   | -72   | -112  | -140  | -154  | -152  | -136  |
| -107  | -68   | -23   | 24    | 68    | 102   | 128   | 140   | 139   | 124   |
| 98    | 62    | 21    | -22   | -61   | -94   | -118  | -129  | -128  | -115  |
| -90   | -58   | -19   | 19    | 56    | 86    | 109   | 119   | 118   | 106   |
| 83    | 53    | 19    | -18   | -52   | -80   | -101  | -110  | -110  | -98   |
| -77   | -49   | -17   | 17    | 48    | 75    | 93    | 103   | 102   | 91    |
| 72    | 45    | 16    | -16   | -45   | -70   | -87   | -96   | -96   | -85   |
| -65   | -43   | -15   | 15    | 42    | 65    | 82    | 90    | 89    | 80    |
| 63    | 40    | 14    | -14   | -40   | -61   | -77   | -84   | -84   | -76   |
| -60   | -38   | -13   | 13    | 38    | 57    | 72    | 80    | 79    | 71    |
| 56    |       |       |       |       |       |       |       |       |       |

\* \* \*



# THE MINIMUM PROBLEM OF THE WAVE RESISTANCE OF THE SURFACE PRESSURE DISTRIBUTION

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## INTRODUCTION

The minimum problem of the wave resistance has no solution in thin ship theory, and this means that singularity distributions exist which have no wave resistance. On the other hand, the wave-free distribution belonging to the usual functional class has no displacement, but wave-free distributions with a finite displacement exist in the theory of the slender ship, although the wave-resistance integral has no finite value in such case (1,2). This apparent contradiction is caused by the confusion of the functional class of the distribution, but the introduction of the function of the wider class or the higher order singularity makes the theory more fruitful (3).

This paper explains such a situation of the problem with respect to the surface pressure distribution (1,4,5). The theory is very similar to the thin and slender ship theory.

By the way, this theory is the case in which the ship surface and the pressure are given in the framework of linearized theory, so that it may be interesting to compare it with the so-called second-order theory.

## PRESSURE DISTRIBUTION

Consider a uniform stream with unit velocity, and Cartesian coordinates, taking the origin at the water surface, the  $x$  axis as positive toward the upstream side, and the  $z$  axis as positive upward. If a pressure  $p(x,y)$  acts over the surface  $S$  at the water surface, some wave motion occurs. Let  $\phi(x,y,z)$  be the velocity potential of this motion; then it must satisfy the conditions (6)

$$\frac{1}{\rho} p(x,y) \approx - \frac{\partial}{\partial x} \phi(x,y,0) - gZ(x,y), \quad (1)$$

$$p(x,y) = 0 \quad \text{outside of } S, \quad (2)$$



$$\frac{\partial}{\partial x} Z(x, y) = \frac{\partial}{\partial z} \phi(x, y, 0), \quad (3)$$

and

$$\frac{\partial^2}{\partial x^2} \phi(x, y, 0) = -g \frac{\partial}{\partial z} \phi(x, y, 0) \text{ outside of } S, \quad (4)$$

where  $\rho$  is the water density,  $g$  is the gravity constant, and  $Z(x, y)$  is the water surface elevation.

Then, it has the well-known (6) representation

$$\phi(x, y, z) = \frac{1}{4\pi\rho g} \iint_S p(x', y') \frac{\partial}{\partial x'} S(x, y, z; x', y', 0) dx' dy', \quad (5)$$

where (7)

$$\begin{aligned} S &= - \lim_{\mu \rightarrow +0} \frac{g}{\pi} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{\exp[kz + ik(x-x') \cos \theta + ik(y-y') \sin \theta]}{k \cos^2 \theta - g + i\mu \cos \theta} dk d\theta \\ &= 4gO_{-2}^{(1)} [g(x-x'), g(y-y'), -z], \end{aligned} \quad (6)$$

and where

$$\frac{\partial^2}{\partial x^2} + g \frac{\partial}{\partial z} S = 2g \frac{\partial(1/r)}{\partial z}, \quad r^2 = (x-x')^2 + (y-y')^2 + z^2, \quad (7)$$

Then the condition on the ship surface  $S$  becomes

$$\frac{\partial}{\partial x} Z = \frac{\partial}{\partial z} \phi = -\frac{1}{\rho g} \frac{\partial}{\partial x} p - \frac{1}{g} \frac{\partial^2}{\partial x^2} \phi, \quad (8)$$

or

$$Z(x, y) = -\frac{1}{\rho g} p(x, y) + \frac{1}{4\pi\rho g} \iint_S p(x', y') \frac{\partial^2}{\partial x'^2} S dx' dy'. \quad (9)$$

The solution of this integral equation has been examined by Maruo for small values (7,8).

For large  $g$  values, it is well known that the second term of the right-hand side of Eq. (9) is small and

$$Z(x, y) \approx -p(x, y)/\rho g \quad (10)$$

except near the periphery of  $S$ .

If the auxiliary function  $m(x, y)$  is defined by the partial differential equation

$$p(x, y) = \left[ \frac{\partial^4}{\partial x^4} + g^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] m(x, y), \quad (11)$$

then it will be determined uniquely except for some arbitrary boundary conditions, say,

$$m(\pm 1, y) = \frac{\partial}{\partial x} m(\pm 1, y) = 0, \quad m(x, \pm b) = 0, \quad (12)$$

where  $S$  is assumed as a rectangle with length 2 and breadth  $2b$ .

Putting Eqs. (11) and (12) into Eq. (5), and integrating partially, yields (10)

$$\begin{aligned} \phi(x, y, z) = & f(x, y, z) + \frac{1}{4\pi\rho g} \int_{-1}^1 \left( \frac{\partial}{\partial y'} m \frac{\partial}{\partial x'} S \right)_{y'=-b}^b dx' \\ & + \frac{1}{4\pi\rho g} \int_{-b}^b \left( \frac{\partial^3}{\partial x'^3} m \frac{\partial}{\partial x'} S - \frac{\partial^2}{\partial x'^2} m \frac{\partial^2}{\partial x'^2} S \right)_{x'=-1}^1 dy', \end{aligned} \quad (13)$$

where (10)

$$f(x, y, z) = \frac{1}{2\pi\rho} \left( \frac{\partial}{\partial z} \right) \left( \frac{\partial^2}{\partial x^2} - g \frac{\partial}{\partial z} \right) \iint_S m(x', y') \frac{\partial}{\partial x'} \left( \frac{1}{r} \right) dx' dy'. \quad (14)$$

Since the first term has no trailing wave, this formula shows that the potential consists of two parts; one is, say, the wave-free potential, and the other is the part having the trailing wave, which is a sum of singularity distributions along its periphery.

If  $m(x, y)$  satisfies also the conditions

$$\frac{\partial^2}{\partial x^2} m(\pm 1, y) = \frac{\partial^3}{\partial x^3} m(\pm 1, y) = 0, \quad \frac{\partial}{\partial y} m(x, \pm b) = 0, \quad (15)$$

along with Eq. (12), then the potential is wave-free.

In this case, integrating Eq. (11) and imposing the conditions of Eqs. (12) and (15), it is found that the total pressure is zero:

$$\iint_S p(x, y) dx dy = 0. \quad (16)$$

This is similar to the conclusion in thin ship theory (10).

For a simple example, the pressure distribution deduced from

$$m(x, y) = (b^2 - y^2)^2 (1 - x^2)^4 \quad (17)$$

is shown in Fig. 1.

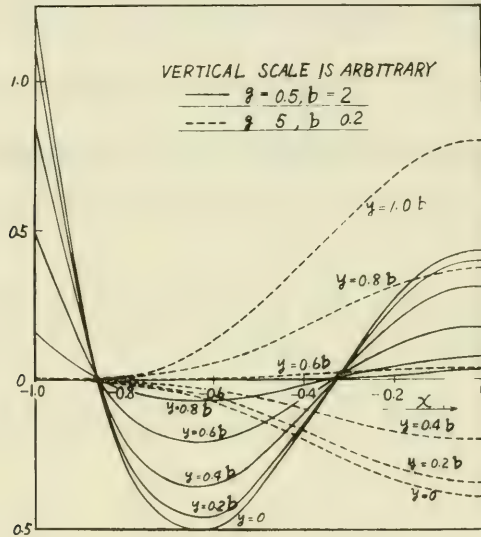


Fig. 1 - Pressure distribution of a wave-free potential

By the way, there are wave-free distributions with finite displacement (payload) in the two-dimensional problem, but their displacement is very small compared with their static buoyancy for high speed. This means that there is a large negative lift at high speed with wave-free distributions (11).

In another way, the potential  $f(x, y, z)$  of Eq. (14) is always wave-free for an arbitrary function  $m(x, y)$  without the conditions of Eqs (12) and (15), because it satisfies the surface condition of Eq. (4), but higher order singularities than the doublet must be introduced.

Thus, an arbitrary large number of pressure distributions exist with the same displacement and wave resistance as the following simple cases:

1. The longitudinal line distribution along the two segments  $|y| = b$ ,  $|x| \leq 1$ , which may be called the twin hull ship type.
2. The slender ship as the limiting case  $b = 0$  of the above.
3. The transversal line distribution along one or two segments  $|y| \leq b$ , which may be called the planing surface type.

For example, the pressure distribution deduced from

$$m(x, y) = (b^2 - y^2)(1 - x^2)^4 \quad (18)$$

belongs to case 1, and the pressure distribution deduced from

$$m(x, y) = (b^2 - y^2)^2 (1 - x)^4 (1 + x)^3 \quad (19)$$

belongs to case 2. Figures 2 and 3 show these examples.

When  $g$  is very small, the first term of Eq. (11) is dominant, but as  $g$  becomes larger and  $b$  smaller, the third term becomes dominant, where Eq. (10) is to be remembered.

The twin hull ship type for large  $g$  (Fig. 2) is especially interesting, for it may be considered as a model of a broad flat stern of a displacement ship (12).

Finally, the wave resistance  $R$  is (6)

$$R = \frac{g^2}{2\pi\rho} \int_{-\pi/2}^{\pi/2} |F(g \sec^2 \theta, \theta)|^2 \sec^5 \theta d\theta, \quad (20)$$

where

$$F(k, \theta) = \iint_S p(x, y) \exp[-ik(x \cos \theta + y \sin \theta)] dx dy, \quad (21)$$

or, interchanging the order of integration, the wave resistance can be written

$$R = \frac{g^2}{\pi} \iint_S p(x, y) G'(x, y) dx dy, \quad (22)$$

where

$$G'(x, y) = \frac{1}{\rho} \iint_S p(x', y') P_{-5}[g(x - x'), g(y - y'), 0] dx' dy', \quad (23)$$

in which (13)

$$\left. \begin{matrix} P_{2n+1} \\ P_{2n} \end{matrix} \right\} (x, y, z) = (-1)^n \int_0^{\pi/2} \cos(y \sec^2 \theta \sin \theta) \left\{ \begin{matrix} -\cos(x \sec \theta) \cos^{2n+1} \theta \\ \sin(x \sec \theta) \cos^{2n} \theta \end{matrix} \right\} e^{-z \sec^2 \theta} d\theta. \quad (24)$$

This function  $G'$  may be called the influence function.

Putting Eq. (11) into Eq. (20), integrating partially, and making use of Eqs. (12) and (15), it is found that the wave-free distribution has no wave resistance.



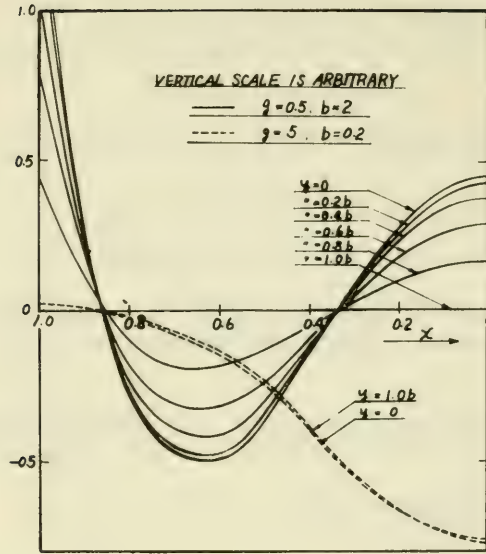


Fig. 2 - Pressure distribution of the twin hull ship type

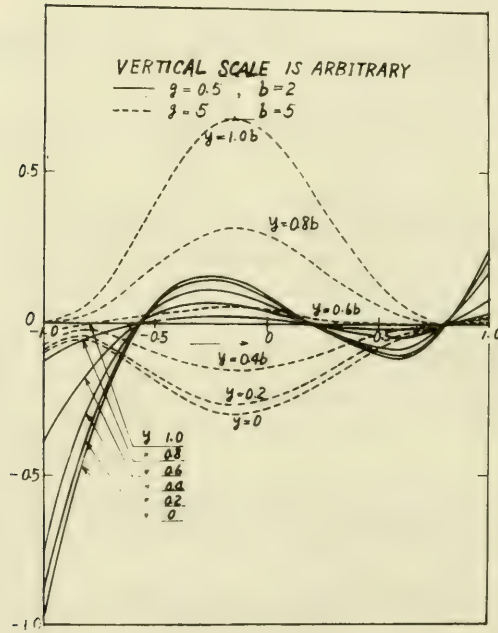


Fig. 3 - Pressure distribution of the planing surface type

## MINIMUM PROBLEM

When the total pressure is given, namely,

$$\iint_S p(x, y) \, dx dy = \Delta, \quad (25)$$

if the influence function becomes constant over  $S$ , that is, if

$$G'(x, y) = C > 0, \quad (26)$$

then the wave resistance is minimum.

On the other hand, differentiating partially,  $G'$  satisfies the differential equation

$$\left[ \frac{\partial^4}{\partial x^4} + g^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] G'(x, y) = 0, \quad (27)$$

assuming the existence of the integral.

Accordingly,  $G'$  may be represented uniquely by some boundary conditions. Hence, the integral equation given by Eq. (26) is regular in the domain, so that it may have a unique solution.

Since the existence of its derivative is assumed, the integral equation obtained by the differentiation of both sides of Eq. (26) may also have a unique solution, and this solution must be identically zero, because the right-hand side is zero.

Hence, the present minimum problem has no definite solution and no minimum value exists for the wave resistance (1,14). This fact may mean that its least value will be zero, because it may be possible to reduce the total wave amplitude as small as necessary, adding the longitudinal and transversal distribution to each other appropriately. However, the minimum solution exists in elementary cases such as thin ship theory.

Thus, the problem may be classified as follows:

1. Twin hull ship type distribution. When the speed is very high and the breadth narrow, this case nearly equals the next. As seen from the preceding section, it is also interesting at low speed, but usually it seems more useful to consider it in combination with case 3 (12), which means case 4.

2. Slender ship. In this well-known case there exists a unique solution except for arbitrary wave-free distributions which have no wave resistance but a finite displacement (1,9).

3. Transversal line distribution. This is the case to be studied in the following section (1,4,5,7).

4. Problem to reduce the wave resistance. This is also a practical problem, that is, how and how much the wave resistance can be reduced by the adequate combination of the elementary distributions, keeping the practical restrictions of the actual ship.

There is another case which has a unique solution, that is, a symmetric distribution about the origin over a circular disc, but it is nearly equal to case 3 for large velocity (1).

#### TRANSVERSAL LINE DISTRIBUTION

Case 3 above will now be studied, namely, a transversal line distribution (1,5,7-9). Suppose the wave-source doublet is along the segment  $|y| < 1$ , of which the total is given as Eq. (25). Introducing a normalized distribution  $H$  as

$$H(y) = \frac{2}{\Delta} \int p(x, y) dx, \quad (28)$$

this is written as

$$\int_{-1}^1 H(y) dy = 2. \quad (29)$$

The wave resistance (Eq. (22)) can be written as

$$r^* = R \left( \frac{\Delta^2}{\rho g B^3} \right) = R / \left( \frac{\Delta^2}{8 \rho g} \right) = 2 \int_{-1}^1 H(y) G(y) dy, \quad (30)$$

where

$$G(y) = \frac{g^3}{2} \left( 1 - \frac{2}{g^2} \frac{d^2}{dy^2} \right) G^*(y) \quad (31)$$

and also

$$G^*(y) = \frac{1}{\pi} \int_{-1}^1 H(y') P_{-1}[0, g(y - y'), 0] dy' \quad (32)$$

or

$$G^*(y) = \frac{1}{2\pi} \int_{-1}^1 H(y') K_0 \left( \frac{g}{2} |y - y'| \right) dy', \quad (33)$$

where  $B$  is the breadth in the usual unit system and  $K_0$  is a modified Bessel function.

Equation (31) can be deduced from the following relations. Since the function  $P_{-1}$  has an expansion (13)

$$P_{-1}(x, y, 0) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{2} K_{2n}(y/2) J_{4n}(x), \quad (34)$$

and since

$$P_{-5}(x, y, 0) = \frac{\partial^4}{\partial x^4} P_{-1}(x, y, 0),$$

then

$$P_{-5}(0, y, 0) = \frac{3}{16} K_0(y/2) - \frac{1}{16} K_2(y/2) = \frac{1}{4} \left( 1 - 2 \frac{d^2}{dy^2} \right) K_0(y/2). \quad (35)$$

To solve the integral equation (33), assume the next expansion in Mathieu functions (15)

$$H(y) = \phi(\theta)/\sin \theta, \quad y = \cos \theta, \quad (36a)$$

$$\phi(\theta) = \sum_{n=0}^{\infty} a_{2n} \operatorname{ce}_{2n}(\theta, -q), \quad q = g^2/16. \quad (36b)$$

Then, since there is the integral

$$\frac{1}{2\pi} \int_0^\pi K_0 \left( \frac{g}{2} |\cos \theta - \cos \theta'| \right) \operatorname{ce}_{2n}(\theta', -q) d\theta' = \lambda_{2n} \operatorname{ce}_{2n}(\theta, -q), \quad (37)$$

where

$$\lambda_{2n} = \frac{\pi}{2} \left[ \frac{A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q)} \right]^2 \frac{\operatorname{Fek}_{2n}(0, -q)}{\operatorname{ce}_{2n}(0, -q)}, \quad (38)$$

because of the representations

$$\frac{1}{\pi} \int_0^\pi \cos(2k \cos \theta \sinh u) \operatorname{ce}_{2n}(\theta, -q) d\theta = \frac{(-1)^n A_0^{(2n)} \operatorname{Ce}_{2n}(u, q)}{\operatorname{ce}_{2n}(0, q)}, \quad (39a)$$

$$\frac{1}{\pi} \int_0^\infty \cos(2k \cosh z \sinh u) \operatorname{Ce}_{2n}(u, q) du = \frac{(-1)^n A_0^{(2n)} \operatorname{Fek}_{2n}(z, -q)}{\operatorname{ce}_{2n}(0, q)}, \quad (39b)$$

and the relation

$$\operatorname{Re} [\operatorname{Fek}_{2n}(-i\theta, -q)] = \frac{\operatorname{Fek}_{2n}(0, -q)}{\operatorname{ce}_{2n}(0, -q)} \operatorname{ce}_{2n}(\theta, -q), \quad (40)$$



$G^*$  can be integrated as (1,15)

$$G^*(\cos \theta) = \sum_{n=0}^{\infty} \lambda_{2n} a_{2n} ce_{2n}(\theta, -q). \quad (41)$$

The minimum solution is a solution such that

$$G(y) = \text{constant}, \quad (42)$$

but, putting this into Eq. (31) results in a differential equation, so that

$$G^*(y) = \text{constant} \quad (43)$$

may be a special solution, but there is also a homogeneous solution:

$$G^*(y) = C \cosh(gy/\sqrt{2}), \quad (44)$$

where  $C$  is an arbitrary constant, for which

$$G(y) = 0 \quad \text{and} \quad r^* = 0. \quad (45)$$

This solution will be called a wave-free solution (1).

Since there are expansions (15)

$$1 = 2 \sum_{n=0}^{\infty} (-1)^n A_0^{(2n)} ce_{2n}(\theta, -q), \quad (46)$$

$$\cosh(g \cos \theta / \sqrt{2}) = \sum_{n=0}^{\infty} (-1)^n C_{2n} ce_{2n}(\theta, -q), \quad (47)$$

where

$$C_{2n} = \frac{2(-1)^n A_0^{(2n)} Ce_{2n}(\sinh^{-1} 1, -q)}{ce_{2n}(0, q)},$$

the general solution can be written in the form

$$H(y) = aH_a(y) - bH_b(y), \quad a - b = 1, \quad (48)$$

where

$$H_a(y) = \phi_a(\theta) / \sin \theta, \quad \phi_a(\theta) = \sum_{n=0}^{\infty} a_{2n}^* ce_{2n}(\theta, -q), \quad (49)$$

$$H_b(y) = \phi_b(\theta)/\sin \theta, \quad \phi_b(\theta) = \sum_{n=0}^{\infty} b_{2n}^* \cos 2n(\theta - q), \quad (50)$$

$$a_{2n}^* = \frac{2A_0^{(2n)}}{\pi A \lambda_{2n}}, \quad A = \sum_{n=0}^{\infty} A_0^{(2n)} \lambda_{2n}, \quad (51)$$

and

$$b_{2n}^* = \frac{2(-1)^n C_{2n}}{\pi D \lambda_{2n}}, \quad D = \sum_{n=0}^{\infty} C_{2n} A_0^{(2n)} \lambda_{2n}. \quad (52)$$

Then

$$G(y) = ag^3/(2\pi A), \quad r^* = 2ag^3/(\pi A). \quad (53)$$

Although this solution becomes infinite at both ends  $y = \pm 1$  in general, taking as  $a$  the value

$$a = \phi_b(0)/[\phi_b(0) - \phi_a(0)], \quad (54)$$

this solution becomes zero there, in which case it will be called  $H_c$ .

For the numerical computation at high speed, it is more convenient to expand in the series of trigonometric functions as follows:

$$\phi_a(\theta) = \sum_{n=0}^{\infty} \alpha_{2n} \cos 2n\theta, \quad \phi_b(\theta) = \sum_{n=0}^{\infty} \beta_{2n} \cos 2n\theta. \quad (55)$$

Figures 4, 5, and 6 and Tables 1 and 2 show the results (4). When the velocity is very large, that is, when  $g$  is very small, the functions become approximately

$$\phi_a(\theta) \approx \frac{2}{\pi} \left[ 1 - \frac{g^2}{16} \log(8/\gamma g) \cos 2\theta \right], \quad (56a)$$

$$\phi_b(\theta) \approx \frac{2}{\pi} \left[ 1 + \frac{3g^2}{16} \log(8/\gamma g) \cos 2\theta \right], \quad (56b)$$

$$\phi_c(\theta) \approx \frac{4}{\pi} \sin^2 \theta, \quad (56c)$$

and

$$H_c(y) = \frac{4}{\pi} \sqrt{1 - y^2}, \quad (56d)$$

where  $\gamma$  means Euler's constant. The wave resistance is, respectively (1,4)

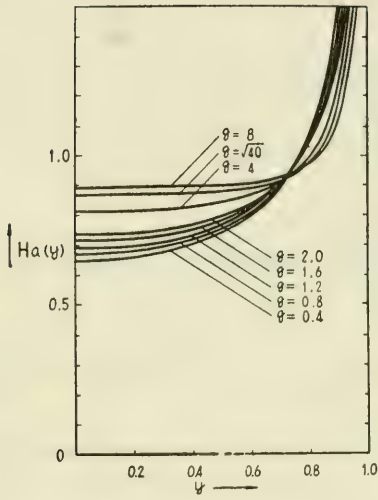


Fig. 4 - Minimum solution  $H_a(y)$

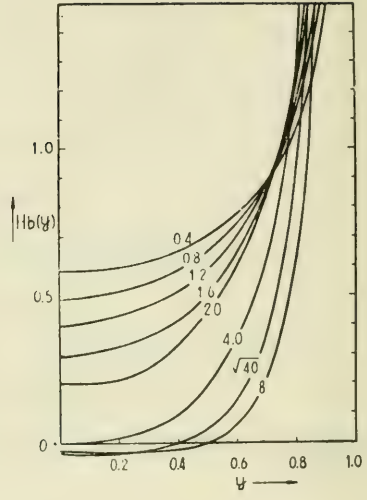


Fig. 5 - Wave-free solution  $H_b(y)$

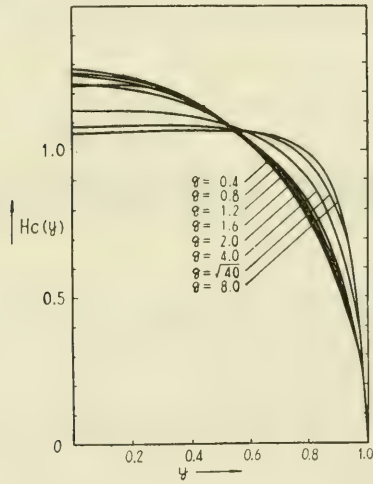


Fig. 6 - Minimum solution  $H_c(y)$

Table 1  
Fourier Coefficients of Trigonometrical Expansion

| g           | $F_r$  | $2\pi/A$ | $r_c^*$ | a     | b     | $x_0$  | $\alpha_2$ | $\alpha_4$ | $\alpha_6$ | $z_R$ | $\phi_a(0)$ | $\beta_0$ | $\beta_2$ | $\beta_4$ | $\beta_6$ | $\beta_8$ | $\beta_{10}$ | $\phi_b(0)$ |
|-------------|--------|----------|---------|-------|-------|--------|------------|------------|------------|-------|-------------|-----------|-----------|-----------|-----------|-----------|--------------|-------------|
| 0.4         | 1.118  | 1.5551   | 1.1051  | 11.10 | 10.10 | 0.6366 | -0.0124    | 0          | 0          | 0     | 0.6242      | 0.6366    | 0.0492    | 0.0002    | 0         | 0         | 0            | 0.6860      |
| 0.8         | 0.7906 | 1.1431   | 2.570   | 4.392 | 3.392 | 0.6366 | -0.0334    | 0          | 0          | 0     | 0.6032      | 0.6366    | 0.1428    | 0.0017    | 0         | 0         | 0            | 0.7811      |
| 1.2         | 0.6455 | 0.9196   | 4.696   | 2.955 | 1.955 | 0.6366 | -0.0565    | 0          | 0          | 0     | 0.5801      | 0.6366    | 0.2377    | 0.0065    | 0.0001    | 0         | 0            | 0.8769      |
| 1.6         | 0.5590 | 0.7732   | 7.052   | 2.227 | 1.227 | 0.6366 | -0.0788    | -0.0003    | 0          | 0     | 0.5575      | 0.6366    | 0.3553    | 0.0199    | 0.0003    | 0         | 0            | 1.0120      |
| 2.0         | 0.5000 | 0.6683   | 10.25   | 1.919 | 0.919 | 0.6366 | -0.0997    | -0.0007    | 0          | 0     | 0.5362      | 0.6366    | 0.4522    | 0.0301    | 0.0008    | 0         | 0            | 1.1196      |
| 4           | 0.3536 | 0.4000   | 35.74   | 1.396 | 0.396 | 0.6366 | -0.1810    | -0.0040    | 0.0001     | 0     | 0.4515      | 0.6366    | 0.7808    | 0.1612    | 0.0137    | 0.0006    | 0            | 1.5923      |
| $\sqrt{40}$ | 0.2812 | 0.2731   | 85.61   | 1.239 | 0.239 | 0.6366 | -0.2389    | -0.0098    | -0.0004    | 0     | 0.3874      | 0.6366    | 0.9509    | 0.3489    | 0.0632    | 0.0068    | 0.0001       | 2.0065      |
| 8           | 0.2500 | 0.2222   | 135.0   | 1.187 | 0.187 | 0.6366 | -0.2667    | -0.0144    | -0.0008    | 0     | 0.3546      | 0.6366    | 1.0164    | 0.4668    | 0.1177    | 0.0175    | 0.0003       | 2.2553      |

Table 2  
Fourier Coefficients in Mathieu Functions

| q    | g           | $F_r$  | $\lambda_0$ | $\lambda_2$ | $\lambda_4$ | $\lambda_6$ | $a_0^*$ | $a_2^*$ | $a_4^*$ | $a_6^*$ | $a_8^*$ | $b_0^*$ | $b_2^*$ | $b_4^*$ | $b_6^*$ | $b_8^*$ | $b_{10}^*$ |
|------|-------------|--------|-------------|-------------|-------------|-------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|------------|
| 0.25 | 2           | 0.5000 | 0.53006     | 0.23202     | 0.12297     | 0.084       | 0.88814 | 0.17815 | 0.00177 | 0.00002 | 0       | 0.93649 | 0.37250 | 0.37250 | 0.02102 | 0       | 0          |
| 1    | 4           | 0.3536 | 0.33596     | 0.19364     | 0.11737     | 0.0814      | 0.80137 | 0.44815 | 0.01776 | 0.00021 | 0       | 1.09883 | 0.47674 | 0.10260 | 0.00684 | 0.00021 | 0          |
| 2.5  | $\sqrt{40}$ | 0.2812 | 0.25608     | 0.15043     | 0.10793     | 0.07789     | 0.64228 | 0.66966 | 0.08237 | 0.00238 | 0.00002 | 1.28569 | 0.38995 | 0.18623 | 0.02797 | 0.00259 | 0.00007    |
| 4    | 8           | 0.2500 | 0.22568     | 0.12770     | 0.10006     | 0.07503     | 0.55116 | 0.73834 | 0.18163 | 0.00821 | 0.00019 | 1.37913 | 0.35853 | 0.20719 | 0.04676 | 0.00547 | 0.00025    |



$$r_0^* \approx (2g^3/\pi) \log (8/\gamma g) \quad \text{for } H_a(y), \quad (57a)$$

$$r^* = 0 \quad \text{for } H_b(y), \quad (57b)$$

and

$$r_c^* \approx 8g/\pi \quad \text{for } H_c(y). \quad (57c)$$

Thus,  $H_c$  represents the elliptic loading of which resistance and minimum character are found by Maruo. He explains that the wave resistance is similar to the induced drag of a wing physically and theoretically (7).

In this respect,  $H_b$  corresponds to the load distribution of a wing:

$$H(y) = 1/\sqrt{1-y^2}. \quad (58)$$

Since the induced velocity of this distribution is zero, there might be no induced drag for such a wing, if such a flow could be realized (16). For the planing surface, however, there may be a possibility to realize such a flow by adding floats at both ends (17).

On the other hand, the similarity of  $H_b$  (Fig. 5) to the wave-free distribution (Fig. 1), especially at low speed, is also to be remarked.

Generally speaking, the situation with respect to the wave-free solution may be similar to that of slender ship theory, in which case there also exist wave-free distributions having a finite displacement, and they correspond to another class of the distribution which has smaller resistance than the slender ship (2).

## CONCLUSION

As explained above, there is a close similarity between the theory of the thin ship and the pressure distribution. Thus, any pressure distribution is composed of line wave sources and wave-free distributions which have no displacement.

A typical elementary wave source is the twin hull ship type, that is, the longitudinal line distribution of the pressure on two parallel lines. Another is the planing surface type, that is, the transversal line distribution of the pressure.

Generally speaking, the minimum problem of the wave resistance has no solution and the least wave resistance may be zero, because some elementary wave sources could be summed up so as to cancel out their amplitude functions with each other. However, there are special cases when the solutions exist.

In this paper, the transversal line distribution of the pressure is treated, and it is found that there exists a unique minimum solution except for the wave-free distribution with a finite displacement. This wave-free distribution corresponds to the inverse elliptic load distribution of a wing and seems difficult to realize practically but suggests that distributions of another class may exist which have smaller wave resistance than the one considered here.

Finally, the larger the velocity, the smaller the wave resistance of the transversal line distribution; namely, this type is essentially preferable for very high speed, but it may be also possible to apply this theory to, say, the design of the destroyer stern.

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## DISCUSSION

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Science is subject to fashion as much as other human activities. Recently the thin ship and surrogates have completely dominated the field, but in the twenties (and earlier) the pressure system has been considered as being an equally important hydrodynamic model (at least in principle) as the Michell ship, especially suitable for picturing fast shallow-draft and planing vessels. By Dr. Bessho's paper a sound equilibrium has been established. The present speaker had emphasized the similarity of the Hogner and the Michell integral (Zamm, 1930) and thus inspired Sir Thomas Havelock to derive the simple relation between source-sink distributions  $\sigma$  and pressure systems  $p$

$$4\pi\rho g\sigma = c \frac{\partial p}{\partial x}$$

with the usual notations ( $c$  = speed of advance) (Havelock, collected papers, p. 373). The line distribution established by Dr. Bessho for a rectangular pressure domain follows from this equation.

Somewhat later (1935) von Karman has shown that the induced resistance of a finite-span wing can be derived from Hogner's integral for vanishing  $gL/c^2$  or vanishing  $g$  in Bessho's formulation, thus anticipating Maruo's result in his splendid papers on planing surfaces. His analysis of pressure systems and corresponding form of planing hulls should be developed. A useful scheme had been developed by H. Wagner, who connected planing surface and wing phenomena.



Dr. Bessho has clarified the conditions of minimum resistance for pressure systems in a similar far-reaching manner, as he and Krein have succeeded in doing it for the Michell ship — now a classical problem which caused so much discussion. Obviously, further work should be done on nonrectangular domains of pressure distributions and on combinations of such domains. Further, his remarks on ship forms and singularity (source-line) distributions open the field for a much needed treatment of the resistance of moderately fast and high-speed forms, including fast displacement ships with transom sterns.

I have two questions: Dr. Bessho's  $g$  equals my speed parameter  $\gamma_0 = 1/2F^2$ . How is this  $g$  defined, and how is the condition obtained given by Eq. (27)?

\* \* \*

## REPLY TO DISCUSSION

Masatoshi Bessho

I thank Prof. Weinblum for his kind remarks and will clarify his questions. The definition of  $g$  in my paper is as follows: Suppose the uniform velocity  $V$  is unity and take the unit length to be a half of the ship length  $L$ ; then the gravity constant  $g$  in this unit system is

$$g = \frac{g^*}{V^2} \left( \frac{L}{2} \right),$$

where  $g^*$  is the gravity constant in the usual unit system, so that it equals Prof. Weinblum's  $\gamma$ . In the latter part, the breadth of the planing surface is taken as twice unity, so that

$$g = \frac{g^*}{V^2} \left( \frac{B}{2} \right).$$

Equation (11) is derived as follows: We can introduce a regular function  $\pi$  such that

$$\frac{\partial}{\partial x} \pi(x, y, z) = - \left( \frac{\partial^2}{\partial x^2} + g \frac{\partial}{\partial z} \right) \phi(x, y, z)$$

having the surface value

$$\pi(x, y, -0) = \frac{1}{\rho} P(x, y).$$

On the other hand, if  $\phi$  can be calculated from a regular function  $M$  by the equation



$$\phi = - \left( \frac{\partial^2}{\partial x^2} - g \frac{\partial}{\partial z} \right) \frac{\partial}{\partial x} M(x, y, z),$$

where  $M(x, y, z) = -M(x, y, z)$ , then  $\phi$  has no radiating wave. Hence, putting the above into the first equation, we have

$$\begin{aligned} \pi(x, y, -0) &= \frac{1}{\rho} P(x, y) = \left( \frac{\partial^4}{\partial x^4} - g^2 \frac{\partial^2}{\partial z^2} \right) m(x, y) \\ &= \left[ \frac{\partial^4}{\partial x^4} - g^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] m(x, y), \end{aligned}$$

where  $m(x, y) = M(x, y, -0)$ . By the assumption, this is a wave-free pressure distribution.

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